TOPOLOGICAL APERIODICITY FOR PRODUCT SYSTEMS OVER SEMIGROUPS OF ORE TYPE

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Abstract. We prove a version of uniqueness theorem for Cuntz-Pimsner algebras of discrete product systems over semigroups of Ore type. To this end, we introduce Doplicher-Roberts picture of Cuntz-Pimsner algebras, and the semigroup dual to a product system of 'regular' C^*-correspondences. Under a certain aperiodicity condition on the latter, we obtain the uniqueness theorem and a simplicity criterion for the algebras in question. These results generalize the corresponding ones for crossed products by discrete groups, due to Archbold and Spielberg, and for Exel's crossed products, due to Exel and Vershik. They also give interesting conditions for topological higher rank graphs and P-graphs, and apply to the new Cuntz C^*-algebra $\mathcal{Q}_{\mathbb{N}}$ arising from the "$ax+b$"-semigroup over $\mathbb{N}$.

1. Introduction

A fundamental problem in every theory dealing with C^*-algebras generated by operators satisfying prescribed relations is the uniqueness of such objects. More specifically, suppose $\mathcal{R}$ is a set of C^*-algebraic relations on a set of generators $\mathcal{G}$, and suppose there is a mapping $\pi : \mathcal{G} \to \mathcal{B}(H)$ such that $\{\pi(g)\}_{g\in\mathcal{G}}$ are non-zero bounded operators on a Hilbert space H which satisfy relations $\mathcal{R}$. We call such π faithful representation of $(\mathcal{G},\mathcal{R})$, and we denote by $C^*(\pi)$ the C^*-algebra generated by $\{\pi(g)\}_{g\in\mathcal{G}}$. The pair $(\mathcal{G},\mathcal{R})$ has *uniqueness property* if for any two faithful representations π_1, π_2 of $(\mathcal{G},\mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

extends to the (necessarily unique) isomorphism $C^*(\pi_1) \cong C^*(\pi_2)$. Results stating that a certain class of relations possesses the above property are called *uniqueness theorems*. For reasonable pairs $(\mathcal{G},\mathcal{R})$, see for instance [5], there exists a universal C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ for the above defined representations of $(\mathcal{G},\mathcal{R})$. Clearly, $(\mathcal{G},\mathcal{R})$ has the uniqueness property if and only if $C^*(\mathcal{G},\mathcal{R})$ exists and for any faithful representation π of $(\mathcal{G},\mathcal{R})$ the natural epimorphism from $C^*(\mathcal{G},\mathcal{R})$ onto $C^*(\pi)$ is actually an isomorphism.

Among the oldest and best studied uniqueness theorems are those related to C^*-dynamical systems. Recall that such a system (A,α,G), consists of a C^*-algebra A and a group action $\alpha : G \to \mathrm{Aut}(A)$. Uniqueness result in this context applies to the associated crossed product. Starting at least from the sixties, uniqueness theorems for crossed products began to appear in connection with various problems such as properties of the Connes spectrum, proper outerness, ideal structure, or spectral analysis of functional-differential operators, see [2, p. 225, 226], [4] and [33] for relevant surveys. One of the most popular conditions of this kind, known today as *topological freeness*, was probably for the first time explicitly stated in [36] for $\mathbb{Z}$-actions. O'Donovan proved in [36] that if the set of periodic points for the dual action $\widehat{\alpha}$ on the spectrum $\widehat{A}$ of A has empty interior then the crossed product $A \rtimes_\alpha \mathbb{Z}$

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has intersection property, which is equivalent to the uniqueness property as defined above. This result was generalized to the case of amenable discrete groups [2] and then to arbitrary discrete groups [4]. More specifically, by [4, Theorem 1] topological freeness of $\widehat{\alpha}$ implies intersection property for $A \rtimes_\alpha G$, and this is equivalent to the uniqueness property if and only if action α is amenable in the sense that the full crossed product $A \rtimes_\alpha G$ and the reduced crossed product $A \rtimes_{\alpha,r} G$ are naturally isomorphic. This formulation is very convenient as it allows to investigate amenability and topological freeness of α independently. Moreover, it can be used to study the structure of the reduced crossed product $A \rtimes_{\alpha,r} G$. We recall that for a separable A and $G = \mathbb{Z}$, or if A is commutative and G amenable discrete, topological freeness of $\widehat{\alpha}$ is equivalent to the uniqueness property for $A \rtimes_\alpha G$, see [37, Theorem 10.4] and [4, Theorem 2], respectively. However, it is known that already for $\mathbb{Z}^2$ actions topological freeness is only sufficient but not necessary for the uniqueness property, [4, Remark on page 123].

Another line of research leading towards numerous uniqueness theorems was initiated by the seminal work of Cuntz and Krieger, [13]. In particular, [13, Theorem 2.13] states that the Cuntz-Krieger relations possess the uniqueness property if the underlying matrix A satisfies condition (I). Since then, similar results concerning various generalizations of the algebra $\mathcal{O}_A$ are usually called Cuntz-Krieger uniqueness theorems. The diagram in Figure 1 presents certain such theorems relevant to the present paper; each item contains the name of universal algebras, the condition which is (at present known to be) equivalent to uniqueness property for the corresponding defining relations, and the names of authors who introduced the condition. An arrow from A to B indicates that algebras in question and the condition in B can be viewed as generalizations of the ones in A. We provide more details and explanations in Section 6.

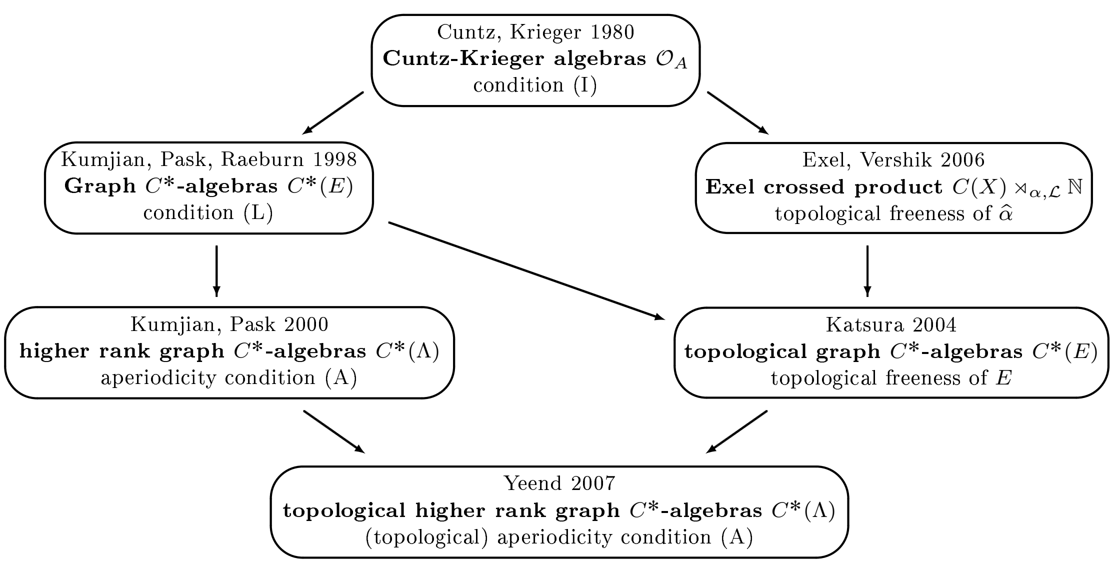

FIGURE 1. Cuntz-Krieger uniqueness theorems

The C^*-algebras associated with topological graphs were introduced in [27] as a generalization of both graph C^*-algebras and crossed products of commutative C^*-algebras by $\mathbb{Z}$-actions. Similarly, C^*-algebras arising from topological higher rank graphs [47] include as examples crossed products of commutative C^*-algebras by $\mathbb{Z}^k$-actions. Algebras associated to topological higher rank graphs provide interesting examples of a general, intensively investigated but still largely undeveloped theory of algebras associated with product systems over semigroups, [8]. One of the main aims of the present article is initialization of a systematic and unified approach to the study of uniqueness properties for universal C^*-algebras $C^*(\mathcal{G},\mathcal{R})$. To this end, we establish certain general results for Cuntz-Pimsner algebras associated with product systems over a large class of semigroups, and with coefficients in an arbitrary (not necessarily commutative) C^*-algebra A.

Uniqueness theorems are often studied via the associated gauge action of a dual group $\widehat{G}$ or a coaction of a relevant group G, e.g. see [26], [8]. In general, existence of such an additional structure on a universal C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ can be thought of as arising from a symmetry in relations $\mathcal{R}$. It establishes a Fell bundle structure $\{B_t\}_{t\in G}$ on $C^*(\mathcal{G},\mathcal{R})$. If $G=\mathbb{Z}$ and the Fell bundle $\{B_k\}_{k\in\mathbb{Z}}$ is semisaturated then C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ is naturally isomorphic to the crossed product $B_0\rtimes_{B_1}\mathbb{Z}$, [1], where B_1 is treated as a Hilbert bimodule over B_0. The first named author proved in [33] a uniqueness theorem for $B_0\rtimes_{B_1}\mathbb{Z}$ under the assumption that a partial homeomorphism of $\widehat{B}_0$ given by Rieffel's induced representation functor $B_1-\mathrm{Ind}$ is topologically free. It seems plausible that similar techniques may lead to a generalization of [33, Theorem 2.2] to Fell bundles over arbitrary discrete groups. However, in many important cases (e.g. those listed in Figure 1 above) the initial data correspond to semigroups rather than groups. The analysis in [32] shows that in the context of Cuntz-Pimsner algebras associated with product systems over semigroups P, passing from the initial algebra A to the core B_0 is a very nontrivial procedure even in the case $A\cong\mathbb{C}^n$ and $P=\mathbb{N}$. That is why we pursue here a more ambitious program focused on semigroups rather than groups.

Our initial object is a product system of C^*-correspondences X over a discrete semigroup P and with coefficients in an arbitrary C^*-algebra A, as defined in [22]. We impose two critical restrictions on the product systems in question, one on the underlying semigroup P and one on the structure of fibers X_p, $p\in P$. Namely, we assume that P is an Ore semigroup. (Actually, we consider slightly more general semigroups, satisfying only one-sided cancellation, see Subsection 2.5 below.) Such semigroups arise naturally in many contexts, including dilations, [34], interactions, [18], and skew rings, [3]. Among examples one finds all groups and all commutative cancellative semigroups. About the fibers X_p, $p\in P$, we assume that the left action of A is given by an injective homomorphism into the compacts $\mathcal{K}(X_p)$. We call such an X *regular product system.*

In the present paper, we are primarily focused on investigations of the Cuntz-Pimsner algebra $\mathcal{O}_X$ associated to a regular product system X, as in [22]. Under our assumptions on X and P, Fowler's definition seems to work particularly well. For instance, when P is a positive cone in an ordered quasi-lattice group (G,P), then $\mathcal{O}_X$ coincides with the Cuntz-Nica-Pimsner algebra $\mathcal{N}\mathcal{O}_X$, [22], [44], [8]. However, we stress that the very definition of the Cuntz-Nica-Pimsner algebra $\mathcal{N}\mathcal{O}_X$ puts severe restrictions on the class of semigroups P one may consider. In particular, P itself cannot be a group and this excludes many interesting examples. By contrast, the algebra $\mathcal{O}_X$ does not have this drawback and our results reinforce the perception that (under our assumptions) it is the right object to study.

In Section 3, we analyze the structure of the algebra $\mathcal{O}_X$ associated to a regular product system X. We show (see Theorem 3.8 below) that $\mathcal{O}_X$ can be constructed in the spirit of the Doplicher-Roberts algebras arising in the abstract duality theory for compact groups, [15]. More precisely, we show that X gives rise to a right tensor C^*-precategory $\mathcal{K}_X$ over the semigroup P, cf. [15], [31], and $\mathcal{O}_X$ is a completion of a graded $*$-algebra whose fibers are direct limits of elements of $\mathcal{K}_X$. In the case $P=\mathbb{N}$ (and with no further assumptions on X) such an approach was elaborated in [31]. This description immediately implies that the universal representation of X in $\mathcal{O}_X$ is injective, thus answering a question going back to

Fowler's original paper [22, Remark 2.10]. It also allows us to view and study $\mathcal{O}_X$ as a cross sectional algebra of a certain Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ over the enveloping group $G(P)$ of P. Taking advantage of this picture, we define the reduced Cuntz-Pimsner algebra $\mathcal{O}_X^r$ of X as the reduced cross sectional algebra of $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ [16], [38]. In the case $\mathcal{O}_X = \mathcal{N}\mathcal{O}_X$, our $\mathcal{O}_X^r$ coincides with the co-universal algebra $\mathcal{N}\mathcal{O}_X^r$ defined in [8].

In Section 4, we present a novel construction of a semigroup $\widehat{X}$ dual to a regular product system X. Elements of $\widehat{X}$ are multivalued maps on the spectrum of the coefficient algebra A. When $P = G$ is a group, these maps are honest homeomorphisms arising through Rieffel's induction, cf. [33]. The semigroup $\widehat{X}$ is particularly well suited for the study of uniqueness property and related questions.

In Section 5, we formulate a topological aperiodicity condition in terms of the semigroup $\widehat{X}$. This is the key ingredient entering our *uniqueness theorem*, see Theorem 5.6 below. We prove that if $\widehat{X}$ is topologically aperiodic, then for any faithful Cuntz-Pimsner representation ψ of X there exists a conditional expectation from the C^*-algebra generated by $\psi(X)$ onto its core C^*-subalgebra. Such conditional expectations are main tools in analysis of representations and ideal structure of C^*-algebras under consideration. In particular, they are of critical importance in various gauge-invariant uniqueness theorems, see, for instance, [26], [8], [39, Chapter 3]. When $\mathcal{O}_X = \mathcal{O}_X^r$, our uniqueness theorem states that a representation of $\mathcal{O}_X$ is faithful if and only if it is faithful on the algebra of coefficients A. As a corollary to Theorem 5.6, we obtain the following simplicity criterion. If $\widehat{X}$ is topologically aperiodic then $\mathcal{O}_X^r$ is simple if and only if X is minimal, see Theorem 5.10 below.

Applications and examples of our main results are presented in Section 6. Logical relationships between the topological aperiodicity of $\widehat{X}$ and other aperiodicity conditions mentioned above, when applied to particular examples, are presented schematically on Figure 2. More specifically, in Subsection 6.1 we consider product systems X whose fibers are Hilbert bimodules. We show that under this assumption the semigroup $\widehat{X} = \{\widehat{X}_p\}_{p\in P}$ consists of partial homeomorphisms and generates a partial action of $G(P)$ on $\widehat{A}$, see Proposition 6.4 below. In this setting, topological freeness implies topological aperiodicity. As a bonus, we obtain uniqueness theorems and simplicity criteria for cross sectional algebras of saturated Fell bundles (Corollary 6.5) and for twisted crossed products by semigroups of injective endomorphisms with hereditary ranges (Proposition 6.9).

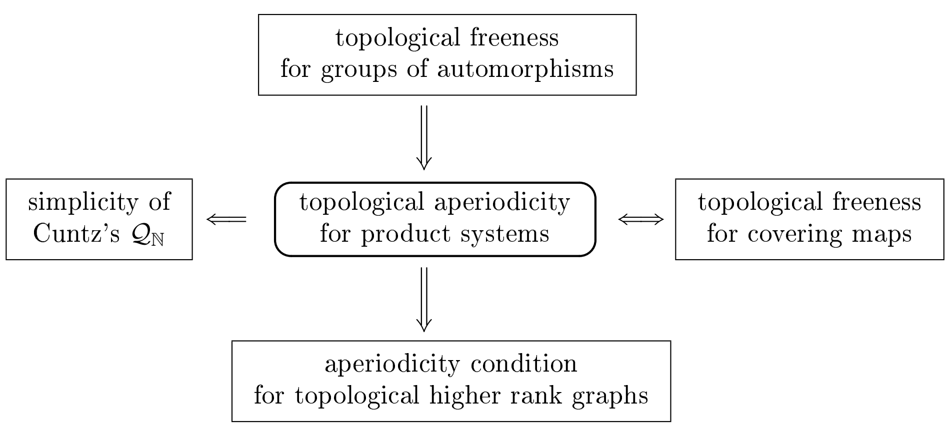

FIGURE 2. Relationship between aperiodicity conditions

Another motivation for our work comes from theory of graph algebras and their generalizations. Any topological graph E gives rise to a product system X over the semigroup of natural numbers, [27]. In this context, topological aperiodicity of $\widehat{X}$ turns out to be strictly stronger than topological freeness of E, but these notions coincide when the range map of E is injective. For example, this latter condition holds for topological graphs arising from Exel's crossed products by covering maps, [20], [6]. Our results give necessary and sufficient conditions for uniqueness and simplicity of such crossed products, see Example 6.13 below. In Subsection 6.4, we look at topological higher rank graphs, [47], and their corresponding product systems over $P = \mathbb{N}^k$, [8]. In fact, we consider certain topological P-graphs where P is an arbitrary semigroup of Ore type and thus we obtain completely new uniqueness and simplicity conditions for the associated C^*-algebras (discrete P-graphs where (G, P) is a quasi-lattice ordered group were considered in [40], [7]). As a final example we show that our results yield a quick and elegant way to see simplicity of the Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$, [12], which has a nice representation as $\mathcal{O}_X$ where X is a natural product system as described in [24], see subsection 6.5.

Finally, we would like to point out two additional applications of our general structural result for $\mathcal{O}_X$, Theorem 3.8. Firstly, we use it to reveal group grading and establish non-degeneracy of the twisted crossed product by a semigroup action of injective endomorphisms, see Proposition 6.6 below. Secondly, we give a natural definition of the C^*-algebra $C^*(\Lambda, d)$ and the reduced C^*-algebra $C^*_r(\Lambda, d)$ associated to a product system of topological graphs over P, [23], see Subsection 6.4. These constructions generalize C^*-algebras associated to topological higher rank graphs and discrete P-graphs [7]. Significantly, the Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$ can be modeled as a C^*-algebra $C^*(\Lambda, d)$ associated to a topological P-graph (Λ, d) where $P = \mathbb{N}^*$, see Remark 6.19 below.

