Afinizacja produktu Segre zanurzalnych przestrzeni Grassmanna: zastosowania i przykłady

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Motivations and objectives

- The goal is to give examples and show applications of the theory developed in

  K. Petelczyc, M. Żynel
  Affinization of Segre products of partial linear spaces
  to appear in Bulletin of the Iranian Mathematical Society

- We focus on the Segre product which factors are Grassmann spaces embeddable into projective spaces.

Grassmann spaces

- V – a vector space
- $\mathrm{Sub}_k(V)$ – the set of all k-dimensional subspaces of V
- We assume that $0 < k < \dim(V)$

For $H \in \mathrm{Sub}_{k-1}(V)$, $B \in \mathrm{Sub}_{k+1}(V)$ with $H \subset B$ a *k-pencil* is a set of the form

$$\mathbf{p}(H, B) := \{U \in \mathrm{Sub}_k(V)\colon H \subset U \subset B\}.$$

The point-line structure

$$\mathbf{P}_k(V) = \langle \mathrm{Sub}_k(V), \mathcal{P}_k(V) \rangle,$$

where $\mathcal{P}_V(k)$ is the family of all k-pencils, is a *Grassmann space*.

- $\mathbf{P}_k(V)$ is a *partial linear space*.
- If $k = 1$ or $k = \dim(V) - 1$ (when $\dim(V) < \infty$), then $\mathbf{P}_k(V)$ is a *projective space*.

Polar Grassmann spaces

- ξ – a reflexive bilinear form on V
- $Q_k(\xi) = \{U \in \mathrm{Sub}_k(V) \colon \xi(U,U) = 0\}$

For $H \in \mathrm{Sub}_{k-1}(V)$, $B \in Q_{k+1}(\xi)$ with $H \subset B$ an *isotropic k-pencil* is a set of the form

$$\mathbf{p}_\xi(H,B) := \mathbf{p}(H,B) \cap Q_k(\xi).$$

The point-line structure

$$\mathbf{P}_k(\xi) := \langle Q_k(\xi), \mathcal{G}_k(\xi) \rangle,$$

where $\mathcal{G}_k(\xi)$ is the family of all isotropic k-pencils, is a *polar Grassmann space*.

- $\mathbf{P}_1(\xi)$ is a *polar space*

Hyperplanes in embeddable Grassmann spaces

Theorem (E. E. Shult / B. De Bruyn)

If D is a field, i.e. $\mathbf{P}_k(V)$ is embeddable into a projective space, then $\mathcal{H}$ is a hyperplane in $\mathbf{P}_k(V)$ iff

$$\mathcal{H} = \left\{ U \in \mathrm{Sub}_k(V) \colon \mu(u_1, \ldots, u_k) = 0 \text{ for all } u_1, \ldots, u_k \in U \right\}$$

for some non-zero k-linear alternating form μ.

E. E. Shult
Geometric hyperplanes in embeddable Grassmannians
J. Algebra **145** (1992), 55–82

B. De Bruyn
Hyperplanes of embeddable Grassmannians arise from projective embeddings: a short proof
Linear Algebra Appl. **430** (2009), 418–422

The Segre product

- $V_1, \dots, V_n$ – vector spaces over a field D of characteristic not 2
- $k = k_1 + \cdots + k_n$ for some positive integers $k_1, \dots, k_n$
- $V := \times_{i=1}^{n} V_i^{k_i} = \underbrace{\underbrace{V_1, \dots, V_1}_{k_1}, \dots, \underbrace{V_n, \dots, V_n}_{k_n}}_{k}$

Convention

$$u^i = [u^i_1, \dots, u^i_{k_i}] \in V_i^{k_i} \quad \text{and} \quad u = (u^1, \dots, u^n) \quad \text{for} \quad u \in V$$

Our Segre product

$$\mathfrak{M} = \mathfrak{M}_{k_1,\dots,k_n}(V_1, \dots, V_n) := \mathbf{P}_{k_1}(V_1) \otimes \cdots \otimes \mathbf{P}_{k_n}(V_n)$$

Semilinear and segment-wise alternating map

Consider a mapping $\mu\colon V \to D$ with the following properties:

