Mathematical Notes, vol. 0, no. 0, 1966, p. 1.
Translated from Matematicheskie Zametki, vol. 0, *no.* 0, 1966, *pp.* 12–26.

Extensions of C^*-dynamical systems to systems with complete transfer operators

B. K. Kwaśniewski

Received March 20, 2005; in final form, January 17, 2017

Abstract—Starting from an arbitrary endomorphism α of a unital C^*-algebra $\mathcal{A}$ we construct in a canonical way a bigger algebra $\mathcal{B}$ and extend α onto $\mathcal{B}$ in such a way that $\alpha : \mathcal{B} \to \mathcal{B}$ possess a unique non-degenerate transfer operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ called complete transfer operator. The pair $(\mathcal{B}, \alpha)$ is universal with respect to a suitable notion of a covariant representation and in general depends on a choice of an ideal in $\mathcal{A}$.

1. INTRODUCTION

The crossed-product of a C^*-algebra $\mathcal{A}$ by an automorphism $\alpha : \mathcal{A} \to \mathcal{A}$ is defined as a universal C^*-algebra generated by a copy of $\mathcal{A}$ and a unitary element U satisfying the relations

$$\alpha(a) = UaU^*, \qquad \alpha^{-1}(a) = U^*aU, \qquad a \in \mathcal{A}.$$

Algebras arising in this way (or their versions adapted to actions of groups of automorphisms) are very well understood and became one of the standard constructions in C^*-theory. On the other hand, the natural desire to adapt this kind of constructions to endomorphisms (or semigroups of endomorphisms) encounters, from the very beginning, serious obstacles. Roughly speaking, it is caused by the irreversibility of the system $(\mathcal{A}, \alpha)$ - the lack of α^{-1} in the case α is an endomorphism.

The difficulty of the matter manifests itself in a variety of approaches, see, for example, [1], [2], [3], [4], [5], [6], [7], [8], which do however have a certain nontrivial intersection. They mostly agree, and simultaneously boast their greatest successes, in the case when the dynamics is implemented by a monomorphism with hereditary range. In view of [9], [7], see also [10], this coincidence seems to be well understood. Namely, it was noticed in [9], [7] that for a class of endomorphisms α (as shown in [10] consisting of endomorphisms with complemented kernels and hereditary ranges) there exist unique non-degenerate transfer operators $\mathcal{L}$ [6], called by authors of [9] *complete transfer operators*. In this case the theory goes smooth, in the spirit very similar to that of crossed products by automorphisms, as $\mathcal{L}$ takes over the role classically played by α^{-1}. The authors of [7] showed that all of the aforementioned constructions can be reduced to crossed product for systems $(\mathcal{A}, \alpha)$ with complete transfer operators, and at the end of [7] they argue that a general crossed product construction should consist of two steps:

1) “initial object and extension procedure”;
2) “crossed product for systems with complete transfer operator”.

Our goal is to provide the missing first step in the above scheme where the initial object is an arbitrary endomorphism of a unital C^*-algebra. The previous preprint version of the present article together with [7] enabled the authors of [11] to develop a general approach to crossed products by arbitrary endomorphisms. We also hope to use the elaborated C^*-algebraic extension method in the further more detailed analysis of objects of this type, cf., e.g., [11, problem on p. 1830]. Such an approach already proved to be very useful in the case the initial system is commutative, see [8], [12], [13].

The idea behind our construction is very similar to that of dilation of endomorphisms up to automorphisms, see for instance [14] and references therein. However, the aforementioned method applies only to injective endomorphisms and yield crossed products that are equivalent only up to the Morita equivalence, whereas our extension procedure is general (in the case the underlying semigroup is $\mathbb{N} = \{0, 1, 2, 3, ...\}$) and yields isomorphic crossed products, see Theorem 2 ii) below.

This article is a journal version of the e-print arXiv:math.OA/0703800

2. THE PROBLEM AND THE SPATIAL OPERATOR-ALGEBRAIC CONSIDERATIONS

Throughout this paper we let $\mathcal{A}$ be a C^*-algebra with an identity 1. By a *C^*-dynamical system* we mean a pair $(\mathcal{A}, \alpha)$ where $\alpha : \mathcal{A} \to \mathcal{A}$ is an endomorphism of $\mathcal{A}$ (all the morphisms appearing in the text are assumed to be *-preserving). To explain the problem let us suppose that the system $(\mathcal{A}, \alpha)$ is faithfully represented on a Hilbert space H; that is we assume $\mathcal{A}$ is a C^*-subalgebra of the algebra $L(H)$ of all bounded linear operators on H, 1 is the identity operator, and there is $U \in L(H)$ such that

$$\alpha(a) = UaU^*, \qquad a \in \mathcal{A}.$$

By [15, Proposition 2.2] and [11, Lemma 1.2] the multiplicativity of α is equivalent to the conditions

$$U \text{ is a (power) partial isometry and } U^*U \in \mathcal{A}',$$

where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. Hence by [15, Proposition 2.2. and Proposition 3.10]

$$\mathcal{B} := \overline{span}\{U^{*n}aU^n : a \in \mathcal{A},\ n \in \mathbb{N}\},$$

is a minimal C^*-algebra containing $\mathcal{A}$ and such that the following relations hold

$$U\mathcal{B}U^* \subset \mathcal{B}, \qquad U^*\mathcal{B}U \subset \mathcal{B}, \qquad U^*U \in Z(\mathcal{B}) = \mathcal{B}' \cap \mathcal{B}.$$

Therefore, see also [9, 3.1] or [7, 2.5], putting

$$\alpha(b) := UbU^*, \qquad \mathcal{L}(b) := U^*bU, \qquad b \in \mathcal{B},$$

we obtain an endomorphism $\alpha : \mathcal{B} \to \mathcal{B}$ that extends $\alpha : \mathcal{A} \to \mathcal{A}$,[1] and a linear operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ which is a complete transfer operator for the extended system $(\mathcal{B}, \alpha)$ (we recall a definition of a complete transfer operator in section 3). In the present paper, we give a positive answer to the following question.

> **Question:** Does there exist an efficient description of the triple $(\mathcal{B}, \alpha, \mathcal{L})$ in terms of the initial C^*-dynamical system $(\mathcal{A}, \alpha)$, independent of the representation in $B(H)$?