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2. Preliminaries

This section contains the necessary preliminaries. In addition to more standard material, we discuss multivalued maps in Subsection 2.1 and semigroups of Ore type in Subsection 2.5.

2.1. **Multivalued maps.** We follow standard conventions, cf. for instance [42, Chapter 5], apart from notion of continuity which will not play any important role in the sequel. Let M and N be sets and 2^N be the family of all subsets of N. A *multivalued mapping* from M to N is by definition a mapping from M to 2^N. We denote such a multivalued mapping f by $f : M \to N$. Also, we identify the usual (single-valued) mappings with multivalued mappings taking values in singletons. We denote

$$D(f) := \{x \in M : f(x) \neq \emptyset\}, \qquad f(M) := \{y \in N : y \in f(x) \text{ for some } x \in M\} = \bigcup_{x \in M} f(x)$$

the domain and the image of f respectively. We put $f(A) := \bigcup_{x \in A} f(x)$ for a subset A of M, and define preimage of $B \subseteq N$ to be the set

$$f^{-1}(B) := \{x \in M : f(x) \cap B \neq \emptyset\}.$$

This goes perfectly well with the natural definition of the multivalued inverse f^{-1} of f, where

$$y \in f^{-1}(x) \overset{def}{\Longleftrightarrow} x \in f(y).$$

For two multivalued mappings $f, g : M \to N$ we write $f \subset g$ whenever $f(x) \subset g(x)$ for all $x \in M$. Composition of two multivalued maps $f : M \to N$ and $g : N \to L$ is the multivalued map $g \circ f : M \to L$ given by

$$(g \circ f)(x) := \bigcup_{y \in f(x)} g(y).$$

One checks that the obvious rule $(f \circ g)^{-1} = g^{-1} \circ f^{-1}$ holds. However note that

$$(f \circ f^{-1})(x) = \bigcup_{x \in f(y)} f(y) \tag{1}$$

is either empty or it is a subset containing x, possibly larger than $\{x\}$. The former happens when x does not belong to the range of f and the latter otherwise.

If M and N are topological spaces, we say that a multivalued map $f : M \to N$ is continuous if $f^{-1}(U)$ is open for every open subset U of N. In the literature, this is usually taken as a definition of lower semi-continuity. But since we will not make use of upper semi-continuity we do not make a distinction.

2.2. **Hilbert modules, C^*-correspondences and induced representations.** Throughout this section, A, B and D are C^*-algebras. We adhere to the convention that $\beta(A, B) = \overline{\operatorname{span}}\{\beta(a, b) \in D \mid a \in A, b \in B\}$ for maps $\beta \colon A \times B \to D$ such as inner products, multiplications or representations. By homomorphism, epimorphism, etc. we always mean an involution preserving map. All ideals in C^*-algebras are assumed to be closed and two-sided.

We adopt the standard notations and definitions of objects related to Hilbert modules, cf. for instance [41]. A right Hilbert B-module is a Banach space X which is a right B-module equipped with an B-valued inner product $\langle \cdot, \cdot \rangle_B : X \times X \to B$. If X, Y are right Hilbert B-modules then $\mathcal{L}(X, Y)$ stands for the space of adjointable operators from X into Y. Also, the space of "compact" operators from X to Y is defined as

$$\mathcal{K}(X, Y) = \overline{\operatorname{span}}\{\Theta_{y,x} : x \in X, y \in Y\} \subseteq \mathcal{L}(X, Y),$$

where

$$\Theta_{y,x}(z) = y\langle x, z\rangle_B, \quad z \in X.$$

In particular, $\mathcal{K}(X) := \mathcal{K}(X, X)$ is an ideal in the C^*-algebra $\mathcal{L}(X) := \mathcal{L}(X, X)$.

A *C^*-correspondence* from A to B is a right Hilbert B-module X equipped with a homomorphism $\phi_X : A \to \mathcal{L}(X)$. We refer to ϕ_X as to the left action of A on X and write $a \cdot x = \phi_X(a)x$, for $a \in A$, $x \in X$. If $A = B$ then we call X a C^*-correspondence with coefficients in A. A *Hilbert A-B-bimodule* is a C^*-correspondence X from A to B equipped with a left A-valued inner product ${}_A\langle \cdot, \cdot \rangle : X \times X \to A$ such that

$$x\langle y, z\rangle_B = {}_A\langle x, y\rangle z, \qquad x, y, z \in X.$$

Equivalently, X is both a left Hilbert A-module and a right Hilbert B-module satisfying the above condition. If, in addition, ${}_A\langle X, X\rangle = A$ and $\langle X, X\rangle_B = B$, then X is an imprimitivity A-B-bimodule. For instance, every C^*-algebra A can be considered a C^*-correspondence (actually, an imprimitivity A-A-bimodule), denoted ${}_AA_A$, where $\langle a, b\rangle_A = a^*b$, ${}_A\langle a, b\rangle = ab^*$, and both left and right action is simply multiplication in A.

We note that there is a one-to-one correspondence between representations $\pi : A \to \mathcal{B}(H)$ of A on a Hilbert space H and C^*-correspondences $X = H$ from A to $\mathbb{C}$ (where left action is induced by π). We say that such C^*-correspondences associated to the representation π.

Furthermore, any right Hilbert A-module can be considered a Hilbert $\mathcal{K}(X)$-A-bimodule, where ${}_{\mathcal{K}(X)}\langle x, y\rangle = \Theta_{x,y}$.

If X is a right Hilbert A-module and Y is a Hilbert A-C-bimodule, then the internal tensor product $X \otimes_A Y = \overline{\operatorname{span}}\{x \otimes_A y : x \in X, y \in Y\}$ (balanced over A) is a right Hilbert C-module with the right action induced from Y and the C-valued inner product given by

$$\langle x_1 \otimes_A y_1, x_2 \otimes_A y_2\rangle_C = \langle y_1, \phi_Y(\langle x_1, x_2\rangle_A)y_2\rangle_C, \quad \text{for } x_i \in X \text{ and } y_i \in Y,\ i = 1, 2.$$

If, in addition, X is a C^*-correspondence from B to A, then $X \otimes_A Y$ is a C^*-correspondence from B to C with the left action implemented by the homomorphism $B \ni a \mapsto \phi_X(a) \otimes_A 1_Y \in \mathcal{L}(X \otimes Y)$, where 1_Y is the unit in $\mathcal{L}(Y)$. In the sequel, in order not to overload notation, we will often write simply $X \otimes Y$ and $x \otimes y$ for tensor products, when A is understood.

In the above scheme, a particularly important special case occurs when Y is a C^*-correspondence from A to $\mathbb{C}$ associated to a representation $\pi : A \to \mathcal{B}(H)$. Then for any C^*-correspondence X from B to A the C^*-correspondence $X \otimes_A Y$ is associated to a certain representation of B which we denote by $X\text{-Ind}(\pi)$ and call *representation induced from π by X*. More precisely, let $X \otimes_\pi H = \overline{\operatorname{span}}\, X \otimes H$ be a Hilbert space equipped with the inner product

$$\langle x_1 \otimes_\pi h_1, x_2 \otimes_\pi h_2\rangle_{\mathbb{C}} = \langle h_1, \pi(\langle x_1, x_2\rangle_A)h_2\rangle_{\mathbb{C}}.$$

Then $X\text{-Ind}(\pi)$ is a representation of B on $X \otimes_\pi H$ such that

$$X\text{-Ind}(\pi)(b)(x \otimes_\pi h) = (bx) \otimes_\pi h, \qquad b \in B. \tag{2}$$

In particular, if X is an imprimitivity B-A-bimodule, then by the celebrated Rieffel's result, cf. e.g. [41, Theorem 3.29, Corollaries 3.32 and 3.33], the induced representation functor $X\text{-Ind}$ factors through to the homeomorphism $[X\text{-Ind}] : \widehat{A} \to \widehat{B}$ between the spectra of A and B. The inverse of this homeomorphism is given by induction with respect to a Hilbert module dual to X. Here, a dual to a right Hilbert A-module X means a left Hilbert A-module $\widetilde{X}$ for which there exists an antiunitary $\flat : X \to \widetilde{X}$. A natural model for $\widetilde{X}$ is $\mathcal{K}(X, {}_A A_A)$ where $\flat(x)y = \langle x, y\rangle_A$. In particular, if X is a Hilbert A-B-bimodule then $\widetilde{X}$ is a Hilbert B-A-bimodule.

2.3. **Product systems, their representations and Cuntz-Pimsner algebras.** Let A be a C^*-algebra and P a discrete semigroup with identity e. A *product system* over P with coefficients in A is a semigroup $X = \bigsqcup_{p \in P} X_p$, equipped with a semigroup homomorphism $d\colon X \to P$ such that

- (P1) $X_p = d^{-1}(p)$ is a C^*-correspondence with coefficients in A for each $p \in P$.
- (P2) X_e is the standard bimodule ${}_A A_A$.
- (P3) The multiplication on X extends to isomorphisms $X_p \otimes_A X_q \cong X_{pq}$ for $p, q \in P\backslash\{e\}$ and the right and left actions of $X_e = A$ on each X_p.

For each $p \in P$, we denote by $\langle\cdot,\cdot\rangle_p$ the A-valued inner product on X_p and by ϕ_p the homomorphism from A into $\mathcal{L}(X_p)$ which implements the left action of A on X_p. Given $p, q \in P$ with $p \neq e$, there is a homomorphism $\iota_p^{pq}\colon \mathcal{L}(X_p) \to \mathcal{L}(X_{pq})$ characterised by

$$\iota_p^{pq}(T)(xy) = (Tx)y, \quad \text{where } x \in X_p,\ y \in X_q \text{ and } T \in \mathcal{L}(X_p). \tag{3}$$

We recall that the map

$$X_p \ni x \to t_x \in \mathcal{K}(A, X_p) \qquad \text{where } t_x(a) = xa, \tag{4}$$

yields a C^*-correspondence isomorphism $X_p \cong \mathcal{K}(A, X_p)$. Here $\mathcal{K}(A, X_p)$ is a C^*-correspondence with A-valued inner product $\langle T, S\rangle_A = T^*S$ and point-wise actions. Thus we may define $\iota_e^p\colon \mathcal{K}(X_e) \to \mathcal{L}(X_p)$ simply by letting $\iota_e^p(t_a) = \phi_p(a)$ for $p \in P$, $a \in A$, [44, §2.2].

A map ψ from X to a C^*-algebra B is a Toeplitz *representation of X in B* if the following conditions hold:

(T1) for each $p \in P\backslash\{e\}$, $\psi_p := \psi|_{X_p}$ is linear, and ψ_e is a homomorphism,
(T2) $\psi_p(x)\psi_q(y) = \psi_{pq}(xy)$ for $x \in X_p$, $y \in X_q$, $p, q \in P$,
(T3) $\psi_p(x)^*\psi_p(y) = \psi_e(\langle x, y\rangle_p)$ for $x, y \in X_p$.

It is well known that, for each $p \in P$ there exists a $*$-homomorphism $\psi^{(p)} : \mathcal{K}(X_p) \longrightarrow B$ such that $\psi^{(p)}(\Theta_{x,y}) = \psi_p(x)\psi_p(y)^*$, for $x, y \in X_p$. The representation ψ is called *Cuntz-Pimsner covariant* if

(CP) $\psi^{(p)}(\phi_p(a)) = \psi_e(a)$ for all $a \in A$ and $p \in P$.

As introduced by Fowler [22], the Cuntz-Pimsner algebra $\mathcal{O}_X$ of a product system X is a universal C^*-algebra for the Cuntz-Pimsner covariant representations. We denote by j_X the universal representation of X in $\mathcal{O}_X$. Hence for any Cuntz-Pimsner representation $\psi : X \to B$ there is a unique epimorphism $\Pi_\psi : \mathcal{O}_X \to C^*(\psi(X))$ such that $j_X(x) = \psi(x)$ for all $x \in X$. We call Π_ψ 'the integrated representation'.

It is well known and not hard to see that a necessary condition for j_X to be injective (and hence for $\mathcal{O}_X$ to be nondegenerate) is that all of the homomorphisms ϕ_p, $p \in P$, are injective. It is known that this condition is also sufficient [44, Corollary 5.2] when P is a directed positive cone in a quasi-lattice ordered group (G, P) and each ϕ_p acts by compacts. In this case, $\mathcal{O}_X$ coincides with the so-called Cuntz-Nica-Pimsner algebra $\mathcal{NO}_X$ introduced in [44]. We recall that a partially ordered group (G, P), consisting of a group G and its subsemigroup $P \subseteq G$ such that $P \cap P^{-1} = \{e\}$, is a *quasi-lattice ordered group* if, under the partial order $g \leqslant h \iff g^{-1}h \in P$, any two elements p, q in G with a common upper bound in P have a least common upper bound $p \vee q$ in P, [35] and [11, Lemma 7]. The semigroup P is *directed* if each pair of elements in P has an upper bound:

$$(\forall p, q \in P)\ (\exists s \in P)\ \ p, q \leqslant s.$$

2.4. **Co-actions, Fell bundles and their C^*-algebras.** Let G be a discrete group. The shortest definition of a *Fell bundle* (also called C^*-algebraic bundle) over G is that it is a collection $\mathcal{B} = \{B_g\}_{g\in G}$ of closed subspaces of a C^*-algebra B such that $B_g^* = B_{g^{-1}}$ and $B_gB_h \subseteq B_{gh}$ for all $g, h \in G$. Then the direct sum $\bigoplus_{g\in G} B_g$ is a $*$-algebra. In general, there are many different C^*-norms on $\bigoplus_{g\in G} B_g$. However, it is well known that there always exists a maximal such norm and it satisfies, cf. [38, Lemma 1.3] or [16], the inequality

$$\|a_e\| \leqslant \|\sum_{g\in G} a_g\|, \qquad \text{for all } \sum_{g\in G} a_g \in \bigoplus_{g\in G} B_g,\ a_g \in B_g,\ g \in G. \tag{5}$$

The completion of $\bigoplus_{g\in G} B_g$ in this maximal C^*-norm is called *cross sectional algebra* of $\mathcal{B}$ and it is denoted $C^*(\mathcal{B})$. Moreover, it follows from [16, Theorem 3.3] that there is also a minimal C^*-norm on $\bigoplus_{g\in G} B_g$ satisfying (5) and a completion of $\bigoplus_{g\in G} B_g$ in this minimal C^*-norm is naturally isomorphic to the *reduced cross sectional algebra* $C_r^*(\mathcal{B})$, as introduced in [16, Definition 2.3] or [38, Definition 3.5]. Both algebras $C^*(\mathcal{B})$ and $C_r^*(\mathcal{B})$ are equipped with natural coactions of G.

We recall (see, for example, [38]) that a coaction of a discrete group G on a C^*-algebra B is an injective and nondegenerate homomorphism $\delta\colon B \to B \otimes C^*(G)$ satisfying the coaction identity $(\delta \otimes id_{C^*(G)}) \circ \delta = (id_B \otimes \delta_G) \circ \delta$, where $\delta_G\colon C^*(G) \to C^*(G) \otimes C^*(G)$ is given by $\delta_G(g) = i_G(g) \otimes i_G(g)$ and $i_G : G \to M(C^*(G))$ is the universal representation of G. The spectral subspaces $B_g^\delta := \{a \in B \mid \delta(a) = a \otimes i_G(g)\}$, $g \in G$, form a Fell bundle $\mathcal{B} = \{B_g^\delta\}_{g\in G}$ and yield a G-gradation of B such that $B = \overline{\bigoplus_{g\in G} B_g}$. Moreover, the norm on $\bigoplus_{g\in G} B_g$ inherited from B satisfies inequality (5).

2.5. **Semigroups of Ore type.** A (left-reversible) Ore semigroup is a cancellative semigroup P which is *left reversible*, that is $sP \cap tP \neq \emptyset$, for all $s, t \in P$. Usually one considers right-reversible Ore semigroups but the left version is more appealing for our purposes. It is well known that a semigroup P is Ore precisely when it can be embedded in a group G in such a way that $G = PP^{-1}$, cf. [10, 34, 3]. For further reference, we include an elementary proof of a slightly more general statement.

We let P be a left reversible and left cancellative semigroup with identity e. We call such a P *semigroup of Ore type* (it is Ore if and only if it is right cancellative as well). The semigroup structure induces a left-invariant preorder on P defined as:

$$p \leqslant q \overset{def}{\iff} pr = q \quad \text{for some } r \in P. \tag{6}$$

If (G, P) is ordered group the (pre)order on P coincides with the one inherited from (G, P). In terms of preorder (6), left reversibility of P simply means that P is directed.

If $p, q \in P$ then left cancellativity implies that relation $pr = q$ determines $r \in P$ uniquely. Thus, we introduce the notation

$$p^{-1}q := r \quad \text{ whenever } \quad pr = q.$$

The *enveloping group* or a *group of fractions* of P is the universal group with the set of generators equal to P and relations $xy = z$, whenever such identity holds in P. To construct the group of fractions explicitly, we first introduce a relation $\sim$ on $P \times P$ as:

$$(p_1, p_2) \sim (q_1, q_2) \overset{def}{\iff} p_1 p = q_1 q, \quad p_2 p = q_2 q \quad \text{for some } p, q \in P. \tag{7}$$

Lemma 2.1. *Relation* (7) *is an equivalence relation on* $P \times P$.

Proof. Reflexivity and symmetry are obvious. To show transitivity, assume that in addition to (7) we also have $(q_1, q_2) \sim (r_1, r_2)$, where $q_1 s = r_1 r$, $q_2 s = r_2 r$ for some $s, r \in P$. Then for any $t \geqslant q, s$ we have

$$p_1 p(q^{-1}t) = q_1 q(q^{-1}t) = q_1 t = q_1 s(s^{-1}t) = r_1 r(s^{-1}t).$$

Similarly, one shows that $p_2 p(q^{-1}t) = r_1 r(s^{-1}t)$. Hence $(p_1, p_2) \sim (r_1, r_2)$. □

We use square brackets to denote the equivalence classes of relation (7):

$$[p_1, p_2] := \{(q_1, q_2) \in P \times P : (p_1, p_2) \sim (q_1, q_2)\},$$

and denote the quotient set by $G(P) := P \times P/\sim$. We define a product on $G(P)$ by the formula

$$[p_1, p_2] \circ [q_1, q_2] := [p_1(p_2^{-1}s), q_2(q_1^{-1}s)] \quad \text{for some} \quad s \geqslant p_2, q_1. \tag{8}$$

This definition is correct due to left cancellativity of P.