- linear w.r.t. vector addition:

$$\mu(u_1, \dots, u_i + w_i, \dots, u_k) = \mu(u_1, \dots, u_i, \dots, u_k) + \mu(u_1, \dots, w_i, \dots, u_k)$$

- semilinear w.r.t. scalar multiplication:

$$\mu(u_1, \dots, \alpha u_i, \dots, u_k) = \alpha^{\sigma_i} \mu(u_1, \dots, u_i, \dots, u_k)$$

for some automorphism σ_i of D and all $\alpha \in D$, $i = 1, \dots, k$

- alternating on every of n segments w.r.t. $k_1, \dots, k_n$:

$$\mu(u_1, \dots, u_{j_1}, \dots, u_{j_2}, \dots, u_k) = -\mu(u_1, \dots, u_{j_2}, \dots, u_{j_1}, \dots, u_k)$$

for all $i = 1, \dots, n$ and j_1, j_2 such that

$$k_1 + \dots + k_{i-1} < j_1 < j_2 \leq k_1 + \dots + k_i$$

Remark

We have $\sigma_{j_1} = \sigma_{j_2}$ for j_1, j_2 within one segment. Thus there could be up to n field automorphisms σ_i associated with μ.

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The map μ on i-th segment

- For $u \in V$ define a map $\mu_i^{[u]}\colon V_i^{k_i} \longrightarrow D$ by setting

$$\mu_i^{[u]}(x^i) := \mu(u^1, \ldots, u^{i-1}, x^i, u^{i+1}, \ldots, u^n).$$

- It is an alternating k_i-semilinear form on V_i associated with some field automorphism σ_i.

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The hyperplane determined by μ

- Let us define

$$\mathcal{H}_{k_1,\dots,k_n}(\mu) := \Big\{ \big(\langle u^1\rangle,\dots,\langle u^n\rangle\big) \in \mathrm{Sub}_{k_1}(V_1)\times\dots\times\mathrm{Sub}_{k_n}(V_n) \colon \mu(u^1,\dots,u^n)=0 \Big\}$$

- Notation: $\mathcal{H}_{k_1,\dots,k_n}(\mu) = \mathcal{H}(\mu)$
- For all $u \in V$ such that

$$U := \big(\langle u^1\rangle,\dots,\langle u^n\rangle\big) \in \mathrm{Sub}_{k_1}(V_1)\times\dots\times\mathrm{Sub}_{k_n}(V_n)$$

we have

$$\mathcal{H}(\mu_i^{[u]}) = \Big\{ \langle x^i\rangle \in \mathrm{Sub}_{k_i}(V_i) \colon \mu_i^{[u]}(x^i) = 0 \Big\} = \Big\{ X \in \mathrm{Sub}_{k_i}(V_i) \colon U[i/X] \in \mathcal{H}(\mu) \Big\} = \mathcal{H}(\mu)_i^{[U]}$$

Proposition

The set $\mathcal{H}(\mu)$ is either a hyperplane in $\mathfrak{M}$ or all of $\mathfrak{M}$.

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Non-zero forms

We say that μ is *non-zero on i-th segment* when for all $u \in V$ such that

$$u^j \text{ is a linearly independent system in } V_j, \text{ where } 1 \leq j \leq n,\ j \neq i \quad (*_i)$$

there is $x^i \in V_i^{k_i}$ with $\mu_i^{[u]}(x^i) \neq 0$.

Proposition

If the form μ is non-zero on at least one of n segments, then $\mathcal{H}(\mu)$ is a hyperplane in $\mathfrak{M}$. If μ is non-zero on all n segments, then $\mathcal{H}(\mu)$ is a non-degenerate hyperplane in $\mathfrak{M}$.

Corollary

There is a non-degenerate hyperplane in $\mathfrak{M}$.

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Fact

Let $n = k = 1$, so $\mathfrak{M}$ is a projective space, and let ξ be a bilinear reflexive form on V. Then $\mathcal{H}$ is a hyperplane in the polar space $\mathbf{P}_1(\xi)$ iff $\mathcal{H} = \mathcal{H}(\mu) \cap Q_1(\xi)$ for some non-zero $\mu \in V^*$ such that $Q_1(\xi) \nsubseteq \mathcal{H}(\mu)$.