Remark. By [15, Proposition 4.1] if $\mathcal{A}$ is commutative, then $\mathcal{B}$ is also commutative. Hence in this case the C^*-dynamical systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \alpha)$ correspond to topological dynamical systems (X, γ) and $(\widetilde{X}, \widetilde{\gamma})$ respectively (consisting of compact Hausdorff spaces and partial mappings). The description of $(\widetilde{X}, \widetilde{\gamma})$ only in terms of (X, γ) was obtained in [12] under the additional assumption that $U^*U \in \mathcal{A}$. In general, $(\widetilde{X}, \widetilde{\gamma})$ is described by (X, γ) in [13] but this description requires additional data encoded in the ideal

$$J := U^*U\mathcal{A} \cap \mathcal{A} = \{a \in \mathcal{A} : U^*Ua = a\}, \tag{2.1}$$

[1] As a rule we use the same symbol for endomorphisms and their extensions.

cf. also [11]. The relationship between (X,γ) and $(\widetilde{X},\widetilde{\gamma})$ is of particular interest. In [13], [12], [8], a number of examples is studied and it is shown that the space $\widetilde{X}$ may be viewed as a generalization of topological inverse limit space. As a rule $\widetilde{X}$ is topologically very complicated. In a typical situation it contains *indecomposable continua*, has a structure of *hyperbolic attractors*, or of a space arising from *substitution tilings*. Thus, among the other things, the construction of the present paper could be considered as a tool to obtain non-commutative counterparts of the aforementioned objects.

We will analyze the structure of $\mathcal{B}$ by means of the following "approximating" algebras

$$\mathcal{B}_n := \overline{\left\{\sum_{i=0}^{n} U^{*i}a_iU^i : a_i \in \mathcal{A}, i=0,...,n\right\}}, \qquad n\in\mathbb{N},$$

see [15, Proposition 3.8 (ii)]. The family $\{\mathcal{B}_n\}_{n\in\mathbb{N}}$ fixes the structure of a direct limit on $\mathcal{B}$:

$$\mathcal{A}=\mathcal{B}_0\subset\mathcal{B}_1\subset ...\subset\mathcal{B}_n\subset ..., \qquad \text{and} \qquad \mathcal{B}=\overline{\bigcup_{n\in\mathbb{N}}\mathcal{B}_n}.$$

The first crucial step is to notice that the algebras $\mathcal{B}_n$ can be canonically identified with direct sums of subalgebras of the C^*-algebra $C^*(\mathcal{A},U^*U)$ generated by $\mathcal{A}$ and U^*U.

Statement 1. *Let $n\in\mathbb{N}$. Every element $a\in\mathcal{B}_n$ can be presented in the form*

$$a = a_0 + U^*a_1U + ... + U^{*n}a_nU^n$$

where

$$a_i \in (1-U^*U)\alpha^i(1)\mathcal{A}\alpha^i(1), \quad i=0,...,n-1, \qquad a_n\in\alpha^n(1)\mathcal{A}\alpha^n(1), \tag{2.2}$$

and this form is unique. Actually, $a\mapsto a_0\oplus a_1\oplus ...\oplus a_n$ establishes the isomorphism

$$\mathcal{B}_n \cong (1-U^*U)\mathcal{A}\oplus(1-U^*U)\alpha^1(1)\mathcal{A}\alpha^1(1)\oplus ...\oplus\alpha^n(1)\mathcal{A}\alpha^n(1). \tag{2.3}$$

Proof. Let $a\in\mathcal{B}_n$. Then $a=\sum_{i=0}^n\mathcal{L}^i(b_i)$ where $b_i\in\mathcal{A}$ and $\mathcal{L}(\cdot)=U^*(\cdot)U$. Without loss of generality we may assume that $b_i\in\alpha^i(1)\mathcal{A}\alpha^i(1)$, because $\mathcal{L}^i(b_i)=\mathcal{L}^i(\alpha^i(1)b_i\alpha^i(1))$. We recall, cf. [15, Proposition 3.6 (iv)], that the family $\{\mathcal{L}^k(1)\}_{k\in\mathbb{N}}\subset Z(\mathcal{B})$ is a decreasing sequence of orthogonal projections. We will construct elements a_i satisfying (2.2) modifying inductively the elements b_i. For a_0 we take $b_0(1-\mathcal{L}(1))$ and 'the remaining part' of b_0 we include in b_1, that is we put $c_1=b_1+\alpha(b_0)$. Then $a=a_0+\mathcal{L}(c_1)+...+\mathcal{L}^n(b_n)$, because $b_0\mathcal{L}(1)=\mathcal{L}(\alpha(b_0))$.
Continuing in this manner we get $k<n$ coefficients $a_0,...,a_{k-1}$ satisfying (2.2) and such that $a=a_0+...+\mathcal{L}^{k-1}(a_{k-1})+\mathcal{L}^k(c_k)+\mathcal{L}^{k+1}(b_{k+1})+...+\mathcal{L}^n(b_n)$ and $c_k\in\alpha^k(1)\mathcal{A}\alpha^k(1)$. We put $a_k=c_k(1-\mathcal{L}(1))\in\mathcal{A}$ and $c_{k+1}=b_{k+1}+\alpha(c_k)$. Then $a_k\in(1-U^*U)\alpha^k(1)\mathcal{A}\alpha^k(1)$ and the following computations

$$\begin{aligned}\mathcal{L}^k(c_k)&=\mathcal{L}^k(c_k)\mathcal{L}^k(1)=\mathcal{L}^k(c_k)\big(\mathcal{L}^k(1)-\mathcal{L}^{k+1}(1)\big)+\mathcal{L}^{k+1}(1)\mathcal{L}^k(c_k)\\&=\mathcal{L}^k\big(c_k(1-\mathcal{L}(1))\big)+\mathcal{L}^{k+1}(\alpha(c_k))=\mathcal{L}^k(a_k)+\mathcal{L}^{k+1}(\alpha(c_k))\end{aligned}$$

show that $a=a_0+...+\mathcal{L}^k(a_k)+\mathcal{L}^{k+1}(c_{k+1})+...+\mathcal{L}^n(b_n)$.
Thus we may assume that (2.2) holds. These conditions imply that

$$\mathcal{L}^i(a_i)\in\big(\mathcal{L}^i(1)-\mathcal{L}^{i+1}(1)\big)\mathcal{A}, \qquad i=0,...,n-1.$$