Proposition 2.2 (Ore's Theorem)**.** *For the left cancellative and directed semigroup P the quotient set $G(P)$ with the product* (8) *is a group such that $G(P) = \iota(P)\iota(P)^{-1}$, where*

$$P \ni p \overset{\iota}{\longmapsto} [p, e] \in G(P)$$

is a semigroup homomorphism. This homomorphism is injective if and only if P is right cancellative.

Proof. Clearly, $[e, e]$ is a neutral element for product $\circ$ and $[q, p]$ is the inverse of $[p, q]$. To show associativity of $\circ$, let $p_i, q_i, r_i \in P$, $i = 1, 2$, and choose any $s \geqslant p_2, q_1$ and $t \geqslant q_2(q_1^{-1}s), r_1$. Then

$$t = q_2(q_1^{-1}s)z \quad \text{and} \quad s = p_2 y \quad \text{for some} \quad y, z \in P.$$

Thus we have

$$\begin{aligned}\big([p_1,p_2]\circ[q_1,q_2]\big)\circ[r_1,r_2] &= [p_1(p_2^{-1}s), q_2(q_1^{-1}s)]\circ[r_1,r_2]\\ &= [p_1(p_2^{-1}s)\Big(\big(q_2(q_1^{-1}s)\big)^{-1}t\Big), r_2(r_1^{-1}t)]\\ &= [p_1(p_2^{-1}s)z), r_2(r_1^{-1}t)].\end{aligned}$$

On the other hand, putting $u := t$ and

$$w := q_1(q_2^{-1}u) = q_1(q_2^{-1}t) = q_1(q_2^{-1}q_2(q_1^{-1}s))z = q_1(q_1^{-1}s)z = sz = p_2yz$$

we get $u \geqslant q_2, r_1$ and $w \geqslant q_1(q_2^{-1}u), p_2$. Consequently,

$$\begin{aligned}[p_1,p_2]\circ\big([q_1,q_2]\circ[r_1,r_2]\big) &= [p_1,p_2]\circ[q_1(q_2^{-1}u), r_2(r_1^{-1}u)]\\ &= [p_1(p_2^{-1}w), r_2(r_1^{-1}u)\big(q_2(q_1^{-1}u)\big)^{-1}w]\\ &= [p_1(p_2^{-1}s)z, r_2(r_1^{-1}t)],\end{aligned}$$

which proves associativity of $\circ$. As $[p,e]\circ[q,e] = [pq,e]$, because $q \geqslant q, e$, we see that ι is a semigroup homomorphism. Moreover, $[p,e] = [q,e]$ if and only if $pt = qt$ for some $t \in P$, and therefore ι is injective if and only if P is right cancellative. □

Remark 2.3. It follows from the above that the relation $p \sim_R q \iff pr = qr$, for some $r \in P$, is a semigroup congruence on P and the quotient semigroup $P/\sim_R$ is an Ore semigroup whose enveloping group is naturally isomorphic to $G(P)$.

3. Regular product systems of C^*-correspondences and their C^*-algebras

In this section, we first introduce and discuss certain product systems of C^*-correspondences satisfying additional regularity conditions, and then construct their associated Cuntz-Pimsner algebras and their reduced versions in the spirit of the Doplicher-Roberts algebras [15]. Our construction involves an object that may be viewed as a right tensor C^*-precategory over P, see [31]. Regular product systems introduced in this section and their C^*-algebras will play a central role in the remainder of this article.

3.1. **Regular product systems and their right tensor C^*-precategories.**

Definition 3.1. Let X be a C^*-correspondence with coefficients in A. We say X is *regular* if its left action is injective and via compact operators, that is

$$\ker\phi = \{0\} \qquad \text{and} \qquad \phi(A) \subseteq \mathcal{K}(X). \tag{9}$$

We say that a product system $X := \bigsqcup_{p\in P} X_p$ over a semigroup P is *regular* if each fiber X_p, $p \in P$, is a regular C^*-correspondence.

The notions of regularity and tensor product are compatible in the sense that the tensor product of two regular C^*-correspondences is automatically regular, see Proposition 3.3 below.

Before proceeding further we need a technical Lemma 3.2 whose assertion is probably well known to experts, but we include a proof for the sake of completeness.

Lemma 3.2. *Let Y be a regular C^*-correspondence with coefficients in A and let X, Z be right Hilbert A-modules.*

i) *For each $x \in X$, the mapping*

$$Y \ni y \xrightarrow{T_x} x \otimes y \in X \otimes Y$$

is compact, that is $T_x \in \mathcal{K}(Y, X\otimes Y)$. Furthermore, we have $\|T_x\| = \|x\|$.

ii) *For each* $S \in \mathcal{K}(X,Z)$*, we have* $S \otimes 1_Y \in \mathcal{K}(X\otimes Y, Z\otimes Y)$ *and the mapping*

$$\mathcal{K}(X,Z) \ni S \longmapsto S\otimes 1_Y \in \mathcal{K}(X\otimes Y, Z\otimes Y) \tag{10}$$

is isometric. It is surjective whenever $\phi_Y : A \to \mathcal{K}(Y)$ *is.*

Proof. Ad (i). Note that $T_x \in \mathcal{L}(Y, X\otimes Y)$ and $T_x^*(x_0\otimes y_0) = \langle x, x_0\rangle_A y_0$. Let $x = x_0 a$ for some $x_0 \in X$ and $a \in A$. Then $\phi_Y(a) = \lim_{n\to\infty}\sum_i \Theta_{\eta_i^n,\mu_i^n}$ for some η_i^n, $\mu_i^n \in Y$. Thus

$$T_x = T_{x_0}\phi_Y(a) = \lim_{n\to\infty}\sum_i T_{x_0}\Theta_{\eta_i^n,\mu_i^n} = \lim_{n\to\infty}\sum_i \Theta_{x_0\otimes\eta_i^n,\mu_i^n} \in \mathcal{K}(Y, X\otimes Y).$$

As ϕ_Y is isometric, $\phi_Y(\langle x,x\rangle_A) = S^*S$ for some $S\in\mathcal{L}(Y)$ with $\|S\| = \|x\|$, and hence

$$\|T_x\|^2 = \sup_{y\in Y, \|y\|=1} \|\langle y, \phi_Y(\langle x,x\rangle_A)y\rangle_A\| = \sup_{y\in Y, \|y\|=1}\|\langle Sy, Sy\rangle_A\| = \|S^2\| = \|x\|^2.$$

Ad (ii). Let $x\in X$, $z\in Z$ and consider $T_x \in \mathcal{K}(Y, X\otimes Y)$ and $T_z\in\mathcal{K}(Y, Z\otimes Y)$ as in item i). Since

$$\Theta_{z,x}\otimes 1_Y = T_z T_x^* \in \mathcal{K}(X\otimes Y, Z\otimes Y),$$

we have $\mathcal{K}(X,Z)\otimes 1_Y \subseteq \mathcal{K}(X\otimes Y, Z\otimes Y)$. To show that mapping (10) is isometric, we first consider the case $Z = X$. Then (10) is a homomorphism of C^*-algebras and therefore it suffices to show it is injective. To this end, let $S\in\mathcal{K}(X)$ be non-zero. Take $x\in X$ such that $Sx\neq 0$ and $y\in Y$ such that $\phi_Y(\langle Sx, Sx\rangle_A)y \neq 0$. Then

$$\langle (S\otimes 1_Y)x\otimes y, Sx\otimes \phi_Y(\langle Sx,Sx\rangle_A)y\rangle = \langle \phi_Y(\langle Sx,Sx\rangle_A)y, \phi_Y(\langle Sx,Sx\rangle_A)y\rangle_A \neq 0,$$

which implies $S\otimes 1_Y \neq 0$. Consequently, $\|S\otimes 1_Y\| = \|S\|$. Now getting back to the general case (when Z is arbitrary), for $S\in\mathcal{K}(X,Z)$ we have

$$\|S\otimes 1_Y\|^2 = \|S^*S\otimes 1_Y\| = \|S^*S\| = \|S\|^2.$$

If the homomorphism $\phi_Y : A\to\mathcal{K}(Y)$ is surjective, then it is an isomorphism and simple computations show that for $x\in X$, $y_1, y_2\in Y$ and $z\in Z$ we have

$$\Theta_{z\otimes y_1, x\otimes y_2} = \Theta_{z\phi_Y^{-1}(\Theta_{y_1,y_2}),x}\otimes 1_Y.$$

This implies that mapping (10) is surjective. □

Proposition 3.3. *Tensor product of regular* C^**-correspondences is a regular* C^**-correspondence.*

Proof. If X and Y are C^*-correspondences over A then the left action of A on $X\otimes Y$ is $\phi_{X\otimes Y} = \phi_X\otimes 1_Y$. Hence if X and Y are regular, then $\phi_{X\otimes Y}$ is injective and acts by compacts, by Lemma 3.2 part (ii). □

Now, let X be a regular product system over P. The family

$$\mathcal{K}_X := \{\mathcal{K}(X_q, X_p)\}_{p,q\in P}$$

forms in a natural manner a C^*-precategory, [31, Definition 2.2]. We will describe a right tensoring structure on $\mathcal{K}_X$ by introducing a family of mappings $\iota_{p,q}^{pr,qr} : \mathcal{K}(X_q, X_p) \to \mathcal{K}(X_{qr}, X_{pr})$, $p,q,r\in P$, cf. [31, Example 3.2], which extends the standard family of diagonal homomorphisms ι_q^{qp} defined in Subsection 2.3 (when restricted to compact operators). If $q\neq e$ we put

$$\iota_{p,q}^{pr,qr}(T)(xy) := (Tx)y, \qquad \text{where } x\in X_q,\ y\in X_r \text{ and } T\in\mathcal{K}(X_q,X_p).$$

Note that under the canonical isomorphism $X_{pq}\cong X_p\otimes_A X_q$ operator $\iota_{p,q}^{pr,qr}(T)$ corresponds to $T\otimes 1_{X_r}$. Hence by part (ii) of Lemma 3.2, $\iota_{p,q}^{pr,qr}(T)\in\mathcal{K}(X_{qr},X_{pr})$ and $\iota_{p,q}^{pr,qr}$ is isometric. Similarly, in the case $q = e$, using (4), the formula

$$\iota_{p,e}^{pr,r}(t_x)(y) := xy, \qquad \text{where } y\in X_r \text{ and } t_x\in\mathcal{K}(X_e, X_p), x\in X_p,$$

yields a well defined map. By Lemma 3.2 part (i), this is an isometry from $\mathcal{K}(X_e, X_p)$ into $\mathcal{K}(X_r, X_{pr})$. Note that $\iota_{p,p}^{pr,pr} = \iota_p^{pr}$.

Definition 3.4. The C^*-precategory $\mathcal{K}_X := \{\mathcal{K}(X_q, X_p)\}_{p,q\in P}$ equipped with the family of maps $\{\iota_{p,q}^{pr,qr}\}_{p,q,r\in P}$ defined above is called *a right tensor C^*-precategory associated to the regular product system X.*

Remark 3.5. $\mathcal{K}_X$ is a C^*-precategory in the sense of [31, Definition 2.2] whose objects are elements of P. One readily sees that the isometric linear maps $\iota_{p,q}^{pr,qr} : \mathcal{K}(X_q, X_p) \to \mathcal{K}(X_{qr}, X_{pr})$, $p, q, r \in P$, satisfy

$$\iota_{p,q}^{pr,qr}(T)^* = \iota_{q,p}^{qr,pr}(T^*), \qquad \iota_{p,q}^{pr,qr}(T)\iota_{q,s}^{qr,sr}(S) = \iota_{p,s}^{pr,sr}(TS), \tag{11}$$

$$\iota_{pr,qr}^{prs,qrs}(\iota_{p,q}^{pr,qr}(T)) = \iota_{p,q}^{prs,qrs}(T), \tag{12}$$

for all $T \in \mathcal{K}(X_q, X_p)$, $S \in \mathcal{K}(X_s, X_q)$, $p, q, r, s \in P$. Thus, if we adopt the notation

$$T \otimes 1_r := \iota_{p,q}^{pr,qr}(T), \qquad T \in \mathcal{K}(X_q, X_p),\ p, q \in P,$$

then (11) means that $\otimes 1_r : \mathcal{K}_X \to \mathcal{K}_X$ is a C^*-precategory monomorphism sending p to pr, see [31, Definition 2.8], and (12) states that $\otimes 1_r \circ \otimes 1_s = \otimes 1_{rs}$, that is $\{\otimes 1_r\}_{r\in P}$ is a semigroup action on $\mathcal{K}_X$. In particular, the pair $(\mathcal{K}_X, \{\otimes 1_r\}_{r\in P})$, which is another presentation of $(\mathcal{K}_X, \{\iota_{p,q}^{pr,qr}\}_{p,q,r\in P})$, is a (strict) right tensor C^*-category (cf. e.g., [15]) when each of the algebra $\mathcal{K}(X_p)$, $p \in P$, is unital.

The following lemma could be considered a counterpart of [31, Proposition 3.14].

Lemma 3.6. *Let ψ be a representation of a regular product system X over a semigroup P in a C^*-algebra B. For each $p, q \in P$ we have a contractive linear map $\psi_{p,q} : \mathcal{K}(X_q, X_p) \longrightarrow B$ determined by the formula*

$$\psi_{p,q}(\Theta_{x,y}) = \psi_p(x)\psi_q(y)^* \quad \text{for } x \in X_p,\ y \in X_q. \tag{13}$$

Mappings $\{\psi_{p,q}\}_{p,q\in P}$ satisfy

$$\psi_{p,q}(S)\psi_{q,r}(T) = \psi_{p,r}(ST) \quad \text{for } S \in \mathcal{K}(X_q, X_p),\ T \in \mathcal{K}(X_r, X_q),\ p, q, r \in P, \tag{14}$$

and are all isometric if ψ is injective. If ψ is Cuntz-Pimsner covariant, then

$$\psi_{p,q}(S) = \psi_{pr,qr}(\iota_{p,q}^{pr,qr}(S)) \qquad \text{for all } p, q, r \in P \text{ and } S \in \mathcal{K}(X_q, X_p). \tag{15}$$

Proof. It is not completely trivial but quite well known that (13) defines a linear contraction which is isometric if ψ_e is injective, see for instance the proof of Lemma 2.2 in [25]. One readily sees that (14) holds for 'rank one' operators $S = \Theta_{u,w}$, $T = \Theta_{v,z}$, and thus it holds in general. To see (15), suppose that ψ is a Cuntz-Pimsner covariant representation on a Hilbert space H and $C^*(\psi(X))H = H$. Since $\psi : X \to \mathcal{B}(H)$ is a semigroup homomorphism, the essential spaces

$$H_p := \psi^{(p)}(\mathcal{K}(X_p))H = \psi_p(X_p)H$$

of algebras $\psi^{(p)}(\mathcal{K}(X_p))$, $p \in P$, form a decreasing family with respect to pre-order (6):

$$p \leqslant q \implies H_p \supseteq H_q.$$

In particular, $H = H_e = \psi_e(A)H$ and actually $H = H_p$ for all $p \in P$, since $\psi_e(A) \subseteq \psi^{(p)}(\mathcal{K}(X_p))$ by Cuntz-Pimsner covariance. Hence the linear span of elements of the form

$\psi_{qr}(x_0y_0)h$, $x_0 \in X_q$, $y_0 \in X_r$, $h \in H$, is dense in H and (15) follows from the following computation:

$$\begin{aligned}\psi_{pr,qr}(\iota_{p,q}^{pr,qr}(\Theta_{x,y}))\psi_{qr}(x_0y_0) &= \psi_{pr}(\iota_{p,q}^{pr,qr}(\Theta_{x,y})x_0y_0) = \psi_{pr}((\Theta_{x,y}x_0)y_0)\\ &= \psi_{pr}(x\langle y,x_0\rangle y_0) = \psi_p(x)\psi_q(y)^*\psi_q(x_0)\psi_r(y_0)\\ &= \psi_{p,q}(\Theta_{x,y})\psi_{qr}(x_0y_0).\end{aligned}$$

□

3.2. **Doplicher-Roberts picture of a Cuntz-Pimsner algebra and its reduced version.** Throughout this subsection we assume that X is a regular product system over a semigroup of Ore type, see Subsection 2.5. For the proof of the main result of this section we need the following lemma, cf. [22, Proposition 5.10].

Lemma 3.7. *Suppose ψ is a Cuntz-Pimsner covariant representation of a regular product system X over a semigroup P of Ore type.*

i) *For all $x \in X_p$, $y \in X_q$ and $s \geqslant p,q$ we have*
$$\psi_p(x)^*\psi_q(y) \in \overline{\operatorname{span}}\{\psi(f)\psi(h)^* : f \in X_{p^{-1}s}, h \in X_{q^{-1}s}\}.$$
ii) *We have the equality*
$$\begin{aligned}&\overline{\operatorname{span}}\{\psi(x)\psi^*(y) : x,y \in X,\ [d(x),d(y)] = [p,q]\}\\ &= \overline{\operatorname{span}}\{\psi(x)\psi^*(y) : x \in X_{pr}, y \in X_{qr},\ r \in P\}.\end{aligned}$$
iii) $C^*(\psi(X)) = \overline{\operatorname{span}}\{\psi(x)\psi(y)^* : x,y \in X\}$. *Furthermore, there is a dense subspace of $C^*(\psi(X))$ consisting of elements of the form*
$$\psi^{(q)}(S_q) + \sum_{p\in F}\psi_{p,q}(S_{p,q}) \tag{16}$$
where $q \in P$ and $F \subseteq P$ is a finite set such that $q \nsim_R p$ for all $p \in F$, cf. Remark 2.3.