Proposition

Let ξ_i be a bilinear reflexive form on V_i for $i = 1, \ldots, n$. Assume that μ and $\xi_1, \ldots, \xi_n$ satisfy the following condition:

$$\text{if } \langle u^j \rangle \in Q_{k_j}(\xi_j) \text{ for } j \neq i, \text{ then } \mathcal{H}(\mu_i^{[u]}) \cap Q_{k_i}(\xi_i)$$
$$\text{is neither empty nor a single point for all } u \in V,\ i = 1, \ldots, n.$$

If μ is non-zero on all n segments and $Q_{k_1}(\xi_1) \times \cdots \times Q_{k_n}(\xi_n) \nsubseteq \mathcal{H}(\mu)$, then

$$\mathcal{H}(\mu, \xi_1, \ldots, \xi_n) = \mathcal{H}(\mu) \cap \left(Q_{k_1}(\xi_1) \times \cdots \times Q_{k_n}(\xi_n)\right)$$

is a non-degenerate hyperplane in $\mathbf{P}_{k_1}(\xi_1) \otimes \cdots \otimes \mathbf{P}_{k_n}(\xi_n)$.

The Segre product of polar Grassmann spaces

Fact

Let $n = k = 1$*, so* $\mathfrak{M}$ *is a projective space, and let* ξ *be a bilinear reflexive form on* V*. Then* $\mathcal{H}$ *is a hyperplane in the polar space* $\mathbf{P}_1(\xi)$ *iff* $\mathcal{H} = \mathcal{H}(\mu) \cap Q_1(\xi)$ *for some non-zero* $\mu \in V^*$ *such that* $Q_1(\xi) \not\subseteq \mathcal{H}(\mu)$*.*

Proposition

Let ξ_i *be a bilinear reflexive form on* V_i *for* $i = 1, \ldots, n$*. Assume that* μ *and* $\xi_1, \ldots, \xi_n$ *satisfy the following condition:*

$$\text{if } \langle u^j \rangle \in Q_{k_j}(\xi_j) \text{ for } j \neq i, \text{ then } \mathcal{H}(\mu_i^{[u]}) \cap Q_{k_i}(\xi_i)$$
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If μ *is non-zero on all* n *segments and* $Q_{k_1}(\xi_1) \times \cdots \times Q_{k_n}(\xi_n) \not\subseteq \mathcal{H}(\mu)$*, then*

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Non-degenerate forms: Gelfand, Kapranov, Zelevinsky

The form μ is *GKZ non-degenerate* if for all $u \in V$ linearly independent on all n segments there is $i \in \{1, \ldots, k\}$ and $v \in V$ such that

$$\mu(u_1, \ldots, u_{i-1}, v_i, u_{i+1} \ldots, u_k) \neq 0.$$

The form μ is GKZ non-degenerate iff the hyperdeterminant of the multidimensional matrix associated with μ is non-zero.

I. M. Gelfand, M. M. Kapranov, A. V. Zelevinsky
Discriminants, resultants and multidimensional determinants
Birkhäuser, Boston, 1994

Remark

If $k_1 = \cdots = k_n = 1$, i.e. if $\mathfrak{M}$ is the Segre product of projective spaces, then the form μ is GKZ non-degenerate iff $\mathcal{H}(\mu)$ is spiky.

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Non-degenerate forms: Hall

The form μ is *non-degenerate on i-th segment in the sense of Hall* if for all $u \in V$ satisfying $(*_i)$ any non-zero $x_1^i \in V_i$ can be completed with $x_2^i, \ldots, x_{k_i}^i \in V_i$ so that $\mu_i^{[u]}(x^i) \neq 0$.

J. I. Hall
Alternating forms and transitive locally grid geometries
European J. Combin. **28** (2007), 1473–1492

Non-degenerate forms: our approach

We say that μ is *non-degenerate on i-th segment* when for all $u \in V$ satisfying $(*_i)$ any linearly independent system $x^i_1, \ldots, x^i_{k_i-1} \in V_i$ can be completed with $x^i_{k_i} \in V_i$ so that $\mu^{[u]}_i(x^i) \neq 0$.

