

Functional Analysis
Exam topics, 30-06-2025

1. Normed spaces and Banach spaces, completeness of closed subspaces (proof), completion, convergence of series in Banach spaces.
2. Characterizations of equivalence of norms (proof), equivalence of norms in finite-dimensional spaces (proof) and conclusions (completeness of finite-dimensional spaces and automatic boundedness of operators on such spaces).
3. Spaces of continuous functions (bounded and vanishing at infinity): completeness, uniform convergence, spaces of sequences.
4. L^p -spaces, for $p \in [1, \infty]$, Hölder's inequality (proof), Minkowski's inequality (proof), completeness (proof), spaces of sequences.
5. Bounded operators: definition and examples, continuity of operators (proof), operator norm, space of bounded operators (completeness proof).
6. Hilbert spaces: definition, Schwartz inequality (proof), characterization as Banach spaces satisfying the parallelogram identity, examples.
7. Orthogonal projection: definition, existence and uniqueness (proof), projection as an operator.
8. Orthonormal bases: existence and characterizations of an orthonormal basis (proof), characterization of a Hilbert space as $\ell_2(I)$, separable Hilbert spaces, dimension of a Hilbert space.
9. Adjoint operators: characterisation of bounded linear functionals on Hilbert spaces (proof), existence and uniqueness of an adjoint operator (proof), properties of adjoints (proof), various classes of operators defined in terms of adjoints.
10. Hahn–Banach theorem (proof): relation between functionals over $\mathbb{R}$ and $\mathbb{C}$ (proof), corollaries of Hahn–Banach theorem (proof), notion of reflexive space.
11. Dual spaces to Banach spaces: dual spaces to L^p -spaces, examples of reflexive and nonreflexive spaces.
12. Banach–Steinhaus theorem (proof): Baire theorem, boundedness of a pointwise limit of a bounded operator (proof). Weak topology and boundedness of a weakly convergent sequence, examples.
13. Open operator theorem (proof): conclusions (proof), Closed graph theorem (proof).
14. Banach algebras and invertible elements: definition and examples, Neumann's lemma (proof), openness of the set of invertible elements
15. Spectrum and spectral radius: definition and examples, basic properties, formula for the spectral radius