Proof. Ad (i). Write $x = Sx'$ with $S \in \mathcal{K}(X_p)$ and $x' \in X_p$, and similarly $y = Ty'$ with $T \in \mathcal{K}(X_q)$, $y' \in X_q$. Then using (13), (15) and (11) we get
$$\psi_p(x)^*\psi_q(y) = \psi_p(x')^*\psi^{(p)}(S^*)\psi^{(q)}(T)\psi_q(y') = \psi_p(x')^*\psi^{(s)}(\iota_p^s(S^*)\iota_p^s(T))\psi_q(y').$$
Since $\iota_p^s(S^*)\iota_p^s(T) \in \mathcal{K}(X_s)$ we may approximate $\psi^{(s)}(\iota_p^s(S^*)\iota_p^s(T))$ with finite sums of operators of the form $\psi_s(f'f)\psi_s(h'h)^*$, where $f' \in X_p$, $f \in X_{p^{-1}s}$ and $h' \in X_q$, $h \in X_{q^{-1}s}$. Hence $\psi_p(x)^*\psi_q(y)$ can be approximated by finite sums of elements of the form
$$\psi_p(x')^*\psi_s(f'f)\psi_s(h'h)^*\psi_q(y') = \psi_{p^{-1}s}(\langle x',f'\rangle_p f)\psi_{q^{-1}s}(\langle y',h'\rangle h)^*.$$
This proves claim (i).
Ad (ii). Clearly, $\overline{\operatorname{span}}\{\psi(x)\psi^*(y) : x,y \in X,\ [d(x),d(y)] = [p,q]\}$ contains $\overline{\operatorname{span}}\{\psi(x)\psi^*(y) : x \in X_{pr}, y \in X_{qr},\ r \in P\}$. To see the converse inclusion, we use the mappings introduced in Lemma 3.6 and assume that $[p',q'] = [p,q]$, that is $p'r' = pr$ and $q'r' = qr$ for some $r,r' \in P$. Then by (15) for $T \in \mathcal{K}(X_{q'},X_{p'})$ we have
$$\psi_{p',q'}(T) = \psi_{p'r',q'r'}(\iota_{p',q'}^{p'r',q'r'}(T)) = \psi_{pr,qr}(\iota_{p',q'}^{p'r',q'r'}(T)) \in \overline{\operatorname{span}}\{\psi(x)\psi^*(y) : x \in X_{pr}, y \in X_{qr}\},$$
which proves our claim.
Ad (iii). Part (i) implies that $C^*(\psi(X))$ is the closure of elements of the form
$$\sum_{i=1}^n \psi_{p_i}(x_i)\psi_{q_i}(y_i)^*, \tag{17}$$

where $p_i, q_i \in P$, $x_i \in X_{p_i}$, $y_i \in X_{q_i}$, $i = 1, ..., n$. Moreover, taking any $q_0 \in P$ that dominates all q_i, $i = 1, ..., n$, and writing $y_i = y_i' a_i$ with $y_i' \in X_{q_i}$, $a_i \in A$, we get

$$\psi_{p_i}(x_i)\psi_{q_i}(y_i)^* = \psi_{p_i}(x_i)\psi^{(q_i^{-1}q_0)}(\phi_{q_i^{-1}q_0}(a_i^*))\psi_{q_i}(y_i')^*, \qquad i = 1, ..., n.$$

Approximating $\psi^{(q_i^{-1}q_0)}(\phi_{q_i^{-1}q_0}(a_i^*))$ by finite sums of elements of the form $\psi_{q_i^{-1}q_0}(u_i)\psi_{q_i^{-1}q_0}(v_i)^*$ we see that $\psi_{p_i}(x_i)\psi_{q_i}(y_i)^*$ can be approximated by finite sums of elements of the form

$$\psi_{p_i}(x_i)\psi_{q_i^{-1}q_0}(u_i)\psi_{q_i^{-1}q_0}(v_i)^*\psi_{q_i}(y_i')^* = \psi_{p_i q_i^{-1}q_0}(x_i u_i)\psi_{q_0}(y_i' v_i)^*.$$

Thus we see that the element (17) can be presented in the form

$$\sum_{p \in F'} \psi_{p,q_0}(S_{p,q_0}) \tag{18}$$

where $F' = \{p_i q_i^{-1} q_0 : i = 1, ..., n\} \subseteq P$ is a finite set. Let $F_0 = \{p \in F' : q_0 \sim_R p\}$ and for each $p \in F_0$ choose $r_p \in P$ such that $pr_p = q_0 r_p$. Let $r \in P$ be such that $r \geqslant r_p$ for all $p \in F_0$, and put

$$q := q_0 r \quad \text{ and } \quad F := \{pr : p \in F' \backslash F_0\}.$$

Then $pr = q$ for all $p \in F_0$, and $p \nsim_R q$ for all $p \in F$. By (14) we have $\psi_{p,q_0}(S_{p,q_0}) \in \psi_{pr,q_0r}(\mathcal{K}(X_{q_0r}, X_{pr})) = \psi_{pr,q}(\mathcal{K}(X_q, X_{pr}))$ and hence the element (18) can be presented in the form (16). □

Now, we are ready to prove the main theorem of this section. It gives a direct construction of the Cuntz-Pimsner algebra $\mathcal{O}_X$ of a regular product system X as the full cross-sectional C^*-algebra of a suitable Fell bundle corresponding to the limits of directed systems of the compact operators arising from X.

Theorem 3.8. *Let X be a regular product system over a semigroup P of Ore type and let $G(P)$ be the enveloping group of P. For each $[p, q] \in G(P)$ we define*

$$B_{[p,q]} := \varinjlim \mathcal{K}(X_{qr}, X_{pr})$$

to be the Banach space direct limit of the directed system $\left(\{\mathcal{K}(X_{qr}, X_{pr})\}_{r \in P}, \{\iota_{pr,qr}^{ps,qs}\}_{\substack{r,s \in P \\ r \leqslant s}}\right)$. *The family $\mathcal{B} = \{B_t\}_{t \in G(P)}$ is in a natural manner equipped with the structure of a Fell bundle over $G(P)$ and we have a canonical isomorphism*

$$\mathcal{O}_X \cong C^*(\{B_g\}_{g \in G(P)})$$

from the Cuntz-Pimsner algebra $\mathcal{O}_X$ onto the full cross-sectional C^*-algebra $C^*(\{B_g\}_{g \in G(P)})$. In particular,

i) *the universal representation $j_X : X \to \mathcal{O}_X$ is injective,*
ii) *$\mathcal{O}_X$ has a natural grading $\{(\mathcal{O}_X)_g\}_{g \in G(P)}$ over $G(P)$, such that*

$$(\mathcal{O}_X)_g = \overline{\operatorname{span}}\{j_X(x) j_X(y)^* : x, y \in X,\ [d(x), d(y)] = g\}. \tag{19}$$

iii) *for every injective representation ψ of X, the integrated representation Π_ψ of $\mathcal{O}_X$ is isometric on each Banach space $(\mathcal{O}_X)_g$, $g \in G(P)$, and thus it restricts to an isomorphism of the core C^*-subalgebra of $\mathcal{O}_X$, namely*

$$(\mathcal{O}_X)_e = \overline{\operatorname{span}}\{j_X(x) j_X(y)^* : x, y \in X,\ d(x) = d(y)\}.$$

Proof. As the direct limit $\varinjlim \mathcal{K}(X_{qr}, X_{pr})$ depends only on 'sufficiently large r', it follows immediately from (7) that the limit does not depend on the choice of a representative of $[p, q]$ and thus $B_{[p,q]}$ is well defined. Let $\varphi_{p,q} : \mathcal{K}(X_q, X_p) \to B_{[p,q]}$ denote the natural embedding of $\mathcal{K}(X_q, X_p)$ into $B_{[p,q]}$. It is isometric because all the connecting maps $\iota_{pr,qr}^{ps,qs}$, $r \leqslant s$, are.

Using the (inductive) properties of the mappings $\varphi_{p,q}$ and (right tensoring) properties (11), (12) of the mappings $\iota_{p,q}^{pr,qr}$, one sees that the formula

$$\varphi_{p_1,p_2}(S)\circ\varphi_{q_1,q_2}(T):=\varphi_{p_1(p_2^{-1}s),q_2(q_1^{-1}s)}\Big(\iota_{p_1,p_2}^{p_1(p_2^{-1}s),s}(S)\iota_{q_1,q_2}^{s,q_2(q_1^{-1}s)}(T)\Big),$$

where $s\geqslant p_2,q_1$, $S\in\mathcal{K}(X_{p_2},X_{p_1})$, $T\in\mathcal{K}(X_{q_2},X_{q_1})$, yields well defined bilinear maps

$$\circ: B_{[p_1,p_2]}\times B_{[q_1,q_2]}\to B_{[p_1,p_2]\circ[q_1,q_2]}.$$

These maps establish an associative multiplication $\circ$ on $\{B_t\}_{t\in G(P)}$, satisfying

$$\|a\circ b\|\leqslant\|a\|\cdot\|b\|.$$

Hence $\{B_t\}_{t\in G(P)}$ becomes a Banach algebraic bundle, cf. e.g. [17, Definition 2.2. parts (i)–(iv)]. Similarly, formula

$$\varphi_{p_1,p_2}(S)^*:=\varphi_{p_2,p_1}(S^*),\qquad S\in\mathcal{K}(X_{p_2},X_{p_1}),$$

defines a '$*$' operation that satisfies axioms [17, Definition 2.2. parts (v)–(xi)] and hence we get a Fell bundle structure on $\{B_g\}_{g\in G(P)}$ (we omit straightforward but tedious verification of the details).

Now, we view $C^*(\{B_g\}_{g\in G(P)})$ as a maximal C^*-completion of the direct sum $\bigoplus_{g\in G(P)}B_g$. Using the maps (4), we define mappings

$$\Psi: X=\bigsqcup_{p\in P}X_p\to C^*(\{B_g\}_{g\in G(P)})$$

by

$$X_p\ni x\longmapsto\varphi_{p,e}(t_x),\qquad p\in P. \tag{20}$$

Since (4) is an isomorphism of C^*-correspondences, it follows that Ψ restricted to each summand X_p is an injective representation of a C^*-correpondence. Moreover, for $x\in X_p$, $y\in X_q$ we have $t_{xy}=i_{p,e}^{pq,q}(t_x)t_y$ and thus

$$\Psi(x)\Psi(y)=\varphi_{p,e}(t_x)\circ\varphi_{q,e}(t_y)=\varphi_{pq,e}(i_{p,e}^{pq,q}(t_x)t_y)=\varphi_{pq,e}(t_{xy})=\Psi(xy).$$

Hence Ψ is a faithful representation of the product system X in $C^*(\{B_t\}_{t\in G(P)})$. We recall that $\iota_{e,e}^{p,p}(t_a)=\iota_e^p(a)=\phi_p(a)$ and hence

$$\Psi(a)=\varphi_{e,e}(t_a)=\varphi_{p,p}(\iota_{e,e}^{p,p}(t_a))=\varphi_{p,p}(\phi_p(a))=\Psi(\phi_p(a)),\qquad a\in A,\ p\in P,$$

that is Ψ is Cuntz-Pimsner covariant. Since Ψ is injective, so is j_X and claim (i) holds. Now, considering the integrated representation $\Pi_\Psi:\mathcal{O}_X\to C^*(\{B_g\}_{g\in G(P)})$, for $x\in X_p$, $y\in X_q$ we have

$$\Pi_\Psi(j_X(x)j_X(y)^*)=\Psi(x)\circ\Psi(y)^*=\varphi_{p,e}(t_x)\circ\varphi_{e,q}(t_y^*)=\varphi_{p,q}(t_xt_y^*)=\varphi_{p,q}(\Theta_{x,y}). \tag{21}$$

It follows that Π_Ψ maps

$$(\mathcal{O}_X)_{[p,q]}:=\overline{\operatorname{span}}\{j_X(x)j_X(y)^*: x\in X_{pr}, y\in X_{qr},\ r\in P\}$$

onto $B_{[p,q]}$. Putting $g=[p,q]$ and using Lemma 3.7 part (iii), we see that $(\mathcal{O}_X)_g$ is given by (19). We claim that Π_Ψ is injective on $(\mathcal{O}_X)_g$. To see this, let $j_{p,q}$ denote the mappings from Lemma 3.6 associated to the universal representation j_X and note that we have

$$j_{ps,qs}\circ\iota_{pr,qr}^{ps,qs}=j_{pr,qr}\qquad\text{for } r\leqslant s$$

by (15). By the universal property of inductive limits, there is a mapping

$$B_{[p,q]}\ni\phi_{pr,qr}(T)\mapsto j_{pr,qr}(T)\in(\mathcal{O}_X)_{[p,q]},$$

which is inverse to $\Pi_\Psi|_{[p,q]}$. Accordingly, Π_Ψ is an epimorphism injective on each $(\mathcal{O}_X)_g$. Since the spaces B_g, $g \in G(P)$, are linearly independent, so are $(\mathcal{O}_X)_g$, $g \in G(P)$. Consequently, in view of Lemma 3.7 we have

$$\mathcal{O}_X = \overline{\bigoplus_{g\in G(P)} (\mathcal{O}_X)_g}$$

and claim (ii) follows. In particular, $\Pi_\Psi : \bigoplus_{g\in G(P)}(\mathcal{O}_X)_g \to \bigoplus_{g\in G(P)} B_g$ is an isomorphism and as $C^*(\{B_g\}_{g\in G(P)})$ is the closure of $\bigoplus_{g\in G(P)} B_g$ in a maximal C^*-norm we see that Π_Ψ actually yields the desired isomorphism $\mathcal{O}_X \cong C^*(\{B_g\}_{g\in G(P)})$.

For the proof of part (iii), notice that we have just showed that $(\mathcal{O}_X)_{[p,q]}$ is the closure of the increasing union $\bigcup_{r\in P} j_{pr,qr}(\mathcal{K}(X_{qr}, X_{pr}))$, where $j_{pr,qr} : \mathcal{K}(X_{qr}, X_{pr}) \to (\mathcal{O}_X)_{[p,q]}$ are isometric maps. Similarly, if ψ is an injective covariant representation of X, then $\Pi_\psi((\mathcal{O}_X)_{[p,q]})$ is the closure of the increasing union $\bigcup_{r\in P} \psi_{pr,qr}(\mathcal{K}(X_{qr}, X_{pr}))$, and by Lemma 3.6 mappings $\psi_{pr,qr} : \mathcal{K}(X_q, X_p) \to \Pi_\Psi((\mathcal{O}_X)_{[p,q]})$ are isometric. Since $\Pi_\psi \circ j_{pr,qr} = \psi_{pr,qr}$, $p, q, r \in P$, it follows that surjection $\Pi_\psi : (\mathcal{O}_X)_{[p,q]} \to \psi((\mathcal{O}_X)_{[p,q]})$ is an isometry, since it is isometric on a dense subset. □

Remark 3.9. Theorem 3.8 has a number of remarkable consequences.

(i) The Cuntz-Pimsner algebra $\mathcal{O}_X$ can be constructed in a natural manner as the full cross-sectional algebra $C^*(\mathcal{B})$ of the Fell bundle $\mathcal{B} = \{B_t\}_{t\in G(P)}$. Thus it is justified to call the reduced cross-sectional algebra

$$\mathcal{O}_X^r := C_r^*(\mathcal{B})$$

the *reduced Cuntz-Pimsner algebra of* X. In particular, $\mathcal{O}_X^r$ is the C^*-algebra $C^*(j_X^r(X))$ generated by an injective Cuntz-Pimnser representation $j_X^r : X \to \mathcal{O}_X^r = C_r^*(\{B_g\}_{g\in G(P)})$ acting according to (20). When P is Ore and $(G(P), \iota(P))$ is a quasi-lattice ordered group then $\mathcal{O}_X^r$ coincides with the co-universal C^*-algebra $\mathcal{N}\mathcal{O}_X^r$ introduced and investigated in [8].

(ii) Our construction yields a faithful Cuntz-Pimsner representation of X and thus the Cuntz-Pimsner algebra $\mathcal{O}_X$ does not degenerate (it contains an isomorphic copy of X). This addresses the problem raised already by Fowler in [22, Remark 2.10]. Until now, this problem was solved positively in the case P is Ore and $(G(P), \iota(P))$ is a quasi-lattice ordered group, in which case $\mathcal{O}_X$ coincides with the Cuntz-Nica-Pimsner algebra $\mathcal{N}\mathcal{O}_X$.

(iii) When P is Ore and $(G(P), \iota(P))$ is a quasi-lattice ordered group then part (iii) of Theorem 3.8 coincides with [8, Theorem 3.8]. In general, this result leads to (or actually could be considered as a version of) the so-called gauge invariant uniqueness theorem, cf. Proposition 5.1 below.

4. Dual objects

In essence, the dual objects we investigate are relations. However, we would like to think of them in dynamical terms and therefore we will consider relations as multivalued maps, see subsection 2.1 for the relevant terminology and conventions.

4.1. **Multivalued maps dual to homomorphisms of C^*-algebras.** Let A be a C^*-algebra. We denote by $\simeq$ the unitary equivalence relation between representations of A, and by $[\pi]$ the corresponding equivalence class of $\pi : A \to \mathcal{B}(H)$. Spectrum $\widehat{A} = \{[\pi] : \pi \in \mathrm{Irr}(A)\}$ consists of the equivalence classes of all irreducible representations of A, equipped with the Jacobson topology. The relation $\leqslant$ of being a subrepresentation factors through $\simeq$

to a relation $\preceq$ on $\widehat{A}$. Namely, if $\pi : A \to \mathcal{B}(H_\pi)$ and $\rho : A \to \mathcal{B}(H_\rho)$ are representations of A, then

$$[\pi] \preceq [\rho] \iff \exists \text{ isometry } U : H_\pi \to H_\rho \text{ s. t. } (\forall a \in A)\ \pi(a) = U^*\rho(a)U.$$

Let $\alpha : A \to B$ be a homomorphism between two C^*-algebras. It is useful to think of the dual map we aim to define as a factorization of a multivalued map $\widehat{\alpha}_0 : \mathrm{Irr}(B) \to \mathrm{Irr}(A)$ given by

$$\widehat{\alpha}_0(\pi_B) = \{\pi_A \in \mathrm{Irr}(A) : \pi_A \leqslant \pi_B \circ \alpha\}. \tag{22}$$

The set $[\widehat{\alpha}_0(\pi_B)] := \{[\pi_A] \in \widehat{A} : \pi_A \leqslant \pi_B \circ \alpha\}$ does not depend on the choice of a representative of the class $[\pi_B]$ and thus the following definition make sense.