Remark

In case $n = 1$, i.e. for Grassmann spaces, if μ is non-degenerate, then μ is non-degenerate in the sense of Hall, while the inverse is true only for $k \leq 2$.

Lemma

If μ is non-degenerate on i-th segment, then $\mathcal{H}(\mu^{[u]}_i)$ is a flappy hyperplane in $\mathbf{P}_{k_i}(V_i)$ for all $u \in V$ satisfying $(*_i)$.

Proposition

If μ is non-degenerate on all n segments, then $\mathcal{H}(\mu)$ is a flappy hyperplane in $\mathfrak{M}$.

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If μ is non-degenerate on all n segments, then $\mathcal{H}(\mu)$ is a flappy hyperplane in $\mathfrak{M}$.

The Segre product of polar spaces

Proposition

Assume that $k = n$ i.e. $k_1 = \cdots = k_n = 1$ or $\mathfrak{M}$ is the Segre product of projective spaces.
If ξ_i is a non-degenerate symplectic bilinear form on V_i for $i = 1, \ldots, n$ and μ is non-zero (or equivalently non-degenerate in this case) on all n segments, then $\mathcal{H}(\mu, \xi_1, \ldots, \xi_n)$ is a flappy hyperplane in the Segre product $\mathbf{P}_1(\xi_1) \otimes \cdots \otimes \mathbf{P}_1(\xi_n)$.

Proposition

If μ is a non-degenerate symplectic bilinear form on V (i.e. $n = 1$, $k = 2$), then the set $Q_2(\mu)$ of all isotropic 2-subspaces of V w.r.t. μ is a flappy hyperplane in $\mathbf{P}_2(V)$.

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The Segre product of two projective spaces

Proposition

Let V_1, V_2 be vector spaces over a division ring D. Then the following conditions are equivalent:

(i) $\mathcal{H}$ *is a hyperplane in* $\mathfrak{M}_{1,1}(V_1, V_2)$.

(ii) *There is a sesquilinear form* $\xi\colon V_1 \times V_2 \longrightarrow D$ *which determines a conjugacy* $\perp$ *by the condition that*

$$\langle u_1 \rangle \perp \langle u_2 \rangle \quad \textit{iff} \quad \xi(u_1, u_2) = 0 \quad \textit{for all} \quad u_1 \in V_1, u_2 \in V_2$$

and $\mathcal{H} = \{(p, q)\colon p \perp q\}$ *(actually* $\mathcal{H} = \perp$*).*

Some remarks

- If $k = n > 2$ i.e. $\mathfrak{M}$ is the Segre product of more than 2 projective spaces and if μ is alternating, then $\mathcal{H}(\mu)$ is non-spiky and thus non-flappy.
- In the Segre product of two projective spaces a hyperplane is flappy iff it is given by some non-degenerate alternating 2-semilinear form μ.
- It is no longer true if the number of factors is more than two.

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Hyperplanes in general Grassmann spaces

- V – finite-dimensional vector space
- W – a fixed subspace of V with $\operatorname{codim}(W) = k$
- $\mathcal{H}(W) := \{U \in \mathrm{Sub}_k(V)\colon U \cap W \neq 0\}$ – a hyperplane in $\mathbf{P}_k(V)$ regardless of whether it is embeddable or non-embeddable

J. I. Hall, E. E. Shult
Geometric hyperplanes of non-embeddable Grassmannians
European J. Combin. **14** (1993), 29–35

Lemma

All hyperplanes of the form $\mathcal{H}(W)$ in $\mathbf{P}_k(V)$ are non-spiky.

Proposition

For integers k_1, k_2 such that $1 < k_1 < \dim(V) - 1$ and $k_1 + k_2 = \dim(V)$

$$\mathcal{H}_{k_1,k_2}(V) := \{(U_1, U_2) \in \mathrm{Sub}_{k_1}(V) \times \mathrm{Sub}_{k_2}(V)\colon U_1 \cap U_2 \neq 0\}$$

is a non-degenerate non-spiky hyperplane in $\mathfrak{M}_{k_1,k_2}(V, V)$.

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Thank you for your attention