Hall triple systems

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XIX-th Conference Geometry, Graphics, Computer, Ustroń 2012

Motivations and references

Steiner J.,
Combinatorische Aufgabe,
Journal für die Reine und Angewandte Mathematik, **45** (1853), 181–182.

M. Hall, Jr.,
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Kirkman's schoolgirl problem

The problem (1850)

Fifteen young ladies in a school walk out three side by side for seven days in succession: it is required to arrange them daily so that no two shall walk twice side by side.

One of solutions

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
01, 06, 11	01, 02, 05	02, 03, 06	05, 06, 09	03, 05, 11	05, 07, 13	11, 13, 04
02, 07, 12	03, 04, 07	04, 05, 08	07, 08, 11	04, 06, 12	06, 08, 14	12, 14, 05
03, 08, 13	08, 09, 12	09, 10, 13	12, 13, 01	07, 09, 15	09, 11, 02	15, 02, 08
04, 09, 14	10, 11, 14	11, 12, 15	14, 15, 03	08, 10, 01	10, 12, 03	01, 03, 09
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Definition of a Steiner triple system

Definition (Basic)

A Steiner Triple System (STS) is a structure $\langle S, \mathcal{L} \rangle$, where S is a set whose elements are called *points*, and the set $\mathcal{L}$ consists of triples of S called *lines* (*blocks*) such that every pair of points appears in a unique line.

aaa

A Steiner triple system of n points is said to be of *order* n and is denoted shortly by STS(n).

Definition (General)

A Steiner system with parameters t, k, n, written S(t, k, n), is an n-element set S together with a set of k-element subsets of S called *blocks* with the property that each t-element subset of S is contained in exactly one block.

A Steiner triple system STS(n) is a S$(2, 3, n)$ Steiner system.

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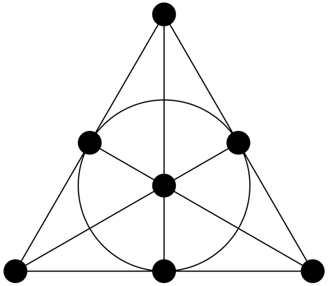
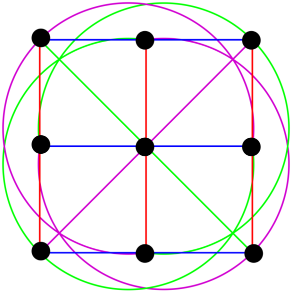
Examples: Finite projective plane



Projective plane of order 2 (Fano configuration), $\mathrm{PG}(2,2)$, $\mathrm{STS}(7)$.

Generally: a finite projective plane of order q, with the lines as blocks, is a Steiner system $\mathrm{S}(2, q+1, q^2+q+1)$.

Examples: Finite affine plane



Affine plane of order 3, AG(2, 3), STS(9).

Generaly: a finite affine plane of order q, with the lines as blocks, is a Steiner system $\mathrm{S}(2, q, q^2)$.

Known facts

Theorem

A Steiner triple system of order n exists if and only if

$$n \equiv 1, 3 \mod 6.$$

Known Steiner triple systems

order	non-isomorphic STSs
3	1
7	1
9	1
13	2
15	80
19	11 084 874 829

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Definition of a Hall triple system

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A Hall Triple System (HTS) is a Steiner triple system where for every point x there is a reflection in x (i.e. an involutory automorphism that fixes exactly the point x).

Theorem

A Steiner triple system is a Hall triple system iff any subsystem generated by three non-collinear points is an affine plane $\mathrm{AG}(2,3)$.

Definition

An Affine Triple System (ATS) is an affine space over GF(3). It is a Steiner triple system.

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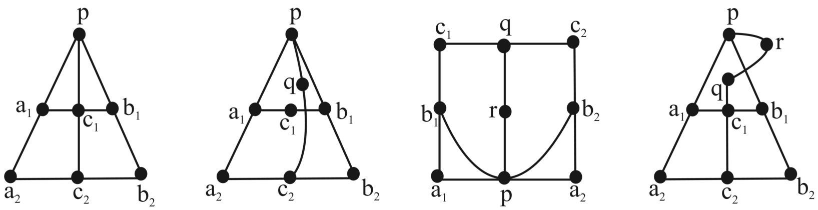
Mitre and variations of anti-mitre configurations

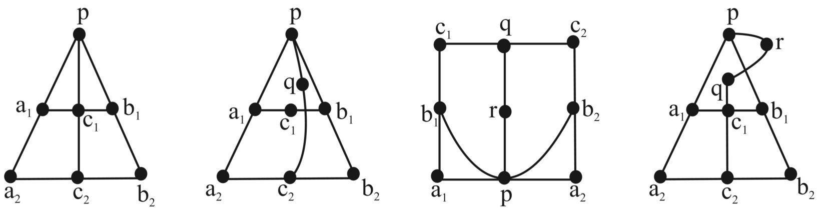


Mitre, anti-mitre C_A, C_S^2, and another visualization of C_S^2.

Theorem

An STS contains C_S^2 iff it contains C_A.