Definition 4.1. The *dual map* to a homomorphism $\alpha : A \to B$ is a multivalued map $\widehat{\alpha} : \widehat{B} \to \widehat{A}$ given by the formula

$$\begin{aligned}\widehat{\alpha}([\pi_B]) :=& \{[\pi_A] \in \widehat{A} : [\pi_A] \preceq [\pi_B \circ \alpha]\} \\ =& \{[\pi_A] \in \widehat{A} : \pi_A \leqslant \pi_B \circ \alpha\}.\end{aligned}$$

The range of $\widehat{\alpha}$ behaves exactly as one would expect. But for non-liminal B the map $\widehat{\alpha}$, and in particular its domain, has to be treated with care. Let us explain it with help of the following proposition and an example.

Proposition 4.2. *For every homomorphism $\alpha : A \to B$ between two C^*-algebras, its image*

$$\widehat{\alpha}(\widehat{B}) = \{[\pi_A] \in \widehat{A} : \ker \pi_A \supseteq \ker \alpha\}$$

is a closed subset of $\widehat{A}$. Its domain $D(\widehat{\alpha})$ is contained in an open subset $\{[\pi_B] \in \widehat{B} : \ker \pi_B \not\supseteq B\alpha(A)B\}$ of $\widehat{B}$. Moreover, if B is liminal, then

$$D(\widehat{\alpha}) = \{[\pi_B] \in \widehat{B} : \ker \pi_B \not\supseteq B\alpha(A)B\}$$

and $\widehat{\alpha} : \widehat{B} \to \widehat{A}$ is continuous.

Proof. If $[\pi_A] \in \widehat{\alpha}(\widehat{B})$, then $\pi_A \leqslant \pi_B \circ \alpha$ for some $\pi_B \in \mathrm{Irr}(B)$, and hence $\ker \pi_A \supseteq \ker \alpha$. Conversely, if $[\pi_A] \in \widehat{A}$ is such that $\ker \pi_A \supseteq \ker \alpha$, then π_A factors through to the irreducible representation of $A/\ker\alpha \cong \alpha(A)$. Thus the formula $\pi(\alpha(a)) := \pi_A(a)$, $a \in A$, yields a well defined element of $\mathrm{Irr}(\alpha(A))$. Extending π to any $\pi_B \in \mathrm{Irr}(B)$ one has $\pi_A \leqslant \pi_B \circ \alpha$.

Now, let J be an ideal of A. Then $\widehat{J} = \{[\pi_A] \in \widehat{A} : \ker \pi \not\supseteq J\}$ is open and we have

$$\begin{aligned}[\pi_B] \in \widehat{\alpha}^{-1}(\widehat{J}) &\iff \exists_{\pi_A \in \mathrm{Irr}(A)}\ \pi_A \leqslant \pi_B \circ \alpha,\ \ \ker \pi_A \not\supseteq J \\ &\implies \ker(\pi_B \circ \alpha) \not\supseteq J \\ &\iff \ker \pi_B \not\supseteq \alpha(J) \\ &\iff \ker \pi_B \not\supseteq B\alpha(J)B.\end{aligned}$$

That is, $\widehat{\alpha}^{-1}(\widehat{J}) \subseteq \{\pi_B \in \widehat{B} : \ker \pi_B \not\supseteq B\alpha(J)B\}$ and in particular $D(\widehat{\alpha}) = \widehat{\alpha}^{-1}(\widehat{A}) \subseteq \{\pi_B \in \widehat{B} : \ker \pi_B \not\supseteq B\alpha(A)B\}$.

If we additionally assume that B is liminal, then for $\pi_B \in \mathrm{Irr}(B)$ the representation $\pi_B \circ \alpha$ decomposes into a direct sum of irreducibles, see for instance [14, §5.4.13]. Namely, there is a subset K of $\widehat{\alpha}_0(\pi_B)$ such that $\pi_B \circ \alpha = \bigoplus_{\pi_A \in K} \pi_A \oplus 0$ (where 0 stands for the zero representation and is vacuous if $\pi_B \circ \alpha$ is nondegenerate). Hence the implication

$$\ker(\pi_B \circ \alpha) \not\supseteq J \implies \exists \pi_A \in K \subseteq \mathrm{Irr}(A) \text{ s. t. } \pi_A \leqslant \pi_B \circ \alpha,\ \ \ker \pi_A \not\supseteq J$$

holds true. This combined with the preceding argument yields $\widehat{\alpha}^{-1}(\widehat{J}) = \{\pi_B \in \widehat{B} : \ker \pi_B \not\supseteq B\alpha(J)B\}$ and the second part of the assertion follows. □

Example 4.3. Let $H = L^2_\mu[0,1]$ with μ the Lebesgue measure. Put $B := \mathcal{B}(H)$, $A := L^\infty[0,1]$ and let $\alpha : A \to B$ be the monomorphism sending $a \in A$ to the operator of multiplication by a. Then $\pi_B = id$ is irreducible and $\pi_B \circ \alpha$ is faithful but $\widehat{\alpha}([\pi_B]) = \emptyset$. Accordingly,

$$D(\widehat{\alpha}) \neq \{[\pi_B] \in \widehat{B} : \ker \pi_B \not\supseteq B\alpha(A)B\} = \widehat{B}.$$

4.2. **Multivalued maps dual to regular C^*-correspondences.** Let X be a regular C^*-correspondence with coefficients in A. We may treat X as a $\mathcal{K}(X)$-$\langle X, X\rangle_A$-imprimitivity bimodule and therefore the induced representation functor $X\text{-Ind} : \mathrm{Irr}(\langle X, X\rangle_A) \to \mathrm{Irr}(\mathcal{K}(X))$ factors through to the homeomorphism $[X\text{-Ind}] : \widehat{\langle X, X\rangle_A} \to \widehat{\mathcal{K}(X)}$, which in turn may be viewed as a multivalued map $[X\text{-Ind}] : \widehat{A} \to \widehat{\mathcal{K}(X)}$ with domain $D([X\text{-Ind}]) = \widehat{\langle X, X\rangle_A}$.

Definition 4.4. Let X be a regular C^*-correspondence over A. We define *dual map* $\widehat{X} : \widehat{A} \to \widehat{A}$ to X as the following composition of multivalued maps

$$\widehat{X} = \widehat{\phi} \circ [X\text{-Ind}],$$

where $\widehat{\phi} : \widehat{\mathcal{K}(X)} \to \widehat{A}$ is dual to the left action $\phi : A \to \mathcal{K}(X)$ of A on X.

Alternatively, $\widehat{X}$ is a factorization of the map $\widehat{X}_0 := \widehat{\phi}_0 \circ X\text{-Ind} : \mathrm{Irr}(A) \to \mathrm{Irr}(A)$, cf. (22).

Proposition 4.5. *The multivalued map dual to a regular C^*-correspondence X is always surjective, that is $\widehat{X}(\widehat{A}) = \widehat{A}$. The domain of $\widehat{X}$ satisfies the following inclusion*

$$D(\widehat{X}) \subseteq \widehat{\langle X, \phi(A)X\rangle_A}. \tag{23}$$

Note here that $\langle X, \phi(A)X\rangle_A$ is an ideal in A. If, in addition, A is liminal, then $\widehat{X}$ is a continuous multivalued map and we have the equality in (23)*; in particular, if X is full and essential, then $\widehat{X} : \widehat{A} \to \widehat{A}$ is a continuous multivalued surjection with the full domain, $D(\widehat{X}) = \widehat{A}$.*

Proof. As $[X\text{-Ind}] : \widehat{A} \to \widehat{\mathcal{K}(X)}$ is surjective and $\ker \phi = \{0\}$ we get $\widehat{X}(\widehat{A}) = \widehat{A}$ by Proposition 4.2. Since $[X\text{-Ind}] : \widehat{\langle X, X\rangle_A} \to \widehat{\mathcal{K}(X)}$ is a homeomorphism, it follows from Proposition 4.2 that

$$D(\widehat{X}) \subseteq [X\text{-Ind}]^{-1}(\mathcal{K}(X)\widehat{\phi(A)}\mathcal{K}(X)) \tag{24}$$

with equality if A is liminal (note that if A is liminal then $\mathcal{K}(X)$ is also liminal being Morita-Rieffel equivalent to the liminal C^*-algebra $\langle X, X\rangle_A \subseteq A$). Hence it suffices to show that the sets in the right hand sides of (23) and (24) coincide. However, for any representation π of A and any C^*-subalgebra $B \subseteq \mathcal{K}(X)$ we have

$$B \subseteq \ker(X\text{-Ind}(\pi)) \iff \pi(\langle BX, BX\rangle_A) = 0 \iff \langle X, BX\rangle_A \subseteq \ker \pi.$$

Thus the assertion follows from the equality

$$\langle X, \mathcal{K}(X)\phi(A)\mathcal{K}(X)X\rangle_A = \langle \mathcal{K}(X)X, \phi(A)\mathcal{K}(X)X\rangle_A = \langle X, \phi(A)X\rangle_A.$$

□

In view of Proposition 3.3, if X and Y are regular C^*-correspondences with coefficients in A, then the tensoring on the right by the identity 1_Y in Y yields a homomorphism $\otimes 1_Y : \mathcal{K}(X) \to \mathcal{K}(X \otimes Y)$. With help of its dual map we are able to analyze the relationship between the spectra of compact operators on the level of spectrum of A.

Proposition 4.6. *Let X and Y be regular C^*-correspondences with coefficients in A. Then we have*

$$[X\text{-Ind}] \circ \widehat{Y} = \widehat{\otimes 1_Y} \circ [(X \otimes Y)\text{-Ind}]. \tag{25}$$

In other words, the diagram of multivalued maps

$$\begin{tikzcd} \widehat{A} \arrow[r, "(X\otimes Y)\text{-Ind}"] \arrow[d, "\widehat{Y}"] & \widehat{\mathcal{K}(X \otimes Y)} \arrow[d, "\widehat{\otimes 1_Y}"] \\ \widehat{A} \arrow[r, "X\text{-Ind}"] & \widehat{\mathcal{K}(X)} \end{tikzcd}$$

is commutative, and in particular

$$D([X\text{-Ind}] \circ \widehat{Y}) = D(\widehat{\otimes 1_Y} \circ [(X \otimes Y)\text{-Ind}]) = \widehat{Y}^{-1}(\widehat{\langle X, X\rangle_A}).$$

Proof. Let $\pi_A : A \to \mathcal{B}(H)$ be an irreducible representation. If $\pi \in \widehat{Y}_0(\pi_A)$, then H_π is a closed subspace of $Y \otimes_{\pi_A} H$ irreducible under the left multiplication by elements of A, or more precisely, irreducible for $(Y\text{-Ind}(\pi_A))(\phi_Y(A))$. Since the tensor product of C^*-correspondences is both associative and distributive with respect to direct sums, we may naturally identify $X \otimes_\pi H_\pi$ with a closed subspace of $X \otimes Y \otimes_{\pi_A} H$. Since for $a \in \mathcal{K}(X)$ we have

$$((X \otimes Y)\text{-Ind}(\pi_A))(a \otimes 1_Y)(x \otimes y \otimes_{\pi_A} h) = ax \otimes y \otimes_{\pi_A} h,$$

we see that the action of $((X \otimes Y)\text{-Ind}(\pi_A))(a \otimes 1_Y)$ on $X \otimes_\pi H_\pi$ coincides with the action of $(X\text{-Ind}(\pi))(a)$. In particular, the subspace $X \otimes_\pi H_\pi$ is either $\{0\}$, when $\pi \notin \widehat{\langle X, X\rangle_A}$, or is irreducible for $((X \otimes Y)\text{-Ind}(\pi_A))(\mathcal{K}(X) \otimes 1_Y)$. Consequently,

$$(X\text{-Ind}) \circ \widehat{Y}_0(\pi_A) \subseteq \widehat{\otimes 1_Y}_0 \circ (X \otimes Y)\text{-Ind}(\pi_A).$$

To show the reverse inclusion, let $\rho \in (\widehat{\otimes 1_Y})_0 \circ (X \otimes Y)\text{-Ind}(\pi_A)$. Then ρ is an irreducible subrepresentation of the representation $\pi_{\mathcal{K}(X)} : \mathcal{K}(X) \to \mathcal{B}(X \otimes Y \otimes_{\pi_A} H)$, where $\pi_{\mathcal{K}(X)}(a) = ((X \otimes Y)\text{-Ind}(\pi_A))(a \otimes 1_Y)$. We may consider the dual C^*-correspondence $\widetilde{X}$ (not to be confused with the dual $\widehat{X}$ to the C^*-correspondence X) as an $\langle X, X\rangle_A$-$\mathcal{K}(X)$-imprimitivity bimodule. Then using the natural isomorphism

$$(\widetilde{X} \otimes_{\mathcal{K}(X)} \otimes X) \otimes Y \otimes_{\pi_A} H \cong Y \otimes_{\pi_A} H,$$

cf. [41, Proposition 2.28], we see that $\widetilde{X}\text{-Ind}(\pi_{\mathcal{K}(X)})$ is equivalent to $Y\text{-Ind}(\pi_A) \circ \phi_Y :$ $A \to \mathcal{B}(Y \otimes_{\pi_A} X)$. Since induction respects direct sums [41, Proposition 2.69], $\widetilde{X}\text{-Ind}(\rho)$ is equivalent to an irreducible subrepresentation π of $Y\text{-Ind}(\pi_A) \circ \phi_Y$. Then π belongs to both $\widehat{\langle X, X\rangle}_A$ and $\widehat{Y}_0(\pi_A)$, and we have

$$\rho \cong X\text{-Ind}(\widetilde{X}\text{-Ind}(\rho)) \cong X\text{-Ind}(\pi).$$

Consequently, $\widehat{\otimes 1_Y}_0 \circ (X \otimes Y)\text{-Ind}(\pi_A) \subseteq X\text{-Ind} \circ \widehat{Y}_0(\pi_A)$. □

Corollary 4.7. *The composition of duals to C^*-correspondences coincides with the dual of their tensor product:*

$$\widehat{X} \circ \widehat{Y} = \widehat{X \otimes Y}.$$

Proof. We showed in the proof of Proposition 4.6 that $X\text{-Ind} \circ \widehat{Y}_0 = \widehat{\otimes 1_Y}_0 \circ (X \otimes Y)\text{-Ind}$, and all subspaces of $X \otimes Y \otimes_{\pi_A} H$ irreducible for $(X \otimes Y)\text{-Ind}(\pi_A)(\mathcal{K}(X) \otimes 1_Y)$ are of the form $X \otimes_\pi H_\pi$, where $\pi \in \widehat{Y}_0(\pi_A) \cap \widehat{\langle X, X\rangle}_A$. Since $\phi_{X \otimes Y}(A) \subseteq \mathcal{K}(X) \otimes 1_Y$, the action of

$(X\otimes Y)$-Ind$(\pi_A)(\phi_{X\otimes Y}(a))$, $a\in A$, coincides on $X\otimes_\pi H_\pi$ with X-Ind$(\pi)(\phi_X(a))$. Thus we have

$$\widehat{X}_0\circ\widehat{Y}_0=(\widehat{\phi_X}_0\circ X\text{-Ind})\circ\widehat{Y}_0=\widehat{\phi_{X\otimes Y}}_0\circ(X\otimes Y)\text{-Ind}=\widehat{X\otimes Y}_0.$$

□

4.3. **Semigroups dual to regular product systems.** Let X be a product system over P. By Corollary 4.7, the family $\{\widehat{X}_p\}_{p\in P}$ of dual maps to C^*-correspondences X_p, $p\in P$, forms a semigroup of multivalued maps on $\widehat{A}$, that is

$$\widehat{X}_e=id,\quad\text{and}\quad\widehat{X}_p\circ\widehat{X}_q=\widehat{X}_{pq},\qquad p,q\in P.$$

If A is liminal then these multivalued maps are continuous by Proposition 4.5.

Definition 4.8. We call the *semigroup* $\widehat{X}:=\{\widehat{X}_p\}_{p\in P}$ *dual to the product system* X.

In the remainder of this subsection we prove certain technical facts concerning the interaction among Cuntz-Pimsner representations, dual maps and the process of induction.

Lemma 4.9. *Let X be a product system over a left cancellative semigroup P. If $p,q,s\in P$ are such that $s\geqslant p,q$, then*

$$\widehat{X}_{q^{-1}s}\widehat{X}_{p^{-1}s}^{-1}=[X_q\text{-Ind}^{-1}]\circ\widehat{i}_q^s\circ\widehat{i}_p^{s^{-1}}\circ[X_p\text{-Ind}].$$

Proof. Applying Proposition 4.6 to $Y=X_{p^{-1}s}$, $X=X_p$ and $Y=X_{q^{-1}s}$, $X=X_q$, respectively, we get

$$[X_s\text{-Ind}]\widehat{X}_{p^{-1}s}=\widehat{i}_p^s[X_s\text{-Ind}]\quad\text{and}\quad[X_s\text{-Ind}]\widehat{X}_{q^{-1}s}=\widehat{i}_q^s[X_s\text{-Ind}].$$

As $[X_s\text{-Ind}]$ is a homeomorphism, this is equivalent to

$$\widehat{X}_{p^{-1}s}=[X_s\text{-Ind}]^{-1}\widehat{i}_p^s[X_s\text{-Ind}]\quad\text{and}\quad\widehat{X}_{q^{-1}s}=[X_s\text{-Ind}]^{-1}\widehat{i}_q^s[X_s\text{-Ind}],$$

and the assertion follows. □

The following Lemma 4.10 is virtually a special case of [33, Lemma 1.3].

Lemma 4.10. *Suppose Y is an imprimitivity Hilbert A-B-bimodule and (π_A,π_Y,π_B) is its representation on a Hilbert space H. Thus $\pi_A:A\to\mathcal{B}(H)$, $\pi_B:B\to\mathcal{B}(H)$ are representations and with the map $\pi_Y:Y\to\mathcal{B}(H)$ they satisfy*

$$\pi_A(a)\pi_Y(y)\pi_B(b)=\pi_Y(ayb),\quad\pi_Y(x)\pi_Y(y)^*=\pi_A({}_A\langle x,y\rangle),\quad\pi_Y(x)^*\pi_Y(y)=\pi_B(\langle x,y\rangle_B),$$

$a\in A$, $b\in B$, $x,y\in Y$. If π is an irreducible subrepresentation of π_B then the restriction $\rho(a):=\pi_A(a)|_{\pi_Y(Y)H_\pi}$ yields an irreducible subrepresentation of π_A such that $[\rho]=[Y\text{-Ind}(\pi)]$.