Since $\{\mathcal{L}^k(1)\}_{k\in\mathbb{N}}\subset Z(\mathcal{B})$ are decreasing orthogonal projections, the projections $1-\mathcal{L}(1)$, $\mathcal{L}(1)-\mathcal{L}^2(1),...,\mathcal{L}^n(1)-\mathcal{L}^{n-1}(1)$, $\mathcal{L}^n(1)$ are pairwise orthogonal and central in $\mathcal{B}_n$. Hence the algebra $\mathcal{B}_n$ is a direct sum of ideals corresponding to these projections and i-th component of such a decomposition is isomorphic to $(1-U^*U)\alpha^i(1)\mathcal{A}\alpha^i(1)$, if $i=0,...,n-1$, and $\alpha^n(1)\mathcal{A}\alpha^n(1)$, if $i=n$. To see the latter it suffices to check that

$$U^iU^{*i}\mathcal{A}U^iU^{*i} = \alpha^i(1)\mathcal{A}\alpha^i(1) \ni a \to \mathcal{L}^i(a) = U^{*i}aU^i \in \mathcal{B}_n, \qquad i=1,...,n,$$

is injective homomorphism which follows immediately from the fact that U^i is a partial isometry. Accordingly, we get the isomorphism (2.3) and the proof is finished.

We note, cf. [13, Proposition 2.2] or [11, Proposition 6.2], that

$$C^*(\mathcal{A}, U^*U) = U^*U\mathcal{A} \oplus (1-U^*U)\mathcal{A} \cong \mathcal{A}/\ker\alpha \oplus \mathcal{A}/J$$

where J is the ideal (2.1). This indicates that the extended system $(\mathcal{B},\alpha,\mathcal{L})$ can be reconstructed from the triple $(\mathcal{A},\alpha,J)$ and we will show that this is indeed the case. This will be achieved in section 4, but first we fix indispensable notation and facts concerning transfer operators and covariant representations of C^*-dynamical systems.

3. TRANSFER OPERATORS AND COVARIANT REPRESENTATIONS

Let us fix a C^*-dynamical system $(\mathcal{A},\alpha)$. A *transfer operator* for $(\mathcal{A},\alpha)$, see [6], is a positive linear map $\mathcal{L}:\mathcal{A}\to\mathcal{A}$ such that

$$\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b), \qquad a,b\in\mathcal{A}. \tag{3.1}$$

If additionally, $\alpha(\mathcal{L}(1)) = \alpha(1)$ the transfer operator $\mathcal{L}$ is said to be *non-degenerate* [6]. The authors of [9] called a transfer operator $\mathcal{L}$ for $(\mathcal{A},\alpha)$ a *complete transfer operator* if it satisfies

$$\alpha(\mathcal{L}(a)) = \alpha(1)a\alpha(1), \qquad a\in\mathcal{A}. \tag{3.2}$$

By [10] it follows that a complete transfer operator exists if and only if $\ker\alpha$ is a complemented ideal in $\mathcal{A}$ and $\alpha(\mathcal{A})$ is hereditary subalgebra of $\mathcal{A}$ (equivalently $\ker\alpha$ is unital and $\alpha(\mathcal{A}) = \alpha(1)\mathcal{A}\alpha(1)$). Then, see [10], such a transfer operator is a unique non-degenerate transfer operator for $(\mathcal{A},\alpha)$ and it is given by the formula $\mathcal{L}(a) = \alpha^{-1}(\alpha(1)a\alpha(1))$ where α^{-1} is the inverse to the isomorphism $\alpha:(\ker\alpha)^\perp\to\alpha(\mathcal{A})$ and $(\ker\alpha)^\perp = \{a\in\mathcal{A}: a\ker\alpha = \{0\}\}$ is the *annihilator* of $\ker\alpha$.

Definition 1 (cf. [11]). A *representation* of $(\mathcal{A},\alpha)$ is a triple (π,U,H) consisting of a unital faithful representation $\pi:\mathcal{A}\to L(H)$ on a Hilbert space H and an operator $U\in L(H)$ satisfying

$$U\pi(a)U^* = \pi(\alpha(a)), \qquad a\in\mathcal{A}. \tag{3.3}$$

Then $J = \{a\in\mathcal{A}: U^*U\pi(a) = \pi(a)\}$ is an ideal in $\mathcal{A}$ contained in $(\ker\alpha)^\perp$, cf. [11, Corollary 1.5] or [13, Proposition 1.16]. We call J the *ideal of covariance* for (π,U,H), and say that (π,U,H) is a *J-covariant representation* of $(\mathcal{A},\alpha)$. If $J=(\ker\alpha)^\perp$, we simply say that (π,U,H) is a *covariant representation* of $(\mathcal{A},\alpha)$.

Remark. By [11, Proposition 1.10] for each system $(\mathcal{A},\alpha)$ and ideal J in $(\ker\alpha)^\perp$ there exists a J-covariant representation of $(\mathcal{A},\alpha)$.

The next two statements explain to some extent the role of covariant representations (without prefix J) and complete transfer operators.

Statement 2. *Let (π,U,H) be a representation of a C^*-dynamical system $(\mathcal{A},\alpha)$ such that* $\ker\alpha$ *has a unit (is a complemented ideal in $\mathcal{A}$). The following conditions are equivalent:*

i) (π,U,H) *is a covariant representation*
ii) $U^*U\in\pi(\mathcal{A})$
iii) $U^*U\in\pi(Z(\mathcal{A}))$ ($Z(\mathcal{A})$ *stands for the center of* $\mathcal{A}$)
iv) U^*U *is the unit in* $\pi((\ker\alpha)^\perp)$

In particular, if α is injective, then (π, U, H) is a covariant representation if and only if U is an isometry.

Proof. It is straightforward, as we know that $\pi(\ker\alpha) = (1-U^*U)\pi(\mathcal{A})\cap\pi(\mathcal{A})$ and $U^*U \in \pi(\mathcal{A})'$, see [11, Proposition 1.9].

