The Bundle Theorem in fragments of projective Grassmann spaces

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Motivations and references

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Locally projective spaces which satisfy the Bundle Theorem
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The complement of a point subset in a projective space and a Grassmann space
J. Appl. Logic (to appear)

Objectives

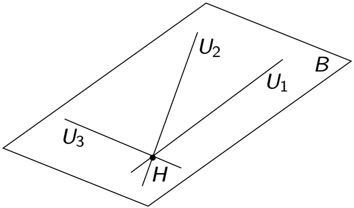
Our goal is to recover a deleted hyperplane from a projective Grassmann space.

Pencils

- V – a vector space of dimension n with $3 \leq n < \infty$
- $\mathrm{Sub}_k(V)$ – the set of all k-dimensional subspaces of V
- Assume that $0 < k < n$.

For $H \in \mathrm{Sub}_{k-1}(V)$, $B \in \mathrm{Sub}_{k+1}(V)$ with $H \subset B$ a k-pencil is a set of the form

$$\mathbf{p}(H, B) := \{U \in \mathrm{Sub}_k(V) \colon H \subset U \subset B\}.$$



Grassmann spaces

The point-line structure

$$\mathfrak{M} = \mathbf{P}_k(V) = \langle \mathrm{Sub}_k(V), \mathcal{P}_k(V) \rangle,$$

where $\mathcal{P}_V(k)$ is the family of all k-pencils, is a Grassmann space.

- If $0 < k < n$, then $\mathfrak{M}$ is a *partial linear space*.
- If $k = 1$ or $k = n - 1$, then $\mathfrak{M}$ is a *projective space*.

Basic properties and facts

- Every strong subspace of $\mathfrak{M}$ is a projective space.
- There are two disjoint classes of maximal strong subspaces in $\mathfrak{M}$.

Stars

For $H \in \mathrm{Sub}_{k-1}(V)$ a star is a set of the form
$$\mathrm{S}(H) = [H)_k = \{U \in \mathrm{Sub}_k(V) \colon H \subset U\}.$$

Tops

For $B \in \mathrm{Sub}_{k+1}(V)$ a top is a set of the form
$$\mathrm{T}(B) = (B]_k = \{U \in \mathrm{Sub}_k(V) \colon U \subset B\}.$$

- Every line of $\mathfrak{M}$ can be uniquely extended to a star and a top.

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Spine spaces

- W – a fixed subspace of V
- m – an integer such that $k - \operatorname{codim}(W) \le m \le k, \dim(W)$
- $\mathcal{F}_{k,m}(W) := \{U \in \mathrm{Sub}_k(V)\colon \dim(U \cap W) = m\}$
- $\mathcal{G}_{k,m}(W) := \{L \cap \mathcal{F}_{k,m}(W)\colon L \in \mathcal{P}_k(V) \text{ and } |L \cap \mathcal{F}_{k,m}(W)| \ge 2\}$

The point-line structure

$$\mathbf{A}_{k,m}(V, W) = \langle \mathcal{F}_{k,m}(W), \mathcal{G}_{k,m}(W) \rangle$$

is called a spine space.

K. Prażmowski, M. Żynel
Automorphisms of spine spaces,
Abh. Math. Sem. Univ. Hamb. **72** (2002), 59–77.

Bundles and planes in partial linear spaces

- The set $[U]$ of all lines through a point U is called a *bundle*.
- Given three lines L_1, L_2, L_3 that form a triangle in a partial linear space, the *plane* $\Pi(L_1, L_2, L_3)$ spanned by this triangle is the set of those points which lie on the lines that intersect two of those three lines L_1, L_2, L_3 in two distinct points.
- In linear spaces where the dimension function can be defined we can say that a plane is a subspace of dimension 2.
- Two lines $L_1, L_2 \in \mathcal{L}$ are said to be *coplanar*, which is written as $L_1 \ \pi \ L_2$, iff there is a plane containing them.

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Hyperplane complement in a Grassmann space

- W – a fixed subspace of V with $\operatorname{codim}(W) = k$
- $\mathcal{H}(W) := \{U \in \mathrm{Sub}_k(V) \colon U \cap W \neq 0\}$ – a hyperplane in $\mathfrak{M}$
- $S_{\mathcal{H}(W)} := \{\mathrm{Sub}_k(V) \setminus \mathcal{H}(W)\}$
- $\mathcal{L}_{\mathcal{H}(W)} := \{L \cap S_{\mathcal{H}(W)} \colon L \in \mathcal{P}_k(V) \text{ and } |L \cap S_{\mathcal{H}(W)}| \geq 2\}$

$$\mathfrak{D} = \mathfrak{D}_{\mathfrak{M}}(\mathcal{H}(W)) = \langle S_{\mathcal{H}(W)},\ \mathcal{L}_{\mathcal{H}(W)} \rangle$$

is the complement of the hyperplane $\mathcal{H}(W)$ in $\mathfrak{M}$.