Proof. Let $\pi\leqslant\pi_B$ be a representation of B on a Hilbert space $H_\pi\subset H$. The Hilbert space $\pi_Y(Y)H_\pi\subset H$ is invariant for elements of $\pi_A(A)$ and therefore $\rho(a):=\pi_A(a)|_{\pi_Y(Y)H_\pi}$, $a\in A$ defines a representation of A. Since

$$\|\sum_{i=1}^n\pi_Y(y_i)h_i\|^2=\sum_{i,j=1}^n\langle\pi_Y(y_i)h_i,\pi_Y(y_j)h_j\rangle=\sum_{i,j=1}^n\langle h_i,\pi_A(\langle y_i,y_j\rangle_A)h_j\rangle=\|\sum_{i=1}^n y_i\otimes_\pi h_i\|^2,$$

the mapping $\pi_Y(y)h\mapsto y\otimes_\pi h$, $y\in Y$, $h\in H_\pi$, extends by linearity and continuity to a unitary operator $V:\pi_Y(Y)H_\pi\to Y\otimes_\pi H_\pi$, which intertwines ρ and Y-Ind(π) because

$$V\rho(a)\pi_Y(y)h=V\pi_Y(ay)h=(ay\otimes_\pi h)=Y\text{-Ind}(\pi)(a)V\pi_Y(y)h.$$

Accordingly, if π is irreducible then ρ, being unitary equivalent to the irreducible representation Y-Ind(π), is also irreducible. □

A counterpart of [33, Lemma 1.3] suitable for our purposes is the following statement.

Lemma 4.11. *Suppose ψ is a Cuntz-Pimsner covariant representation of a regular product system X over P on a Hilbert space H. Let $p, q \in P$ and let π be an irreducible summand of $\psi^{(q)}$ acting on a subspace K of H. Then the restriction*

$$\pi_p(T) := \psi^{(p)}(a)|_{\psi_p(X_p)\psi_q(X_q)^*K}, \qquad T \in \mathcal{K}(X_p), \tag{26}$$

yields a representation $\pi_p : \mathcal{K}(X_p) \to \mathcal{B}(\psi_p(X_p)\psi_q(X_q)^*K)$ which is either zero or irreducible, and such that

$$[\pi_p] = [(X_p\text{-Ind})((X_q\text{-Ind})^{-1}(\pi))].$$

Proof. The dual C^*-correspondence $\widetilde{X}_q$ to X_q is an imprimitivity $\langle X_q, X_q\rangle_A$-$\mathcal{K}(X_q)$-bimodule and $(\psi_e, \widetilde{\psi}_q, \psi^{(q)})$, where $\widetilde{\psi}_q(\flat(x)) = \psi_q(x)^*$, is its representation. Thus, by Lemma 4.10, the restriction $\pi_e(a) := \psi_e(a)|_{\psi_q(X_q)^*K}$, $a \in A$, yields an irreducible subrepresentation $\pi_e : A \to \mathcal{B}(\psi_q(X_q)^*K)$ of ψ_e such that $[\pi_e] = [\widetilde{X}_q\text{-Ind}(\pi)] = [(X_q\text{-Ind})^{-1}(\pi)]$. If $\pi_e(\langle X_p, X_p\rangle_A) = 0$, then (26) is a zero representation. Otherwise we may apply Lemma 4.10 to π_e and the representation $(\psi^{(p)}, \psi_p, \psi_e)$ of the imprimitivity $\mathcal{K}(X_p)$-$\langle X_p, X_p\rangle_A$-bimodule X_p. Then we see that (26) yields an irreducible representation such that $[\pi_p] = [X_p\text{-Ind}(\pi_e)] = [X_p\text{-Ind}((X_q\text{-Ind})^{-1}(\pi))]$. □

5. A uniqueness theorem and simplicity criteria for Cuntz-Pimsner algebras

Throughout this section, we consider a directed, left cancellative semigroup P and a regular product system X over P with coefficients in an arbitrary C^*-algebra A. We recall from Theorem 3.8 that the Cuntz-Pimsner algebra $\mathcal{O}_X$ is graded over the enveloping group $G(P)$ with fibers

$$(\mathcal{O}_X)_g = \overline{\text{span}}\{j_X(x)j_X(y)^* : x, y \in X,\ [d(x), d(y)] = g\}, \qquad g \in G(P).$$

Moreover, cf. Remark 3.9, $\mathcal{O}_X$ may be viewed as a full cross-sectional algebra $C^*(\{(\mathcal{O}_X)_g\}_{g\in G(P)})$ of the Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$, and the *reduced Cuntz-Pimsner algebra*

$$\mathcal{O}_X^r := C_r^*(\{(\mathcal{O}_X)_g\}_{g\in G(P)})$$

is defined as the reduced cross-sectional algebra of $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$. There exists a canonical epimorphism

$$\lambda : \mathcal{O}_X \to \mathcal{O}_X^r. \tag{27}$$

This epimorphism may not be injective. However, λ is always injective whenever group $G(P)$ is amenable or more generally when the Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ has the approximation property defined in [16].

We want to clarify what we mean by a *uniqueness theorem* in this context. By now, several conditions implying amenability of the Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ are known. That is, conditions which guarantee the identity $\mathcal{O}_X = \mathcal{O}_X^r$, see e.g. [26], [8], [16]. These conditions seem to be independent of aperiodicity we want to investigate, and thus we decided not to assume any of them. Accordingly, we seek an intrinsic condition on the product system X (or on the dual semigroup $\widehat{X}$) which would guarantee that every Cuntz-Pimsner representation of X injective on the coefficient algebra A generates a C^*-algebra lying in between $\mathcal{O}_X$ and $\mathcal{O}_X^r$. Before proceeding further, we summarize a few know facts useful in the aforementioned context.

Proposition 5.1. *Suppose that ψ is an injective Cuntz–Pimsner representation of a regular product system X. If the epimorphism λ from (27) is an isomorphism, then the following conditions are equivalent.*

i) *The canonical epimorphism $\Pi_\psi : \mathcal{O}_X \to C^*(\psi(X))$, where $i_X(x) = \psi(x)$, $x \in X$, is an isomorphism.*
ii) *There is a coaction β of $G = G(P)$ on $C^*(\psi(X))$ such that $\beta(\psi(x)) = \psi(x) \otimes i_G(d(x))$, $x \in X$.*
iii) *There is a conditional expectation E_ψ from $C^*(\psi(X))$ onto*

$$\mathcal{F}_\psi = \overline{\operatorname{span}}\{ \psi(x)\psi(y)^* : x, y \in X, d(x) \sim d(y) \},$$

vanishing on elements $\psi(x)\psi(y)^*$ with $d(x) \nsim d(y)$, cf. Remark 2.3.

Not assuming injectivity of λ, we have implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), and (iii) is equivalent to existence of a unique epimorphism $\pi_\psi : C^*(\psi(X)) \to \mathcal{O}^r_X$ such that the following diagram

$$\begin{tikzcd} \mathcal{O}_X \arrow[r, "\Pi_\psi"] \arrow[rr, bend right, "\lambda"'] & C^*(\psi(X)) \arrow[r, "\pi_\psi"] & \mathcal{O}^r_X \end{tikzcd} \tag{28}$$

is commutative.

Proof. It suffices to prove the second part of the assertion. Implication (i) $\Rightarrow$ (ii) is obvious because we know that $\mathcal{O}_X$ is equipped with the coaction in the prescribed form. Suppose (ii) holds. Using the contractive projections onto the spectral subspaces for the coaction β, cf. [38, Lemma 1.3], and the fact that elements of the form $\psi(x)\psi(y)^*$ span a dense subspace of $C^*(\psi(E))$, Lemma 3.7, we get

$$[C^*(\psi(X))]^\beta_g = \{c \in C^*(\psi(X)) : \beta(c) = c \otimes i_G(g)\} = \overline{\operatorname{span}}\{\psi(x)\psi(y)^* : [d(x), d(y)] = g\}.$$

In particular, the projection onto $[C^*(\psi(X))]^\beta_e = \mathcal{F}_\psi$ is the conditional expectation described in (iii). If we assume (iii), then $\{\Pi_\psi((\mathcal{O}_X)_g)\}_{g\in G}$ is a Fell bundle which yields a topological grading of $C^*(\psi(X))$, see [16, Definition 3.4]. Hence by [16, Theorem 3.3] there exists a desired epimorphism $\pi_\psi : C^*(\psi(X)) \to \mathcal{O}^r_X$. Conversely, if such an epimorphism $\pi_\psi : C^*(\psi(X)) \to \mathcal{O}^r_X$ exists, then composing it with the canonical conditional expectation on $\mathcal{O}^r_X$ one gets the conditional expectation described in (ii). $\square$

The authors of [8] call a representation $\psi\colon X \to B$ possessing the property described in part (ii) of Proposition 5.1 *gauge-compatible*. For our purposes the property given in part (iii) of Proposition 5.1 is more relevant, and thus we coin the following definition, cf. [16, Definition 3.4].

Definition 5.2. We say that a representation $\psi\colon X \to B$ of a product system X is *topologically graded* if it has the property described in part (iii) of Proposition 5.1.

Thus, to conclude our discussion, by *uniqueness theorem* for $\mathcal{O}_X$ we understand a result which guarantees that for every injective Cuntz-Pimsner covariant representation ψ of X there is a map π_ψ making the diagram (28) commutative. By Proposition 5.1, this is equivalent to ψ being topologically graded. We now introduce a dynamical condition which entails such a result.

Definition 5.3. We say that a regular product system X, or the dual semigroup $\{\widehat{X}_p\}_{p\in P}$, is *topologically aperiodic* if for each nonempty open set $U \subseteq \widehat{A}$, each finite set $F \subseteq P$ and element $q \in P$ such that $q \nsim_R p$ for $p \in F$, there exists a $[\pi] \in U$ such that for a certain enumeration of elements of $F = \{p_1, ..., p_n\}$ and certain elements $s_1, ..., s_n \in P$ with $q \leqslant s_1 \leqslant ... \leqslant s_n$ and $p_i \leqslant s_i$ we have

$$[\pi] \notin \widehat{X}_{q^{-1}s_i}(\widehat{X}^{-1}_{p_i^{-1}s_i}([\pi])) \qquad \text{for all} \quad i = 1, ..., n. \tag{29}$$

Remark 5.4. Since $(P, \leqslant)$ is a directed preorder, for any $F = \{p_1, ..., p_n\} \subseteq P$ and $q \in P$ there exists an increasing sequence $q \leqslant s_1 \leqslant ... \leqslant s_n$ such that $p_i \leqslant s_i$ for all $i = 1, ..., n$. Therefore the essential part of the condition in Definition 5.3 is existence of a $[\pi]$ satisfying (29), which a priori depends on the choice of the sequence $q \leqslant s_1 \leqslant ... \leqslant s_n$ and enumeration of elements of F.

Proposition 5.5. *If condition* (29) *holds for a certain sequence $q \leqslant s_1 \leqslant ... \leqslant s_n$, then it also holds for any sequence $q \leqslant s_1' \leqslant ... \leqslant s_n'$ such that*

$$p_i \leqslant s_i' \leqslant s_i \quad \text{for all } i = 1, ..., n.$$

Moreover, we have the following.

i) *If $(G(P), P)$ is a quasi-lattice ordered group then in Definition 5.3 one can always take*

$$s_1 = p_1 \vee q \quad \text{and} \quad s_i = p_i \vee s_{i-1} \quad \text{for all } i = 2, ..., n.$$

ii) *Topological aperiodicity of X implies that for any open nonempty set $U \subseteq \widehat{A}$ and any finite set $F \subseteq P$ such that $p \nsim_R e$ for $p \in F$, there is a $[\pi] \in U$ satisfying*

$$[\pi] \notin \widehat{X}_p([\pi]) \quad \text{for all } p \in F. \tag{30}$$

If $(P, \leqslant)$ is linearly ordered then the converse implication also holds.

iii) *In the simplest case of a product system $\{X^{\otimes n}\}_{n \in \mathbb{N}}$ arising from a single regular C^*-correspondence X, the topological aperiodicity is equivalent to that for each $n > 0$ set*

$$F_n = \{[\pi] \in \widehat{A} : \pi \in \widehat{X}^n([\pi])\}$$

has empty interior. (In this case we will say that the C^*-correspondence X is topologically aperiodic.)

Proof. Let us notice that if $q, p_i \leqslant s_i' \leqslant s_i$, then using the semigroup property of $\widehat{X}$ (Corollary 4.7), surjectivity of mappings $\widehat{X}_p$, $p \in P$, (Proposition 4.5) and taking into account (1) we get

$$\begin{aligned}
\widehat{X}_{q^{-1}s_i} \circ \widehat{X}^{-1}_{p_i^{-1}s_i} &= \widehat{X}_{q^{-1}s_i'} \circ \widehat{X}_{s_i'^{-1}s_i} \circ (\widehat{X}_{p_i^{-1}s_i'} \circ \widehat{X}_{s_i'^{-1}s_i})^{-1} \\
&= \widehat{X}_{q^{-1}s_i'} \circ \widehat{X}_{s_i'^{-1}s_i} \circ \widehat{X}^{-1}_{s_i'^{-1}s_i} \circ \widehat{X}^{-1}_{p_i^{-1}s_i'} \\
&\supseteq \widehat{X}_{q^{-1}s_i'} \circ \widehat{X}^{-1}_{p_i^{-1}s_i'}.
\end{aligned}$$

Hence $[\pi] \notin \widehat{X}_{q^{-1}s_i}(\widehat{X}^{-1}_{p_i^{-1}s_i}([\pi]))$ implies $[\pi] \notin \widehat{X}_{q^{-1}s_i'}(\widehat{X}^{-1}_{p^{-1}s_i'}([\pi]))$. This proves the initial part of the assertion.

Ad (i). It follows immediately from what we have just shown.

Ad (ii). If $F = \{p_1, ..., p_n\} \subseteq P$ and $p \nsim_R e$ for all $p \in F$, then putting $q = e$ we see that topological aperiodicity of X implies that for any nonempty open set $U \subseteq \widehat{A}$ there are elements $s_1, ..., s_n \in P$, $p_i \leqslant s_i$, $i = 1, ..., n$ and a point $[\pi] \in U$ such that

$$[\pi] \notin \widehat{X}_{q^{-1}s_i}(\widehat{X}^{-1}_{p_i^{-1}s_i}([\pi])) = \widehat{X}_{s_i}(\widehat{X}^{-1}_{p_i^{-1}s_i}([\pi])) \qquad \text{for all} \quad i = 1, ..., n.$$

By the inclusion noticed above we have $\widehat{X}_{s_i} \circ \widehat{X}^{-1}_{p_i^{-1}s_i} = \widehat{X}_{p_i} \circ \widehat{X}_{p^{-1}s_i} \circ \widehat{X}^{-1}_{p_i^{-1}s_i} \supseteq \widehat{X}_{p_i}$ and thus condition (30) follows.

Conversely, suppose $(P, \leqslant)$ is linearly ordered and the condition described in (ii) is satisfied. Let $U \subseteq \widehat{A}$ be open and nonempty, $F \subseteq P$ finite and $q \in P$ such that $q \not\sim_R p$, for $p \in F$. Enumerating elements of $F = \{p_1, ..., p_n\} \subseteq P$ in a non-increasing order we have

$$p_1 \leqslant p_2 \leqslant ... \leqslant p_{k_0} \leqslant q \leqslant p_{k_0+1} \leqslant ... \leqslant p_n$$

for certain $k_0 \in \{0, 1, ..., n\}$. Defining

$$s_i := \begin{cases} q, & i \leqslant k_0 \\ p_i & i \geqslant k_0 + 1 \end{cases}$$

we see that $q \leqslant s_1 \leqslant ... \leqslant s_n$ and

$$\widehat{X}_{q^{-1}s_i} \circ \widehat{X}^{-1}_{p_i^{-1}s_i} = \begin{cases} \widehat{X}^{-1}_{p_i^{-1}q}, & i \leqslant k_0 \\ \widehat{X}_{q^{-1}p_i} & i \geqslant k_0 + 1 \end{cases}.$$

Put $F' := \{p_i^{-1}q : i = 1, ...k_0\} \cup \{q^{-1}p_i : i = k_0 + 1, ...n\}$ and note that $p \not\sim_R e$ for all $p \in F'$. Thus we may apply condition described in (ii) to F' and then we obtain a $[\pi] \in U$ satisfying (29).

Ad (iii). By part (ii) above, topological aperiodicity implies the condition described in (iii). To see the converse, again by part (ii), it suffices to show (30) for a finite set $F \subseteq \mathbb{N}\backslash\{0\}$. The latter follows from condition described in (iii) applied to $n = m!$ where $m = \max\{k : k \in F\}$. $\square$

Now, we are ready to state and prove the main result of the present paper.

Theorem 5.6 (Uniqueness theorem). *Suppose that a regular product system X is topologically aperiodic. Then every injective Cuntz-Pimsner representation of X is topologically graded. If the canonical epimorphism $\lambda : \mathcal{O}_X \to \mathcal{O}_X^r$ is injective then there is a natural isomorphism*

$$\mathcal{O}_X \cong C^*(\psi(X))$$

for every injective Cuntz-Pimsner representation ψ of X.