Statement 3. *Let (π, U, H) be a representation of a C^*-dynamical system $(\mathcal{A}, \alpha)$ which admits a complete transfer operator $\mathcal{L} : \mathcal{A} \to \mathcal{A}$. Then (π, U, H) is a covariant representation if and only if*

$$\pi(\mathcal{L}(a)) = U^*\pi(a)U, \qquad a \in \mathcal{A}. \tag{3.4}$$

Proof. We recall, cf. [10, Proposition 1.5], that $\mathcal{L}(1)$ is the unit in $(\ker\alpha)^\perp$. Hence if (π, U, H) is satisfies (3.4), then $U^*U = \pi(\mathcal{L}(1))$ is the unit in $\pi(\ker\alpha^\perp)$ and (π, U, H) is the covariant representation of $(\mathcal{A}, \alpha)$ by Statement 2. Conversely, if (π, U, H) is a covariant representation of $(\mathcal{A}, \alpha)$, then $U^*U = \pi(\mathcal{L}(1)) \in \pi(Z(\mathcal{A}))$, again by Statement 2, and using (3.2) for $a \in \mathcal{A}$ we get

$$\begin{aligned} U^*\pi(a)U &= U^*(UU^*\pi(a)UU^*)U = U^*\pi(\alpha(1)a\alpha(1))U = U^*\pi(\alpha(\mathcal{L}(a)))U \\ &= U^*U\pi(\mathcal{L}(a))U^*U = \pi(\mathcal{L}(1))\pi(\mathcal{L}(a)) = \pi(\mathcal{L}(a)), \end{aligned}$$

which finishes the proof.

We can always reduce investigation of J-covariant representations to covariant representations (without prefix J) with the help of the following construction, cf. [11, 6.1], [13, 2.1.1].

Definition 2. Let $(\mathcal{A}, \alpha)$ be a C^*-dynamical system and let J be an ideal in $(\ker\alpha)^\perp$. We treat $\mathcal{A}$ as a C^*-subalgebra of

$$\mathcal{A}_J = \big(\mathcal{A}/\ker\alpha\big) \oplus \big(\mathcal{A}/J\big)$$

using the embedding $\mathcal{A} \ni a \longmapsto \big(a + \ker\alpha\big) \oplus \big(a + J\big) \in \mathcal{A}_J$. We define an extension of α up to $\mathcal{A}_J$, which we will still denote by α, by the formula

$$\mathcal{A}_J \ni (a + \ker\alpha) \oplus (b + J) \longrightarrow (\alpha(a) + \ker\alpha) \oplus (\alpha(a) + J) \in \mathcal{A}_J.$$

We call $(\mathcal{A}_J, \alpha)$ a *C^*-dynamical system obtained from $(\mathcal{A}, \alpha)$ by J-unitization of kernel.*

Remark. The kernel of the endomorphism $\alpha : \mathcal{A}_J \to \mathcal{A}_J$ has the unit given by $(0 + \ker\alpha) \oplus (1 + J)$, and the algebras $\mathcal{A}_J$ and $\mathcal{A}$ coincide if and only if $\ker\alpha$ is unital and $J = (\ker\alpha)^\perp$. This to some extent explains the terminology, cf. [11, Remark 6.1], [13, Remark 2.3]. In the commutative case passing from $(\mathcal{A}, \alpha)$ to $(\mathcal{A}_J, \alpha)$ corresponds to compactification of the complement of the image of a partial mapping described in [13, Proposition 2.4].

Statement 4. *Let $(\mathcal{A}_J, \alpha)$ be a C^*-dynamical system obtained by a J-unitization of the kernel of $\alpha : \mathcal{A} \to \mathcal{A}$. There is a one-to-one correspondence between J-covariant representations (π, U, H) of $(\mathcal{A}, \alpha)$ and covariant representations (π_J, U, H) of $(\mathcal{A}_J, \alpha)$ established by the equality*

$$\pi_J\big((a + \ker\alpha) \oplus (b + J)\big) = U^*U\pi(a) + (1 - U^*U)\pi(b). \tag{3.5}$$

In particular, for every J-covariant representation (π, U, H) of $(\mathcal{A}, \alpha)$ the algebra $\mathcal{A}_J$ is isomorphic to $C^*\big(U^*U, \pi(\mathcal{A})\big)$.

Proof. See [11, Proposition 6.2] or [13, Proposition 2.2].

4. MAIN CONSTRUCTION

For convenience, until Definition 4, we assume that the kernel of $\alpha : \mathcal{A} \to \mathcal{A}$ is unital and let q denote the unit in $\ker\alpha$ (in general situation we will pass through the system $(\mathcal{A}_J, \alpha)$ described in Definition 2). We put

$$\mathcal{A}_n := \alpha^n(1)\mathcal{A}\alpha^n(1), \qquad n \in \mathbb{N},$$

and define algebras $\mathcal{B}_n$ as direct sums of the form

$$\mathcal{B}_n := q\mathcal{A}_0 \oplus q\mathcal{A}_1 \oplus ... \oplus q\mathcal{A}_{n-1} \oplus \mathcal{A}_n, \qquad n \in \mathbb{N}. \tag{4.1}$$

In particular, $\mathcal{B}_0 = \mathcal{A}_0 = \mathcal{A}$. For each $n \in \mathbb{N}$ we let $\alpha_n : \mathcal{B}_n \to \mathcal{B}_{n+1}$ to be a homomorphism schematically presented by the diagram

$$\begin{tikzcd}
\mathcal{B}_n \arrow[d, "\alpha_n"] & = & q\mathcal{A}_0 \arrow[d, "id"] & \oplus & ... & \oplus & q\mathcal{A}_{n-1} \arrow[d, "id"] & \oplus & \mathcal{A}_n \arrow[d, "q"] \arrow[drr, "\alpha"] & & \\
\mathcal{B}_{n+1} & = & q\mathcal{A}_0 & \oplus & ... & \oplus & q\mathcal{A}_{n-1} & \oplus & q\mathcal{A}_n & \oplus & \mathcal{A}_{n+1}
\end{tikzcd}$$

and formally given by the formula

$$\alpha_n(a_0 \oplus ... \oplus a_{n-1} \oplus a_n) = a_0 \oplus ... \oplus a_{n-1} \oplus qa_n \oplus \alpha(a_n),$$