Fact

In every hyperplane of the form $\mathcal{H}(W)$ in $\mathfrak{M}$ there are points collinear with no point outside $\mathcal{H}(W)$.

- $\mathfrak{D} = \langle \mathcal{F}_{k,0}(W), \mathcal{G}_{k,0}(W) \rangle = \mathbf{A}_{k,0}(V, W)$
- $\mathcal{H}(W) = \mathcal{F}_{k,1}(W) \cup \cdots \cup \mathcal{F}_{k,\min(k,n-k)}(W)$

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Key facts about the complement

Lemma

(i) *If* $L \in \mathcal{A}_{k,1}(W)$, *then* $\mathrm{S}(L), \mathrm{T}(L) \subseteq \mathcal{H}(W)$.

(ii) *If* $L \in \mathcal{L}^{\alpha}_{k,1}(W)$, *then* $\mathrm{T}(L) \nsubseteq \mathcal{H}(W)$ *and* $\mathrm{S}(L) \subseteq \mathcal{H}(W)$.

(iii) *If* $L \in \mathcal{L}^{\omega}_{k,1}(W)$, *then* $\mathrm{S}(L) \nsubseteq \mathcal{H}(W)$ *and* $\mathrm{T}(L) \subseteq \mathcal{H}(W)$.

In what follows we assume that $2 < k < n - 2$.

Corollary

For every point $U \in \mathcal{F}_{k,1}(W)$ *there are* $U_1, U_2, U_3 \in \mathcal{F}_{k,0}(W)$ *such that* U, U_1, U_2, U_3 *form a tetrahedron in* $\mathfrak{M}$.

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Bundle Theorem

- Both in the Grassmann space $\mathfrak{M}$ and in the hyperplane complement $\mathfrak{D}$ the following is true.

Theorem (Bundle Theorem)

Let L_1, L_2, L_3, L_4 be lines such that no three of them are coplanar. If five of the six pairs $\{L_i, L_j\}, 1 \leq i < j \leq 4$ are coplanar, then so is the sixth pair.

- In locally-projective linear spaces (cf. [Kreuzer, 1996]) this theorem is used to prove that two lines determine a bundle uniquely.

Maximal π-clique and bundles

- Let $\mathcal{K}$ be a maximal π-clique in $\mathfrak{D}$. There are two possibilities:
  (i) there is a point U in $\mathfrak{M}$ such that $U \in \overline{L}$ for all $L \in \mathcal{K}$, or
  (ii) there is a plane Π in $\mathfrak{D}$ such that $L \subset \Pi$ for all $L \in \mathcal{K}$.
- They can be distinguished by requirement that:
  ($*$) there are three non-coplanar lines in $\mathcal{K}$.
  The π-clique $\mathcal{K}$ is of type (i) iff it satisfies ($*$).
- There is a strong subspace in $\mathfrak{D}$ containing $\mathcal{K}$.
- Every bundle $[U]$ breaks up into π-cliques.

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Gluing π-cliques together

- X_1, X_2 – strong subspaces of different types in $\mathfrak{D}$

$$\mathcal{K}_\pi^{X_1}(U_1)\ \sigma\ \mathcal{K}_\pi^{X_2}(U_2) :\iff U_1, U_2 \in X_1 \cap X_2 \land \big(\exists\ L_1 \in \mathcal{K}_\pi^{X_1}(U_1)\big)\big(\exists\ L_2 \in \mathcal{K}_\pi^{X_2}(U_2)\big)\big[\, L_1, L_2 \neq X_1 \cap X_2 \land \mathrm{T}(L_1) \cap \mathrm{S}(L_2) \text{ or } \mathrm{S}(L_1) \cap \mathrm{T}(L_2) \text{ is a proper line distinct from } X_1 \cap X_2 \,\big]$$

- X_1, X_2 – strong subspaces of the same type in $\mathfrak{D}$

$$\mathcal{K}_\pi^{X_1}(U_1)\ \sigma\ \mathcal{K}_\pi^{X_2}(U_2) :\iff \big(\exists\ U \in \mathcal{F}_{k,0}(W) \cup \mathcal{F}_{k,1}(W)\big) (\exists \text{ a maximal strong subspace } X \text{ of the other type than } X_1, X_2) \big[\, U \in X \land \mathcal{K}_\pi^{X_1}(U_1)\ \sigma\ \mathcal{K}_\pi^{X}(U)\ \sigma\ \mathcal{K}_\pi^{X_2}(U_2) \,\big]$$

- σ is an equivalence relation; its equivalence classes are bundles.
- To every point $U \in \mathcal{F}_{k,0}(W) \cup \mathcal{F}_{k,1}(W)$ we can associate a unique bundle $[U]_{\mathfrak{D}}$ of lines of $\mathfrak{D}$.