Proof. Suppose that ψ is an injective Cuntz-Pimsner representation of X in a C^*-algebra B. Then $\psi^{(p)} : \mathcal{K}(X_p) \to B$ is injective for all $p \in P$. Let us consider an element of the form

$$\psi^{(q)}(S_q) + \sum_{p \in F} \psi_{p,q}(S_{p,q}), \tag{31}$$

where $q \in P$, $F \subseteq P$ is a finite set such that $q \not\sim_R p$ for all $p \in F$, and $S_q \in \mathcal{K}(X_q)$, $S_{p,q} \in \mathcal{K}(X_q, X_p)$. By Lemma 3.7 part (iii), such elements form a dense subspace of $C^*(\psi(X))$. Thus existence of the appropriate conditional expectation will follow from the inequality

$$\|S_q\| = \|\psi^{(q)}(S_q)\| \leqslant \|\psi^{(q)}(S_q) + \sum_{p \in F} \psi_{p,q}(S_{p,q})\|.$$

To prove this inequality, we fix $\varepsilon > 0$ and recall that for any $a \in A$ the mapping $\widehat{A} \ni [\pi] \mapsto \|\pi(a)\|$ is lower semicontinuous and attains its maximum equal to $\|a\|$, cf. e.g. [14, Proposition 3.3.2., Lemma 3.3.6]. Thus, since X_q-Ind $: \widehat{A} \to \widehat{\mathcal{K}(X)}$ is a homeomorphism, we deduce that there is an open nonempty set $U \subseteq \widehat{A}$ such that

$$\|X_q\text{-Ind}(\pi)(S_q)\| > \|S_q\| - \varepsilon \qquad \text{for every } [\pi] \in U.$$

Let $F = \{p_1, ..., p_n\}$. By topological aperiodicity of X, there are elements $s_1, ..., s_n \in P$ such that $q \leqslant s_1 \leqslant ... \leqslant s_n$ and $p_i \leqslant s_i$, $i = 1, ..., n$, and there exists a $[\pi] \in U$ satisfying (29). Let us fix these objects.

We recall that if $p \leqslant s$, then $i_p^s(\mathcal{K}(X_p)) \subseteq \mathcal{K}(X_s)$ and thus $\psi^{(p)}(\mathcal{K}(X_s)) \subseteq \psi^{(s)}(\mathcal{K}(X_s))$, cf. Lemma 3.6. In particular, we have the increasing sequence of algebras

$$\psi^{(q)}(\mathcal{K}(X_q)) \subseteq \psi^{(s_1)}(\mathcal{K}(X_{s_1})) \subseteq ... \subseteq \psi^{(s_n)}(\mathcal{K}(X_{s_n})) \subseteq C^*(\psi(X)).$$

We construct a relevant sequence of representations of these algebras as follows. We put

$$\nu_q : \psi^{(q)}(\mathcal{K}(X_q)) \to \mathcal{B}(H_q) \quad \text{defined as} \quad \nu_q(\psi^{(q)}(S)) = X_q\text{-Ind}(\pi)(S).$$

Then ν_q is an irreducible representation because so is π. We let $\nu_{s_1} : \psi^{(s_1)}(\mathcal{K}(X_{s_1})) \to \mathcal{B}(H_{s_1})$ to be any irreducible extension of ν_q, and for $i = 2, 3, ..., n$ we take $\nu_{s_i} : \psi^{(s_i)}(\mathcal{K}(X_{s_i})) \to \mathcal{B}(H_{s_i})$ to be any irreducible extension of $\nu_{s_{i-1}}$. Finally, we let $\nu : C^*(\psi(X)) \to \mathcal{B}(H)$ to be any extension of ν_{s_n}. In particular, we have

$$H_q \subseteq H_{s_1} \subseteq ... \subseteq H_{s_n} \subseteq H.$$

Let $P_q \in \mathcal{B}(H)$ be the projection onto the subspace H_q. Clearly

$$\|P_q\nu(\psi^{(q)}(S_q))P_q\| = \|\nu_q(\psi^{(q)}(S_q))\| = \|X_q\text{-Ind}(\pi)(S_q)\| \geqslant \|S_q\| - \varepsilon$$

and as ε is arbitrary we can reduce the proof to showing that

$$P_q\nu(\psi_{p,q}(S_{p,q}))P_q = 0 \quad \text{for } p \in F. \tag{32}$$

To this end, we fix a $p_i \in F$. Let P_{s_i} be the projection onto H_{s_i} and consider the space

$$H_{p_i} := \nu(\psi_{p_i}(X_{p_i})\psi_q(X_q)^*)H_q.$$

We claim that $P_{s_i}H_{p_i} = \{0\}$. Since $H_q \subseteq H_{s_i}$, this implies (32) and finishes the proof. Suppose to the contrary that $P_{s_i}H_{p_i} \neq \{0\}$. By Lemma 4.11 and the definitions of ν and H_{p_i}, the mapping

$$\mathcal{K}(X_{p_i}) \ni S \longrightarrow \nu(\psi^{(p_i)}(S))|_{H_{p_i}}$$

is an irreducible representation equivalent to X_{p_i}-Ind(π). In particular, H_{p_i} is irreducible for $\nu(\psi^{(p_i)}(\mathcal{K}(X_{p_i})))$. Since

$$\nu(\psi^{(p_i)}(\mathcal{K}(X_{p_i}))) \subseteq \nu(\psi^{(s_i)}(\mathcal{K}(X_{s_i}))) \quad \text{and} \quad P_{s_i} \in \nu(\psi^{(s_i)}(\mathcal{K}(X_{s_i})))',$$

we see that $P_{s_i}H_{p_i}$ is an irreducible subspace for $\nu(\psi^{(p_i)}(\mathcal{K}(X_{p_i})))$. Thus, since H_{p_i} and $P_{s_i}H_{p_i}$ are both irreducible subspaces for $\nu(\psi^{(p_i)}(\mathcal{K}(X_{p_i})))$, either $H_{p_i} = P_{s_i}H_{p_i}$ or $H_{p_i} \perp P_{s_i}H_{p_i}$. However, (as $P_{s_i}H_{p_i} \neq \{0\}$) the latter is clearly impossible. Thus $H_{p_i} \subseteq H_{s_i}$ and denoting by π_{s_i} the representation

$$\mathcal{K}(X_{s_i}) \ni S \xrightarrow{\pi_{s_i}} \nu(\psi^{(s_i)}(S))|_{H_{s_i}},$$

we get $[\pi_{s_i}] \in \widehat{\iota_{p_i}^{s_i}}^{-1}([X_{p_i}\text{-Ind}(\pi)])$. Denoting by π_q the representation

$$\mathcal{K}(X_q) \ni S \to \nu(\psi^{(q)}(S))|_{H_q},$$

we have $[\pi_q] \in \widehat{\iota_q^{s_i}}([\pi_{s_i}])$ and $\pi_q = X_q\text{-Ind}(\pi)$. Hence we get

$$[\pi] = [(X_q\text{-Ind})^{-1}(\pi_q)] \in [X_q\text{-Ind}^{-1}](\widehat{i_q^{s_i}}([\pi_{s_i}])) \subseteq [X_q\text{-Ind}^{-1}](\widehat{i_q^{s_i}}(\widehat{i_{p_i}^{s_i}}^{-1}([X_{p_i}\text{-Ind}(\pi)]))).$$

Thereby in view of Lemma 4.9 we arrive at

$$[\pi] \in \widehat{X}_{q^{-1}s_i}(\widehat{X}_{p_i^{-1}s_i}^{-1})([\pi]),$$

which contradicts the choice of π. □

As an application of Theorem 5.6, we obtain simplicity criteria for the reduced Cuntz-Pimsner algebra $\mathcal{O}^r_X$. To this end, we first introduce the indispensable terminology.

Definition 5.7. Let X be a regular product system over a semigroup P with coefficients in a C^*-algebra A. We say that an ideal J in A is *X-invariant* if and only if for each $p \in P$ the set

$$X_p^{-1}(J) := \{a \in A : \langle X_p, aX_p\rangle_p \subseteq J\}$$

is equal to J. We say X is *minimal* if there are no nontrivial X-invariant ideals in A, that is if for any ideal J in A we have

$$(\forall p \in P)\ X_p^{-1}(J) = J \quad \Longrightarrow\ J = \{0\} \ \text{ or } J = A.$$

Remark 5.8. When $P = \mathbb{N}$ and A is unital, our Definition 5.7 agrees with [43, Definition 3.7] treating the case of a single C^*-correspondence, see the discussion on page 418 therein.

Remark 5.9. Let X be a regular C^*-correspondence. It is well known, cf. for instance [29, Proposition 1.3], that for any ideal J in A we have

$$XJ = \{xj : x \in X,\ j \in J\} = \{x \in X : \langle x, y\rangle \in J \text{ for all } y \in X\}.$$

Therefore we see that

$$X^{-1}(J) = \{a \in A : aX \subseteq XJ\} = \phi^{-1}(\mathcal{K}_J(X)),$$

where

$$\mathcal{K}_J(X) := \overline{\operatorname{span}}\{\Theta_{x,y} : x \in X, y \in XJ\} = \overline{\operatorname{span}}\{\Theta_{x,y} : x, y \in XJ\}$$

is an ideal in $\mathcal{K}(X)$. In particular, we infer that $X^{-1}(J)$ is an ideal and J is $\{X^{\otimes n}\}_{n\in\mathbb{N}}$-invariant if and only if $X^{-1}(J) = J$, in which case we will say that J is X-invariant.

Theorem 5.10 (Simplicity of $\mathcal{O}^r_X$)**.** *If a regular product system X is topologically aperiodic and minimal, then $\mathcal{O}^r_X$ is simple.*

Proof. Suppose I is an ideal in $\mathcal{O}^r_X$ and put $J = (j^r_X)^{-1}(I) \cap A = \{a \in A : j^r_X(a) \in I\}$. Then J is an ideal in A. We claim that J is X-invariant. Indeed, for $p \in P$ we have

$$j^r_X(\langle X_p, JX_p\rangle_p) = j^r_X(X_p)^* j^r_X(JX_p) = j^r_X(X_p)^* j^r_X(J) j^r_X(X_p) \subseteq I.$$

That is, $\langle X_p, JX_p\rangle_A \subseteq J$ and hence $J \subseteq X_p^{-1}(J)$. On the other hand, if $a \in X_p^{-1}(J)$ then by Remark 5.9 we have

$$\phi_p(a) = \sum_i \Theta_{x_i, y_i j_i} \quad \text{where} \quad x_i, y_i \in X_p \ \text{ and } j_i \in J.$$

Since $j^r_X : X \to \mathcal{O}^r_X$ is Cuntz-Pimsner covariant, we get

$$\begin{aligned} j^r_X(a) = j^{r\,(p)}_X(\phi_p(a)) &= \sum_i j^{r\,(p)}_X(\Theta_{x_i, y_i j_i}) = \sum_i j^r_X(x_i) j^r_X(y_i j_i)^* \\ &= \sum_i j^r_X(x_i) j^r_X(j_i^*) j^r_X(y_i)^* \in I. \end{aligned}$$

Thus $X_p^{-1}(J) \subseteq J$ and this proves our claim. In view of minimality of X, either $J = A$ or $J = \{0\}$. In the former case, $\mathcal{O}^r_X = C^*(j^r_X(X)) = I$ because $j^r_X(X_p) = j^r_X(AX_p) = j^r_x(A) j^r_X(X_p) \subseteq I$ for each $p \in P$. In the latter case, the composition of $j^r_X : X \to \mathcal{O}^r_X$ with the quotient map $\theta : \mathcal{O}^r_X \to \mathcal{O}^r_X/I$ yields a Cuntz-Pimsner representation $k_X := \theta \circ j^r_X$ of X in $\mathcal{O}^r_X/I$ which is injective on A. Thus by Theorem 5.6 we have an epimorphism

$$\pi_{k_X} : \mathcal{O}^r_X/I \to \mathcal{O}^r_X$$

such that $\pi_{k_X}(q(j^r_X(x)) = j^r_X(x)$, $x \in X$. Hence $j^r_X(X) \cap I = \{0\}$ and therefore $I = \{0\}$. $\square$

Schweizer found in [43] a necessary and sufficient condition for simplicity of Cuntz-Pimsner algebras associated with single C^*-correspondences, improving similar results of [25]. Namely, by [43, Theorem 3.9], if X is a left essential and full C^*-correspondence with coefficients in a unital C^*-algebra A, then $\mathcal{O}_X$ is simple if and only if X is minimal and *nonperiodic*, meaning that $X^{\otimes n} \approx {}_AA_A$ implies $n = 0$, where $\approx$ denotes the unitary equivalence of C^*-correspondences. This result suggests that topological aperiodicity of a product system X should imply nonperiodicity of X, and this is indeed the case.

Proposition 5.11. *Suppose that X is a topologically aperiodic regular product system over a semigroup P of Ore type. Then $X_p \approx X_e$ implies $p \sim_R e$, and if in addition $(G(P), P)$ is a quasi-lattice ordered group, then $X_{q^{-1}(p\vee q)} \approx X_{p^{-1}(p\vee q)}$ implies $p = q$.*

Proof. In view of Proposition 5.5 parts (i) and (ii), it suffices to note that $X_p \approx X_q$ implies that $[\pi] \in \widehat{X}_p(\widehat{X}_q^{-1}([\pi]))$ for all $[\pi] \in \widehat{A}$. To this end, let $V : X_p \to X_q$ be a bimodule unitary implementing the equivalence $X_p \approx X_q$. Let $[\pi] \in \widehat{A}$ be arbitrary and take any $[\rho] \in \widehat{X}_q^{-1}([\pi])$ (such ρ exists because $\widehat{X}_q$ is surjective). In other words, $[\pi] \leq [X_q\text{-Ind}(\rho) \circ \phi_q]$. Then V gives rise to a unitary map

$$\widetilde{V} : X_p \otimes_\rho H_\rho \to X_q \otimes_\rho H_\rho, \qquad \text{such that} \quad \widetilde{V}(x \otimes h) = (Vx) \otimes h.$$

Indeed, this follows from the following simple computation:

$$\begin{aligned}
\|\sum_{i=1}^n x_i \otimes h_i\|^2 &= \sum_{i,j=1}^n \langle x_i \otimes_\rho h_i, x_j \otimes_\rho h_j\rangle = \sum_{i,j=1}^n \langle h_i, \rho(\langle x_i, x_j\rangle_A) h_j\rangle \\
&= \sum_{i,j=1}^n \langle h_i, \rho(\langle Vx_i, Vx_j\rangle_A) h_j\rangle = \sum_{i,j=1}^n \langle (Vx_i) \otimes_\rho h_i, (Vx_j) \otimes_\rho h_j\rangle \\
&= \|\sum_{i=1}^n (Vx_i) \otimes_\rho h_i\|^2,
\end{aligned}$$

where $x_i \in X_p$, $h_i \in H_\rho$, $i = 1, ..., n$. Since V is a left A-module morphism, we see that $\widetilde{V}$ establishes a unitary equivalence between $X_p\text{-Ind}(\rho) \circ \phi_p$ and $X_q\text{-Ind}(\rho) \circ \phi_q$. Hence we have both $[\pi] \leq [X_q\text{-Ind}(\rho) \circ \phi_q]$ and $[\pi] \leq [X_p\text{-Ind}(\rho) \circ \phi_p]$. □

6. Applications and examples

In this section, we give several examples and applications of the theory developed above. In particular, we discuss algebras associated with saturated Fell bundles, twisted C^*-dynamical systems, product systems of topological graphs and the Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$.

6.1. **Product systems of Hilbert bimodules, Fell bundles and dual partial actions.** In this subsection, we consider a regular product system X over a semigroup P of Ore type, with the additional property that each C^*-correspondence X_p, $p \in P$, is a Hilbert bimodule equipped with left A-valued inner product ${}_p\langle \cdot, \cdot \rangle : X_p \times X_p \to A$. We call such an X *regular product system of Hilbert bimodules.* With help of for instance [31, Proposition 1.11], one can show that a regular product system is a product system of Hilbert bimodules if and only if each left action homomorphism $\phi_p : A \to \mathcal{K}(X_p)$ is surjective. In this case, $\phi_p : A \to \mathcal{K}(X_p)$ is an isomorphism and

$${}_p\langle x, y\rangle = \phi_p^{-1}(\Theta_{x,y}), \qquad x, y \in X_p.$$

The following Proposition 6.1 gives another characterization of regular product systems of Hilbert bimodules in terms of the Fell bundle structure in $\mathcal{O}_X$ identified in Theorem 3.8 above, cf. [31, Theorem 5.9].

Proposition 6.1. *A regular product system X over a semigroup P of Ore type is a product system of Hilbert bimodules if and only if the algebra of coefficients A embeds into $\mathcal{O}_X$ as the core subalgebra $(\mathcal{O}_X)_{[e,e]}$, that is*

$$j_X(A) = (\mathcal{O}_X)_{[e,e]}.$$

In this case, each space X_p embeds into $\mathcal{O}_X$ as the fiber $(\mathcal{O}_X)_{[p,e]}$. In particular, $j_X(X_p) = (\mathcal{O}_X)_{[p,e]}$, for all $p \in P$, and

$$(\mathcal{O}_X)_{[p,q]} = \overline{\operatorname{span}}\{j_X(x)j_X(y)^* : x \in X_p, y \in X_q\}, \qquad p, q \in P. \tag{33}$$

Proof. If all the maps $\phi_p : A \to \mathcal{K}(X_p)$ are isomorphisms, it follows from Lemma 3.2 part (ii) that all the maps $\iota_{p,q}^{pr,qr} : \mathcal{K}(X_q, X_p) \to \mathcal{K}(X_{qr}, X_{pr})$ are (Banach space) isomorphisms. Hence

$$\varinjlim \mathcal{K}(X_{qr}, X_{pr}) = \varphi_{p,q}(\mathcal{K}(X_q, X_p))$$

where $\varphi_{p,q}$ denotes the natural embedding of $\mathcal{K}(X_q, X_p)$ into the inductive limit $\varinjlim \mathcal{K}(X_{qr}, X_{pr})$. As the isomorphism from Theorem 3.8 sends $j_X(x)j_X(y)^*$ to $\varphi_{p,q}(\Theta_{x,y})$, $x \in X_p, y \in X_q$, we get (33). In particular, we have $j_X(A) = (\mathcal{O}_X)_{[e,e]}$.
Conversely, if we assume that $\phi_p : A \to \mathcal{K}(X_p)$ is not onto for certain $p \in P$. Then

$$\varphi_{e,e}(\mathcal{K}(A)) = \varphi_{p,p}(\phi_p(A)) \subsetneq \varphi_{p,p}(\mathcal{K}(X_p)) \subseteq \varinjlim \mathcal{K}(X_r, X_r),$$

and hence $j_X(A) \subsetneq (\mathcal{O}_X)_{[e,e]}$. $\square$

Remark 6.2. If $\{B_g\}_{g\in G}$ is a saturated Fell bundle, [21], i.e.

$$B_g B_{g^{-1}} = B_e, \qquad \text{for all } g \in G,$$

we may treat $X = \bigsqcup_{g\in G} B_g$ as a regular product system of Hilbert bimodules with the structure inherited in an obvious way from $\{B_g\}_{g\in G}$. Then the Fell bundles $\{(\mathcal{O}_X)_g\}_{g\in G}$ and $\{B_g\}_{g\in G}$ coincide. Accordingly, every cross sectional algebra of a saturated Fell bundle admits a natural realization as Cuntz-Pimsner algebra of a regular product system of Hilbert bimodules. Conversely, by Proposition 6.1, if X is a regular product system of Hilbert bimodules over a semigroup P of Ore type, and each fiber X_p is nondegenerate as right Hilbert module (so it is an imprimitivity bimodule), then the Fell bundle $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ is saturated.