where $a_k \in q\mathcal{A}_k$, $k = 0, ..., n-1$, and $a_n \in \mathcal{A}_n$. Let us note that, since $a_n = qa_n + (1-q)a_n$ and $\alpha : (1-q)\mathcal{A} \to \alpha(\mathcal{A})$ is an isomorphism, homomorphism α_n is injective. We define $\mathcal{B} := \varinjlim \{\mathcal{B}_n, \alpha_n\}$ to be the direct limit of the direct sequence

$$\mathcal{B}_0 \xrightarrow{\alpha_0} \mathcal{B}_1 \xrightarrow{\alpha_1} \mathcal{B}_2 \xrightarrow{\alpha_2} ... \tag{4.2}$$

and denote by $\phi_n : \mathcal{B}_n \to \mathcal{B}$, $n \in \mathbb{N}$, the natural embeddings (ϕ_n are injective since the bonding morphisms α_n are). Thus we have

$$\phi_0(\mathcal{A}) = \phi_0(\mathcal{B}_0) \subset \phi_1(\mathcal{B}_1) \subset ... \subset \phi_n(\mathcal{B}_n) \subset ... \qquad \text{and} \qquad \mathcal{B} = \overline{\bigcup_{n\in\mathbb{N}} \phi_n(\mathcal{B}_n)}.$$

We will identify the algebra $\mathcal{A}$ with the subalgebra $\phi_0(\mathcal{A}) \subset \mathcal{B}$ and under this identification we extend α onto the algebra $\mathcal{B}$. To this end, we consider two sequences (an inverse one and a direct one)

$$\mathcal{B}_0 \xleftarrow{s_1} \mathcal{B}_1 \xleftarrow{s_2} \mathcal{B}_2 \xleftarrow{s_3} ... , \tag{4.3}$$

$$\mathcal{B}_0 \xrightarrow{s_{*,0}} \mathcal{B}_1 \xrightarrow{s_{*,1}} \mathcal{B}_2 \xrightarrow{s_{*,2}} ... , \tag{4.4}$$

where s_n is a "left-shift" and $s_{*,n}$ is a "right-shift":

$$s_n(a_0 \oplus a_1 \oplus ... \oplus a_n) = a_1 \oplus a_2 \oplus ... \oplus a_n$$

$$s_{*,n}(a_0 \oplus ... \oplus a_{n-1} \oplus a_n) = 0 \oplus \big(\alpha(1)a_0\,\alpha(1)\big) \oplus ... \oplus \big(\alpha^{n+1}(1)a_n\alpha^{n+1}(1)\big),$$

$a_k \in q\mathcal{A}_k$, $k = 0, ..., n-1$, $a_n \in \mathcal{A}_n$. Since $\alpha^n(1)$, $n \in \mathbb{N}$, form a decreasing sequence of orthogonal projections, mappings s_n and $s_{*,n}$ are well defined. Moreover the operators s_n are homomorphisms, whereas operators $s_{*,n}$ in general fail to be multiplicative.

Statement 5. *Sequence* (4.3) *induces an endomorphism $\alpha : \mathcal{B} \to \mathcal{B}$ extending the endomorphism $\alpha : \mathcal{A} \to \mathcal{A}$, whereas sequence* (4.4) *induces an operator $\mathcal{L} : \mathcal{B} \to \mathcal{B}$ which is a complete transfer operator for the extended C^*-dynamical system $(\mathcal{B}, \alpha)$.*

The word "induces" means here that α and $\mathcal{L}$ are given on the dense *-subalgebra $\bigcup_{n\in\mathbb{N}} \phi_n(\mathcal{A}_n)$ of $\mathcal{B}$ by the formulae

$$\alpha(a) = \phi_{n-1}(s_n(\phi_n^{-1}(a))), \qquad \mathcal{L}(a) = \phi_{n+1}(s_{*,n}(\phi_n^{-1}(a))), \tag{4.5}$$

where $a \in \phi_n(\mathcal{B}_n)$, $n > 0$.

Proof. Direct computations show that the following diagrams

$$\begin{tikzcd}
\mathcal{B}_0 \arrow[r, "\alpha_0"] & \mathcal{B}_1 \arrow[r, "\alpha_1"] & \mathcal{B}_2 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_{n-1}"] & \mathcal{B}_n \arrow[r, "\alpha_n"] & \dots \\
\mathcal{B}_0 \arrow[r, "\alpha_0"] & \mathcal{B}_1 \arrow[r, "\alpha_1"] \arrow[ul, "s_1"'] & \mathcal{B}_2 \arrow[r, "\alpha_2"] \arrow[ul, "s_2"'] & \dots \arrow[r, "\alpha_{n-1}"] \arrow[ul, "s_3"'] & \mathcal{B}_n \arrow[r, "\alpha_n"] \arrow[ul, "s_n"'] & \dots \arrow[ul, "s_{n+1}"']
\end{tikzcd}$$

$$\begin{tikzcd}
\mathcal{B}_0 \arrow[r, "\alpha_0"] & \mathcal{B}_1 \arrow[r, "\alpha_1"] & \mathcal{B}_2 \arrow[r, "\alpha_2"] & \dots \arrow[r, "\alpha_{n-1}"] & \mathcal{B}_n \arrow[r, "\alpha_n"] & \dots \\
\mathcal{B}_0 \arrow[r, "\alpha_0"] \arrow[ur, "s_{*,0}"] & \mathcal{B}_1 \arrow[r, "\alpha_1"] \arrow[ur, "s_{*,1}"] & \mathcal{B}_2 \arrow[r, "\alpha_2"] \arrow[ur, "s_{*,2}"] & \dots \arrow[r, "\alpha_{n-1}"] \arrow[ur, "s_{*,n-1}"] & \mathcal{B}_n \arrow[r, "\alpha_n"] \arrow[ur, "s_{*,n}"] & \dots
\end{tikzcd}$$