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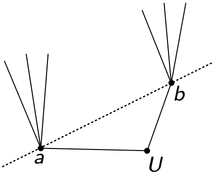
Adjacency of bundles

- We will write $\mathcal{B}(\mathfrak{D})$ for the set of all bundles in $\mathfrak{D}$.
- If $a \in \mathcal{B}(\mathfrak{D})$ we write $\bar{a}$ for the bundle of $\mathfrak{M}$ with $a \subseteq \bar{a}$.
- For a point U of $\mathfrak{D}$ and a bundle $a \in \mathcal{B}(\mathfrak{D})$ we write $U \sim a$ if there is a line $L \in a$ such that $U \in L$, we write $L = \overline{U, a}$.
- Frankly, two bundles $a, b \in \mathcal{B}(\mathfrak{D})$ are adjacent iff they share a line.

$$a \sim b :\iff \big(\exists\ U \in \mathcal{F}_{k,0}(W)\big)\big[\,U \sim a \wedge U \sim b \wedge \overline{U, a}\ \pi\ \overline{U, b}\,\big]$$



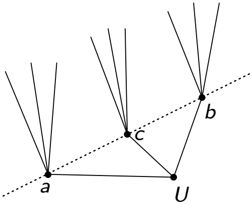
Lemma

For two distinct bundles $a, b \in \mathcal{B}(\mathfrak{D})$ we have $a \sim b$ iff there is a proper line L or $L \in \mathcal{L}^{\alpha}_{k,1}(W) \cup \mathcal{L}^{\omega}_{k,1}(W)$ such that $L \in \bar{a} \cap \bar{b}$.

Lines of bundles

- Given two distinct adjacent bundles $a, b \in \mathcal{B}(\mathfrak{D})$ we can define the new line through a, b:

$$[a, b] := \big\{c \in \mathcal{B}(\mathfrak{D})\colon \text{for all points } U \text{ of } \mathfrak{D} \text{ with } [U]_{\mathfrak{D}} \neq a, b, c \\ \text{if } U \sim a, b, \text{ then } U \sim c \text{ and } \overline{U, a}, \overline{U, b}, \overline{U, c} \text{ are coplanar}\big\}.$$



Lemma

If $a, b \in \mathcal{B}(\mathfrak{D})$, $a \neq b$, $a \sim b$, and $L \in \bar{a} \cap \bar{b}$, then

$$[a, b] = \big\{c \in \mathcal{B}(\mathfrak{D})\colon L \in \bar{c}\big\}.$$

Bundle space

- The set of all new lines will be written as

$$\mathfrak{L}(\mathfrak{D}) := \big\{[a,b]\colon\ a,b \in \mathcal{B}(\mathfrak{D}), a \sim b \text{ and } a \neq b\big\}.$$

- The structure

$$\mathfrak{B}(\mathfrak{D}) := \big\langle \mathcal{B}(\mathfrak{D}), \mathfrak{L}(\mathfrak{D}) \big\rangle$$

  is called the *bundle space* over $\mathfrak{D}$. It is a partial linear space.

Proposition

The bundle space $\mathfrak{B}(\mathfrak{D})$ is definable in terms of $\mathfrak{D}$.

- The bundle space $\mathfrak{B}(\mathfrak{D})$ is isomorphic to

$$\mathfrak{D}' = \big\langle \mathcal{F}_{k,0}(W) \cup \mathcal{F}_{k,1}(W),\ \mathcal{G}_{k,0}(W) \cup \mathcal{L}^{\alpha}_{k,1}(W) \cup \mathcal{L}^{\omega}_{k,1}(W) \big\rangle.$$

- This is a new proof of a know fact that $\mathfrak{D}'$ is definable in $\mathfrak{D}$.

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Affine lines on the horizon

- The affine lines $\mathcal{A}_{k,1}(W)$ are not yet recovered.
- Let Π be a plane in $\mathfrak{D}'$. There are two possibilities:

  (i) Π is a strong subspace of $\mathfrak{D}'$, then
  Π is a proper plane in $\mathfrak{D}$ or it is a projective plane in $\mathbf{A}_{k,1}(V, W)$,

  (ii) Π contains collinear and non-collinear points, then
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Proposition (Prażmowski and Żynel, 2002)

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Affine lines on the horizon

- The affine lines $\mathcal{A}_{k,1}(W)$ are not yet recovered.
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- Continuing this procedure we can recover $\mathbf{A}_{k,2}(V, W)$, then $\mathbf{A}_{k,3}(V, W)$, and so on.
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Summary and final remarks

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Thank you for your attention