Mitre and variations of anti-mitre configurations



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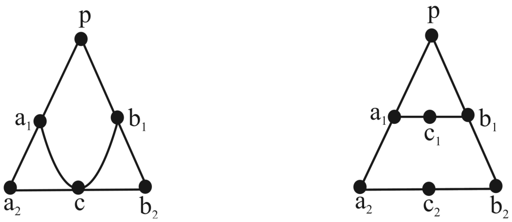
Pasch and anti-Pasch configurations

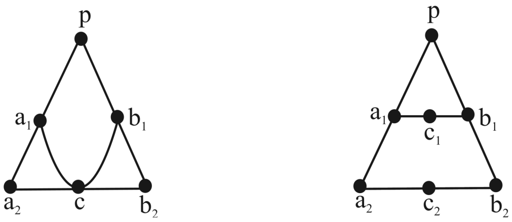


Pasch configuration C_{16} and anti-Pasch configuration C_{14}.

Theorem

An STS is an HTS iff it does not contain any C_{16} or C_S^2 subconfiguration.

Pasch and anti-Pasch configurations



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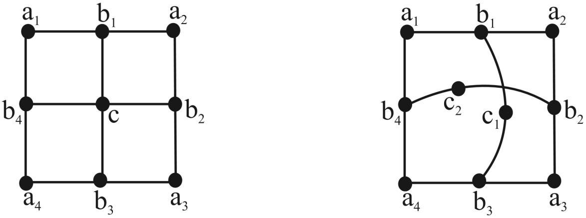
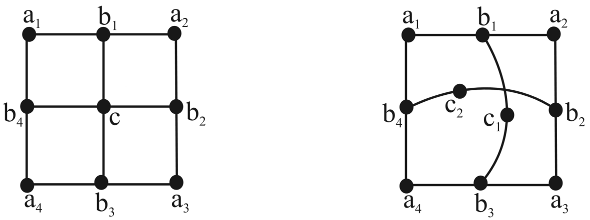


Net configuration and anti-net configuration C^1_S.

Theorem

An HTS is affine iff it does not contain any C^1_S subconfiguration.

Net and anti-net configurations



Net configuration and anti-net configuration C_S^1.

Theorem

An HTS is affine iff it does not contain any C_S^1 subconfiguration.

Steiner quasigroup

The-third-point-on-a-line operation

Using a Steiner triple system $\langle S, \mathcal{L} \rangle$ we can define a multiplication $\circledcirc$ on the set S by setting

- $a \circledcirc a = a$ for all $a \in S$, and
- $a \circledcirc b = c$ if $\{a, b, c\} \in \mathcal{L}$.

The multiplication $\circledcirc$ so defined has the additional property that:

$$a \circledcirc (b \circledcirc a) = b.$$

Hence we get $\langle S, \circledcirc \rangle$ an idempotent, totally symmetric (and thus commutative) quasigroup.

Conversely, every (finite) idempotent, totally symmetric quasigroup arises from a Steiner triple system.

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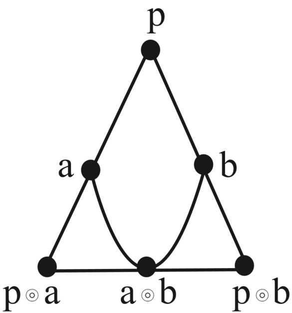
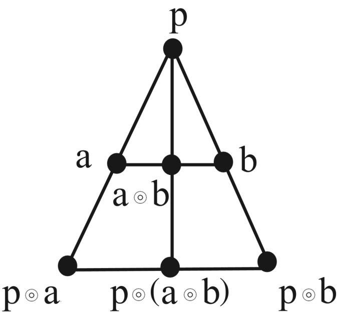
Pasch axiom as an algebraic law



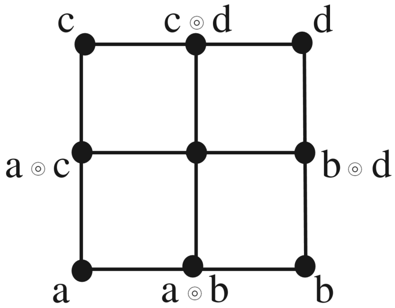
$$a \circledcirc b = (p \circledcirc a) \circledcirc (p \circledcirc b)$$

Hall requirement as the distributivity law



$$p \circledcirc (a \circledcirc b) = (p \circledcirc a) \circledcirc (p \circledcirc b)$$

Net axiom as an algebraic law



$$(a \circledcirc b) \circledcirc (c \circledcirc d) = (a \circledcirc c) \circledcirc (b \circledcirc d) \qquad (*)$$

Proposition

An STS is affine iff it satisfies $(*)$.

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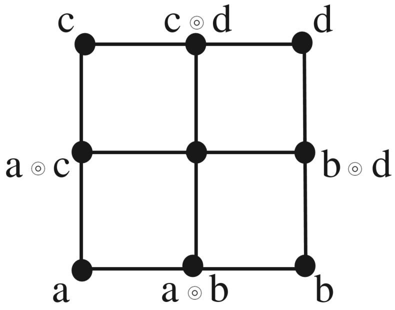

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Characterization of HTSs in terms of trilinear forms

Trilinear form

Let K be a commutative field of characteristic not 2 and let $V = V(n, K)$ be a n-dimensional vector space over K. A mapping

$$t \colon V \times V \times V \longrightarrow K$$

linear in each of its variables is called a *trilinear* form.

A trilinear form t is said to be *symplectic* when

$$t(x, x, y) = t(x, y, x) = t(y, x, x) = 0.$$

Theorem

There is a one-to-one correspondence between non-equivalent trilinear symplectic forms over $V(n, \mathrm{GF}(3))$ *and Hall triple systems of order* 3^{n+1} *and rank* $n + 1$.

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Thank you for your attention