commute. Hence (4.3) and (4.4) induce certain linear mappings on $\mathcal{B}$ (i.e. formulae (4.5) make sense). The former mapping, which for the sake of proof we denote by $\widetilde{\alpha}$, is a homomorphism (since s_n is a homomorphism for all $n \in \mathbb{N}$) and the latter one, which we denote by $\mathcal{L}$, is positive (because s_{*n} posses that property for all $n \in \mathbb{N}$).
We assert that the mapping $\widetilde{\alpha}$ induced by (4.3) agrees with α on $\mathcal{A}$ which we identify with $\phi_0(\mathcal{A})$. Indeed, an element $\phi_0(a)$, $a \in \mathcal{A}$, of the inductive limit $\mathcal{B}$ is represented by the sequence $(a, qa \oplus \alpha(a), qa \oplus q\alpha(a) \oplus \alpha^2(a), \dots)$ and hence $\phi_1^{-1}(\phi_0(a)) = qa \oplus \alpha(a)$. Thus in view of (4.5) we have

$$\widetilde{\alpha}(\phi_0(a)) = \phi_0(s_1(\phi_1^{-1}(\phi_0(a)))) = \phi_0(s_1(qa \oplus \alpha(a))) = \phi_0(\alpha(a)).$$

Thereby our assertion is true and we are justified to denote by α the mapping $\widetilde{\alpha}$ induced by (4.3).
To prove that $\mathcal{L}$ is a complete transfer operator for $(\mathcal{B}, \alpha)$ it suffices to show (3.1) and (3.2). For that purpose we take arbitrary elements $\widetilde{a}, \widetilde{b} \in \bigcup_{n\in\mathbb{N}} \phi_n(\mathcal{B}_n) \subset \mathcal{B}$ and note that there exist $n \in \mathbb{N}$, such that $\widetilde{a} = \phi_{n+1}(a)$ and $\widetilde{b} = \phi_n(b)$ for $a \in \mathcal{B}_{n+1}$ and $b \in \mathcal{B}_n$. Direct computation shows that $s_{*,n}(s_{n+1}(a)b) = a \cdot s_{*,n}(b)$ and thus using formulae (4.5) we have

$$\begin{aligned}
\mathcal{L}(\alpha(\widetilde{a})\widetilde{b}) &= \mathcal{L}(\phi_n(s_{n+1}(a))\,\widetilde{b}) = \phi_{n+1}(s_{*,n}(s_{n+1}(a)b)) \\
&= \phi_{n+1}(a \cdot s_{*,n}(b)) = \phi_{n+1}(a) \cdot \phi_{n+1}(s_{*,n}(b)) = \widetilde{a}\mathcal{L}(\widetilde{b})
\end{aligned}$$

which proves (3.1). Similarly, one checks that $s_{n+1}(s_{*,n}(a)) = s_{n+1}(1)as_{n+1}(1)$ and then we have

$$\begin{aligned}
\alpha(\mathcal{L}(\widetilde{a})) &= \alpha(\phi_{n+1}(s_{*,n}(a)) = \phi_n(s_{n+1}(s_{*,n}(a))) = \phi_n(s_{n+1}(1)as_{n+1}(1)) \\
&= \phi_n(s_{n+1}(1))\phi_n(a)\phi_n(s_{n+1}(1)) = \alpha(1)\widetilde{a}\alpha(1),
\end{aligned}$$

which proves (3.2) and finishes the proof.

The systems $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \alpha)$ considered above coincide if and only if the range of $\alpha : \mathcal{A} \to \mathcal{A}$ is a hereditary subalgebra of $\mathcal{A}$. Indeed, the range of the endomorphism $\alpha : \mathcal{B} \to \mathcal{B}$ is always a hereditary subalgebra of $\mathcal{B}$, as it admits a complete transfer operator. If $\alpha(\mathcal{A}) = \alpha(1)\alpha(1)$ is a hereditary subalgebra of $\mathcal{A}$, then $\mathcal{A}_n = \alpha^n(1)\mathcal{A}\alpha^n(1) = \alpha^n(\mathcal{A})$, for all $n \in \mathbb{N}$. Consequently, the monomomorphisms $\alpha_n : \mathcal{B}_n \to \mathcal{B}_{n+1}$ are isomorphisms, and hence $(\mathcal{A}, \alpha) = (\mathcal{B}, \alpha)$, under our identifications. This justifies the following

Definition 3. If $(\mathcal{A}, \alpha)$ is such that $\ker \alpha$ is a complemented ideal, we call the system $(\mathcal{B}, \alpha)$ described in Statement 5 a C^*-dynamical system obtained from $(\mathcal{A}, \alpha)$ by *hereditation of range.*

Theorem 1. *Suppose that $(\mathcal{B}, \alpha)$ is a C^*-dynamical system obtained from $(\mathcal{A}, \alpha)$ by hereditation of the range. There is a one-to-one correspondence between covariant representations (π, U, H) and $(\widetilde{\pi}, U, H)$ of $(\mathcal{A}, \alpha)$ and $(\mathcal{B}, \alpha)$ respectively, which is established by the relation*

$$\widetilde{\pi}(\phi_n(a_0 \oplus a_1 \dots \oplus a_n)) = \pi(a_0) + U^*\pi(a_1)U + \dots + U^{*n}\pi(a_n)U^n. \tag{4.6}$$

Proof. Let $(\widetilde{\pi}, U, H)$ be a covariant representation of $(\mathcal{B}, \alpha)$. It is straightforward that (π, U, H) where $\pi = \widetilde{\pi}|_{\mathcal{A}}$ is a representation of $(\mathcal{A}, \alpha)$. To see that (π, U, H) is a covariant representation, by Statement 2, it suffices to show that q is the unit not only in the kernel of $\alpha : \mathcal{A} \to \mathcal{A}$ but also in the kernel of its extension $\alpha : \mathcal{B} \to \mathcal{B}$. To see the latter let $n > 0$ and notice that

$$\phi_0(q) = \phi_n(q \oplus 0 \oplus 0... \oplus 0).$$

Thus for $a = \phi_n(a_0 \oplus a_1 \oplus ... \oplus a_n) \in \phi_n(\mathcal{B}_n)$ we have

$$\alpha(a) = \phi_{n-1}(a_1 \oplus ... \oplus a_n) = 0 \iff a_1 = ... = a_n = 0 \iff a = qa.$$

We fix now a covariant representation (π, U, H) of $(\mathcal{A}, \alpha)$ and show that formula (4.6) defines a faithful representation $\widetilde{\pi}$ of $\mathcal{B}$. To this end, we note that in view of Statement 1 for every $n \in \mathbb{N}$, the mapping $\widetilde{\pi}_n : \phi_n(\mathcal{B}_n) \to C^*\big(\bigcup_{k=0}^{n} U^{*n}\pi(\mathcal{A})U^n\big)$ where

$$\widetilde{\pi}_n(\phi_n(a_0 \oplus a_1 ... \oplus a_n)) = \pi(a_0) + U^*\pi(a_1)U + ... + U^{*n}\pi(a_n)U^n$$

is an isomorphism. Consequently, to show that $\widetilde{\pi} : \mathcal{B} \to C^*\big(\bigcup_{n\in\mathbb{N}} U^{*n}\pi(\mathcal{A})U^n\big)$ given by (4.6) is a well defined isomorphism, it suffices to check that the diagram

$$\begin{array}{ccc}
\mathcal{B}_n & \xrightarrow{\alpha_n} & \mathcal{B}_{n+1} \\
\Big\downarrow{\scriptstyle \widetilde{\pi}_n\circ\phi_n} & & \Big\downarrow{\scriptstyle \widetilde{\pi}_{n+1}\circ\phi_{n+1}} \\
\widetilde{\pi}_n(\phi_n(\mathcal{B}_n)) & \xrightarrow{id} & \widetilde{\pi}_{n+1}(\phi_{n+1}(\mathcal{B}_{n+1}))
\end{array}$$

commutes. Let $a = a_0 \oplus a_1 \oplus ... \oplus a_n \in \mathcal{B}_n$. Since $\pi(1-q) = U^*U$ we have

$$\begin{aligned}
U^{*n}\pi(a_n)U^n &= U^{*n}\pi(qa_n + (1-q)a_n)U^n = U^{*n}\pi(qa_n)U^n + U^{*n}(U^*U)\pi(a_n)U^n \\
&= U^{*n}\pi(qa_n)U^n + U^{*n}(U^*U)\pi(a_n)(U^*U)U^n \\
&= U^{*n}\pi(qa_n)U^n + U^{*n+1}\pi(\alpha(a_n))U^{n+1}
\end{aligned}$$

and thus

$$\widetilde{\pi}_n(\phi_n(a)) = \sum_{k=0}^{n-1} U^{*k}\pi(a_k)U^k + U^{*n}\pi(a_n)U^n = \widetilde{\pi}_{n+1}(\phi_{n+1}(\alpha_n(a))).$$

Accordingly, $\widetilde{\pi}$ is a faithful representation of $\mathcal{B}$. Since $U^*U \in \pi(\mathcal{A}) \subset \widetilde{\pi}(\mathcal{B})$, in view of Statement 2, the only thing we need to prove is that $(\widetilde{\pi}, U, H)$ is a representation of $(\mathcal{B}, \alpha)$. Let $a = \phi_n(a_0 \oplus a_1 \oplus ... \oplus a_n) \in \phi_n(\mathcal{B}_n)$, $n > 0$. Using the relations $a_k \in \mathcal{A}_k = \alpha^k(1)\mathcal{A}\alpha^k(1)$, $a_0 \in q\mathcal{A} = \ker\alpha$ and the fact that $\{U^{k*}U^k\}_{k=0}^{\infty}$ is a decreasing sequence of projections lying in the center of $C^*\big(\bigcup_{n\in\mathbb{N}} U^{*n}\pi(\mathcal{A})U^n\big)$, cf. [15, Proposition 3.7], we have

$$\begin{aligned}
U\widetilde{\pi}(a)U^* &= \sum_{k=0}^{n} UU^{*k}\pi(a_k)U^kU^* = \pi(\alpha(a_0)) + \sum_{k=1}^{n} UU^*U^{*k-1}\pi(a_k)U^{k-1}UU^* \\
&= \sum_{k=1}^{n} UU^*(U^{*k-1}U^{k-1})U^{*k-1}\pi(a_k)U^{k-1}(U^{*k-1}U^{k-1})UU^* \\
&= \sum_{k=1}^{n} (U^{*k-1}U^{k-1})(UU^*)U^{*k-1}\pi(a_k)U^{k-1}(UU^*)(U^{*k-1}U^{k-1}) \\
&= \sum_{k=1}^{n} U^{*k-1}(U^kU^{*k})\pi(a_k)(U^kU^{*k})U^{k-1}
\end{aligned}$$

$$= \sum_{k=1}^{n} U^{*k-1}\pi(\alpha^k(1)a_k\alpha^k(1))U^{k-1} = \sum_{k=0}^{n-1} U^{*k}\pi(a_{k+1})U^k$$
$$= \widetilde{\pi}(\phi_{n-1}(a_1 \oplus a_2 \oplus ... \oplus a_n)) = \widetilde{\pi}(\alpha(a)).$$

This finishes the proof.

Putting together constructions from Definitions 2 and 3 we obtain a construction that embraces the general situation.

Definition 4. Suppose $(\mathcal{A}, \alpha)$ is an arbitrary C^*-dynamical system and J is an ideal in $\mathcal{A}$ such that $J \cap \ker\alpha = \{0\}$. Let $(\mathcal{A}_J, \alpha)$ be the C^*-dynamical system obtained from $(\mathcal{A}, \alpha)$ by J-unitization of kernel and let $(\mathcal{B}, \alpha)$ be the system obtained from $(\mathcal{A}_J, \alpha)$ by hereditation of range:

$$\mathcal{A} \subset \mathcal{A}_J \subset \mathcal{B}.$$

We call the system $(\mathcal{B}, \alpha)$ the *natural J-extension* of $(\mathcal{A}, \alpha)$ to a C^*-dynamical system possessing a complete transfer operator. If $J = (\ker\alpha)^\perp$ we call $(\mathcal{B}, \alpha)$ simply the *natural extension* of $(\mathcal{A}, \alpha)$.

Remark. In order to construct $(\mathcal{B}, \alpha)$ directly from $(\mathcal{A}, \alpha, J)$, without passing through $(\mathcal{A}_J, \alpha)$, one may apply our direct limit construction, almost literally, changing the meaning of q from an element of $\mathcal{A}$ to the quotient map $q : \mathcal{A} \to \mathcal{A}/J$.

Remark. If $\ker\alpha$ is unital, then natural extension of $(\mathcal{A}, \alpha)$ coincides with the system obtained by hereditation of range. Morever, $\alpha : \mathcal{B} \to \mathcal{B}$ is an automorphism if and only if $(\mathcal{B}, \alpha)$ is a natural extension of a unital monomorphism $\alpha : \mathcal{A} \to \mathcal{A}$ (to see it note that $\alpha(\mathcal{B}) = \alpha(1)\mathcal{B}\alpha(1)$ and $1 - \mathcal{L}(1)$ is the common unit in both of the kernels of $\alpha : \mathcal{A}_J \to \mathcal{A}_J$ and $\alpha : \mathcal{B} \to \mathcal{B}$).

Within the notation of Definition 4 and denoting by $\mathcal{L}$ the complete transfer operator for $\alpha : \mathcal{B} \to \mathcal{B}$, we have

$$\mathcal{A}_J = C^*(\mathcal{A}, \mathcal{L}(1)) = \mathcal{L}(1)\mathcal{A} \oplus (1 - \mathcal{L}(1))\mathcal{A},$$
$$\mathcal{B} = C^*\left(\bigcup_{n=0}^{\infty} \mathcal{L}^n(\mathcal{A})\right) = \overline{span}\{\mathcal{L}^n(a) : a \in \mathcal{A}, n \in \mathbb{N}\}.$$

In view of Statement 4 and Theorem 1, we get the following theorem.

Theorem 2. *Let $(\mathcal{A}, \alpha)$ be an arbitrary C^*-dynamical system and let $(\mathcal{B}, \alpha)$ be its natural J-extension with the complete transfer operator $\mathcal{L}$. There is a one-to-one correspondence between J-covariant representations (π, U, H) of $(\mathcal{A}, \alpha)$ and covariant representations $(\widetilde{\pi}, U, H)$ of $(\mathcal{B}, \alpha)$, which is established by the relation*

$$\widetilde{\pi}(\sum_{k=0}^{n} \mathcal{L}^k(a_k)) = \sum_{k=0}^{n} U^{*k}\pi(a_k)U^k, \quad a_k \in \mathcal{A}.$$

In particular,

i) for every J-covariant representation (π, U, H) of $(\mathcal{A}, \alpha)$ we have

$$\mathcal{B} \cong C^*\big(\bigcup_{n\in\mathbb{N}} U^{*n}\pi(\mathcal{A})U^n\big) = \overline{span}\{U^{*n}\pi(a)U^n : a \in \mathcal{A}, n \in \mathbb{N}\};$$

ii) the crossed product $C^*(\mathcal{A}, \alpha; J)$ of $\mathcal{A}$ by α associated to J defined in [11, Definition 1.12] *is naturally isomorphic to the crossed product $\mathcal{B} \rtimes_\alpha \mathbb{Z}$ defined in* [7, Definition 2.6] (*cf. Statement* 3).

ACKNOWLEDGMENTS

This research was supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme; project 'OperaDynaDual', and National Science Centre grant number DEC-2011/01/D/ST1/04112.

BIBLIOGRAPHY

1. J. Cuntz and W. Krieger, "A class of C*-algebras and topological Markov chains," *Inventiones Math.*, **56** (1980), 251–268.
2. W. L. Paschke, "The crossed product of a C^*-algebra by an endomorphism," *Proc. Amer. Math. Soc.* (1980), no. 1, 113–118.
3. P. J. Stacey, "Crossed products of C^*-algebras by *-endomorphisms," *J. Austral. Math. Soc. Ser. A*, **54** (1993), 204–212.
4. G. J. Murphy, "Crossed products of C*-algebras by endomorphisms," *Integral Equat. Oper. Theory*, **24** (1996), 298–319.
5. R. Exel, "Circle actions on C^*-algebras, partial automorhisms and generalized Pimsner-Voiculescu exact sequence," *J. Funct. Analysis*, **122** (1994), 361–401.
6. R. Exel, "A new look at the crossed-product of a C^*-algebra by an endomorphism," *Ergodic Theory Dynam. Systems*, **23** (2003), 1733–1750.
7. A. B. Antonevich, V. I. Bakhtin, A. V. Lebedev, "Crossed product of C^*-algebra by an endomorphism, coefficient algebras and transfer operators," *Math. Sbornik*, **202** (2011), no. 9, 1253–1283.
8. B. K. Kwaśniewski, "Covariance algebra of a partial dynamical system," *Central European J. Math.*, **3** (2005), no. 4, 718–765.
9. V. I. Bakhtin, A. V. Lebedev, "When a C^*-algebra is a coefficient algebra for a given endomorphism," *arXiv*:*math.OA*/0502414 *v*1 19 *Feb* 2005.
10. B. K. Kwaśniewski, "On transfer operators for C^*-dynamical systems," *Rocky Mountain J. Math.*, **42** (2012), no. 3, 919–938.
11. B. K. Kwaśniewski, A. V. Lebedev, "Crossed products by endomorphisms and reduction of relations in relative Cuntz-Pimsner algebras," *J. Funct. Anal.*, **264** (2013), no. 8, 1806–1847.
12. B. K. Kwaśniewski, A. V. Lebedev, "Reversible extensions of irreversible dynamical systems: C^*-method," *Math. Sbornik*, **199** (2008), no. 11, 45–74.
13. B. K. Kwaśniewski, "C^*-algebras associated with reversible extensions of logistic maps," *Math. Sbornik*, **203** (2012), no. 10, 1448–1489.
14. M. Laca, "From endomorphisms to automorphisms and back: dilations and full corners," *J. London Math. Soc.*, **61** (2000), 893-904.
15. A. V. Lebedev, A. Odzijewicz, "Extensions of C^*-algebras by partial isometries," *Math. Sbornik*, **195** (2004), no. 7, 37-70.

B. K. Kwaśniewski

Department of Mathematics and Computer Science, The University of Southern Denmark, Campusvej 55, DK–5230 Odense M, Denmark // Institute of Mathematics, University of Bialystok ul. Akademicka 2, PL-15-267 Bialystok, Poland

E-mail: `bartoszk@math.uwb.edu.pl`