Suppose X is a regular product system of Hilbert bimodules over a semigroup P of Ore type. Since all the maps $\phi_p : A \to \mathcal{K}(X_p)$, $p \in P$, are isomorphisms, we infer from Definition 4.4 that the semigroup $\widehat{X} = \{\widehat{X}_p\}_{p\in P}$ dual to X consists of partial homeomorphisms $\widehat{X}_p$ with domain $\widehat{\langle X_p, X_p\rangle_p}$ and range $\widehat{A}$. We show in Proposition 6.4 below that the semigroup $\{\widehat{X}_p\}_{p\in P}$ generates a partial action of the enveloping group $G(P)$. We recall the relevant definitions concerning partial actions, cf. e.g. [19].

Definition 6.3. A *partial action* of a group G on a topological space Ω consists of a pair $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$, where D_g's are open subets of Ω and $\theta_g : D_{g^{-1}} \to D_g$ are homeomorphisms such that

(PA1) $D_e = \Omega$ and $\theta_e = id$,
(PA2) $\theta_t(D_{t^{-1}} \cap D_s) = D_t \cap D_{ts}$,
(PA3) $\theta_s(\theta_t(x)) = \theta_{st}(x)$, for $x \in D_{t^{-1}} \cap D_{t^{-1}s^{-1}}$.

The partial action $(\{D_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ is *topologically free* if for every open nonempty $U \subseteq \Omega$ and finite $F \subseteq G\backslash\{e\}$ there exists $x \in U$ such that $x \in D_{t^{-1}}$ implies $\theta_t(x) \neq x$ for all $t \in F$.

Proposition 6.4. *Suppose X is a regular product system of Hilbert bimodules and the underlying semigroup P is of Ore type. The formulas*

$$D_{[q,p]} := \widehat{X}_q(\widehat{\langle X_p, X_p\rangle_p}),$$

$$\widehat{X}_{[p,q]}([\pi]) := \widehat{X}_p\widehat{X}_q^{-1}([\pi]), \qquad [\pi] \in D_{[q,p]},\ p,q \in P,$$

yield a well defined family of open sets $\{D_g\}_{g\in G(P)}$ and homeomorphisms $\widehat{X}_g : D_{g^{-1}} \to D_g$ such that $(\{D_g\}_{g\in G(P)}, \{\widehat{X}_g\}_{g\in G(P)})$ is a partial action of $G(P)$ on $\widehat{A}$. Moreover,

i) *$\{X_g\}_{g\in G(P)}$ is a semigroup dual to $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$, where we treat $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ as a product system, and $\widehat{X}_g$ are viewed as multivalued maps on $\widehat{A}$ with $\widehat{X}_g(\widehat{A}\backslash D_{g^{-1}}) = \{\varnothing\}$.*

ii) *We have the following implication:*

$$(\{D_g\}_{g\in G(P)}, \{\widehat{X}_g\}_{g\in G(P)}) \text{ is topologically free } \Longrightarrow\ X \text{ is topologically aperiodic,} \tag{34}$$

and if P is both left and right Ore (so for instance it is a group or a cancellative abelian semigroup) then the above implication is actually an equivalence.

Proof. To begin with, let us note that for an ideal I in A and $p \in P$ we have

$$\widehat{X}_p(\widehat{I}) = \widehat{{}_p\langle X_pI, X_p\rangle}, \qquad \widehat{X}_p^{-1}(\widehat{I}) = \widehat{\langle X_p, IX_p\rangle_p}, \tag{35}$$

cf. [33, Remark 2.3], [41, Subsection 3.3]. Now, let $[\pi] \in \widehat{A}$ and $r \in P$ be arbitrary. Natural representatives of the classes $\widehat{X}_p\widehat{X}_q^{-1}([\pi])$ and $\widehat{X}_{pr}\widehat{X}_{qr}^{-1}([\pi])$ act by multiplication from the left on the spaces

$$X_p \otimes \widetilde{X}_q \otimes_\pi H_\pi, \qquad X_{pr} \otimes \widetilde{X}_{qr} \otimes_\pi H_\pi,$$

respectively. The obvious C^*-correspondence isomorphisms

$$X_{pr} \otimes \widetilde{X}_{qr} \cong X_p \otimes (X_r \otimes \widetilde{X}_r) \otimes \widetilde{X}_q \cong X_p \otimes A \otimes \widetilde{X}_q \cong X_p \otimes \widetilde{X}_q$$

yield a unitary equivalence between the aforementioned representations. Hence $\widehat{X}_p\widehat{X}_q^{-1}([\pi]) = \widehat{X}_{pr}\widehat{X}_{qr}^{-1}([\pi])$, and thus $\widehat{X}_p\widehat{X}_q^{-1}$ does not depend on the choice of representatives of $[p,q]$. It follows from (35) that the natural domain of $\widehat{X}_p\widehat{X}_q^{-1}$ is $\widehat{X}_q(\widehat{\langle X_p, X_p\rangle_p})$ which coincides with the spectrum of ${}_q\langle X_q\langle X_p, X_p\rangle_p, X_q\rangle$. This shows that the formulas above indeed define homeomorphisms $\widehat{X}_g : D_{g^{-1}} \to D_g$, $g \in G(P)$.

Condition (PA1) is obvious. To show (PA2), let $t = [t_1,t_2]$, $s = [s_1,s_2]$ and $r \geqslant t_2, s_1$. Putting $q = t_1(t_2^{-1}r)$, $p = s_2(s_1^{-1}r)$, we have $t = [t_1(t_2^{-1}r), t_2(t_2^{-1}r)] = [q,r]$ and $s = [s_1(s_1^{-1}r), s_2(s_1^{-1}r)] = [r,p]$. Hence

$$\widehat{X}_t(D_s) = \widehat{X}_{[q,r]}(D_{[r,p]}) = \widehat{X}_q\widehat{X}_r^{-1}(\widehat{X}_r(D_{[e,p]})) = \widehat{X}_q(D_{[e,p]} \cap D_{[e,r]}).$$

On the other hand, since $st = [t_1,t_2] \circ [s_1,s_2] = [t_1(t_2^{-1}r), s_2(s_1^{-1}r)] = [q,p]$, we have

$$D_{ts} \cap D_t = D_{[q,p]} \cap D_{[q,r]} = \widehat{X}_q(D_{[e,p]}) \cap \widehat{X}_q(D_{[e,r]}) = \widehat{X}_q(D_{[e,p]} \cap D_{[e,r]}).$$

This proves condition (PA2).

To show (PA3), let $t = [t_1,t_2]$, $s = [s_1,s_2]$, $r \geqslant t_1, s_2$ and $[\pi] \in D_{t^{-1}} \cap D_{t^{-1}s^{-1}}$. Then a natural representative of $\widehat{X}_{st}([\pi]) = \widehat{X}_{[s_1,s_2]\circ[t_1,t_2]}([\pi]) = \widehat{X}_{s_1s_2^{-1}r}\widehat{X}^{-1}_{t_2t_1^{-1}r}([\pi])$ acts by left multiplication on the space

$$X_{s_1s_2^{-1}r} \otimes \widetilde{X}_{t_2t_1^{-1}r} \otimes_\pi H_\pi = X_{s_1} \otimes X_{s_2^{-1}r} \otimes \widetilde{X}_{t_1^{-1}r} \otimes \widetilde{X}_{t_2} \otimes_\pi H_\pi,$$

Similarly, a representative of $\widehat{X}_s(\widehat{X}_t([\pi])) = (\widehat{X}_{s_1} \circ \widehat{X}_{s_2}^{-1} \circ \widehat{X}_{t_2} \circ \widehat{X}_{t_2}^{-1})([\pi])$ acts by left multiplication on the space

$$X_{s_1} \otimes \widetilde{X}_{s_2} \otimes X_{t_1} \otimes \widetilde{X}_{t_2} \otimes_\pi H_\pi.$$

The latter can be considered an invariant subspace of the former with help of the following natural isomorphisms of C^*-correspondences:

$$\begin{aligned} X_{s_1} \otimes \widetilde{X}_{s_2} \otimes X_{t_1} \otimes \widetilde{X}_{t_2} &\cong X_{s_1} \otimes \widetilde{X}_{s_2} \otimes (X_r \otimes \widetilde{X}_r) \otimes X_{t_1} \otimes \widetilde{X}_{t_2} \\ &\cong X_{s_1}\langle X_{s_2}, X_{s_2}\rangle_{s_2} \otimes X_{s_2^{-1}r} \otimes \widetilde{X}_{t_1^{-1}r} \otimes \langle X_{t_1}, X_{t_1}\rangle_{t_1} \widetilde{X}_{t_2}. \end{aligned}$$

By the choice of $[\pi]$ and property (PA2), we see that $\widehat{X}_s\widehat{X}_t([\pi])$ is nonzero and thus equals $\widehat{X}_{st}([\pi])$, as irreducible representations have no non-trivial subrepresentations.

Ad (i). This follows from our description of $\widehat{X}_{[p,q]}$ and the form of $\mathcal{O}_{[p,q]}$ given in (33).

Ad (ii). Implication (34) is straightforward. For the converse, let us additionally assume that P is right cancellative and right reversible (then P is both left and right Ore). Take any $g_1,..., g_n \in G(P)\backslash\{[e,e]\}$. Using left reversibility of P we may represent these elements in the form $g_1 = [t, r_1],..., g_n = [t, r_n]$, where $t, r_1, ..., r_n \in P$ and $t \neq r_i$ for $i = 1, ..., n$. By right reversibility of P, one can inductively find elements $q_1, ..., q_n, p_1', p_2', ..., p_n' \in P$ such that

$$\begin{aligned} q_1 t &= p_1' r_1, \\ q_2 q_1 t &= p_2' p_1' r_2, \\ &\dots \\ q_n ... q_2 q_1 t &= p_n' ... p_2' p_1' r_n. \end{aligned}$$

Then defining

$$q := q_n...q_1, \qquad s := qt \quad \text{and} \quad p_i := q_n...q_{i+1}p_i'...p_1' \quad \text{for} \quad i = 1, ..., n,$$

we get $s = p_i r_i$ and $p_i \neq q$ for $i = 1, ..., n$. Hence $q^{-1}s = t$ and $p_i^{-1}s = r_i$ for every $i = 1, ..., n$. Thus

$$\widehat{X}_{g_i} = \widehat{X}_{[t,r_i]} = \widehat{X}_t\widehat{X}_{r_i}^{-1} = \widehat{X}_{q^{-1}s}\widehat{X}_{p_i^{-1}s}^{-1}.$$

Since $\widehat{X}_{q^{-1}s}\widehat{X}_{p_i^{-1}s}^{-1} = \widehat{X}_{[q^{-1}s, p_i^{-1}s]}$ does not depend on the choice of $s \geqslant q, p_i$, we see that the aperiodicity condition applied to q and $p_1, ..., p_n$ yields the topological freeness condition for $g_1, ..., g_n$. □

We do not know if the converse to implication (34) holds in general, see also Remark 6.10 below. Nevertheless, applying Proposition 6.4 and Theorems 5.6 and 5.10, we obtain the following.

Corollary 6.5. *Suppose $\{B_g\}_{g\in G}$ is a saturated Fell bundle. Treating its fibers as imprimitivity Hilbert bimodules over B_e, cf. Remark 6.2, the dual semigroup $\{\widehat{B}_g\}_{g\in G}$ is a group of genuine homeomorphisms of $\widehat{B}_e$.*

i) *The action $\{\widehat{B}_g\}_{g\in G}$ is topologically free if and only if the product system $X = \bigsqcup_{g\in G} B_g$ is topologically aperiodic. If this is the case, then every C^*-norm on $\bigoplus_{g\in G} B_g$ is topologically graded.*

ii) *If the action $\{\widehat{B}_g\}_{g\in G}$ is topologically free and has no invariant non-trivial open subsets then the reduced cross-sectional C^*-algebra $C_r^*(\{B_g\}_{g\in G})$ is simple.*

6.2. **Crossed products of twisted C^*-dynamical systems.** Suppose α is an action of a semigroup P by endomorphisms of A such that each α_s, $s \in P$, extends to a strictly continuous endomorphism $\overline{\alpha}_s$ of the multiplier algebra $M(A)$. Let ω be a circle-valued multiplier on P. That is $\omega : P \times P \to \mathbb{T}$ is such that

$$\omega(p,q)\omega(pq,r) = \omega(p,qr)\omega(q,r), \qquad p,q,r \in P.$$

Then (A, α, P, ω) is called a *twisted semigroup C^*-dynamical system.* A *twisted crossed product* $A \times_{\alpha,\omega} P$, see [22, Definition 3.1], is the universal C^*-algebra generated by $\{i_A(a)i_P(s) : a \in A, s \in P\}$, where (i_A, i_P) is a universal covariant representation of (A, P, α, ω). That is, $i_A : A \to A \times_{\alpha,\omega} P$ is a homomorphism and $\{i_P(p) : p \in P\}$ are isometries in $M(A \times_{\alpha,\omega} P)$ such that

$$i_P(p)i_P(q) = \omega(p,q)i_P(pq) \quad \text{ and } \quad i_P(p)i_A(a)i_P(p)^* = i_A(\alpha_p(a)),$$

for $p, q \in P$ and $a \in A$. A necessary condition for i_A to be injective is that all endomorphisms α_p, $p \in P$, are injective. We apply Theorem 3.8 to show that when P is of Ore type this condition is also sufficient. Additionally, we reveal a natural Fell bundle structure in $A \times_{\alpha,\omega} P$.

Following [22], we associate to (A, α, P, ω) a product system $X = \bigsqcup_{p \in P^{op}} X_p$ over the opposite semigroup P^{op}. We equip the linear space $X_p := \alpha_p(A)A$ with the following C^*-correspondence operations

$$a \cdot x = \alpha_p(a)x, \qquad x \cdot a = xa, \quad \langle x, y\rangle_p = x^*y,$$

$a \in A$, $x, y \in X_p$. The multiplication in X is defined by

$$x \cdot y = \overline{\omega(q,p)}\alpha_q(x)y, \quad \text{ for } \ x \in X_p = \alpha_p(A)A \ \text{ and } \ y \in X_q = \alpha_q(A)A.$$

By [22, Lemma 3.2], X is a product system and the left action of A on each of its fibers is by compacts. Accordingly, X is a regular product system if and only if all the endomorphisms α_p, $p \in P$, are injective. Moreover, by [22, Proposition 3.4] there is an isomorphism

$$A \rtimes_{\alpha,\omega} P \cong \mathcal{O}_X$$

given by the mapping that sends an element $i_P(p)^*i_A(a) \in A \rtimes_{\alpha,\omega} P$ to the image of the element $a \in X_p = \alpha_p(A)A$ in $\mathcal{O}_X$. Using this isomorphism and Theorem 3.8 one immediately gets the following.

Proposition 6.6. *Suppose that (A, α, P, ω) is a twisted semigroup C^*-dynamical system, where P is of Ore type and all the endomorphisms α_p, $p \in P$, are injective. Then the following hold.*

i) *The algebra A embeds via i_A into the crossed product $A \rtimes_{\alpha,\omega} P$.*
ii) *The crossed product $A \rtimes_{\alpha,\omega} P$ is naturally graded over the group of fractions $G(P)$ by the subspaces of the form*

$$B_g := \overline{\operatorname{span}}\{i_P(p)^*i_A(a)i_P(q) : a \in \alpha_p(A)A\alpha_q(A),\ [p,q] = g\}, \qquad g \in G(P).$$

Moreover, $A \rtimes_{\alpha,\omega} P$ can be identified with the cross-sectional C^*-algebra $C^*(\{B_g\}_{g\in G(P)})$.

In the remainder of this subsection we keep the assumptions of Proposition 6.6. It is natural to define a *reduced twisted crossed product* $A \times^r_{\alpha,\omega} P$ to be the reduced cross-sectional algebra of the Fell bundle $\{B_g\}_{g\in G(P)}$. Let $\lambda : A \times_{\alpha,\omega} P \to A \times^r_{\alpha,\omega} P$ be the canonical epimorphism, and

$$I_\lambda := \ker \lambda.$$

We wish to generalize the main results of [4] to the case of twisted semigroup actions. Let X be a product system associated to (A, α, P, ω) as above. One can see, cf., for instance, [31, Example 1.12], that a fiber X_p, $p \in P$, is a Hilbert bimodule if and only if the range of α_p is a hereditary subalgebra of A. If this is the case, then $\alpha_p(A)$ is a corner in A:

$$\alpha_p(A) = \alpha_p(A)A\alpha_p(A) = \overline{\alpha}_p(1)A\overline{\alpha}_p(1),$$

and the left inner product in X_p is defined by

$${}_p\langle x, y\rangle = \alpha_p^{-1}(xy^*), \qquad x, y \in X_p = \alpha_p(A)A.$$

The spectrum of $\alpha_p(A)$ can be identified with an open subset of $\widehat{A}$. Then the homeomorphism $\widehat{\alpha}_p : \widehat{\alpha_p(A)} \to \widehat{A}$ dual to the isomorphism $\alpha_p : A \to \alpha_p(A)$ can be naturally treated as a partial homeomorphism of $\widehat{A}$, cf. [32, Definition 2.16]. The following Lemma 6.7 is based on [32, Proposition 2.18] dealing with interactions on unital algebras.

Lemma 6.7. *If the monomorphism α_p has a hereditary range, then the homeomorphisms $\widehat{\alpha}_p : \widehat{\alpha_p(A)} \to \widehat{A}$ and $\widehat{X}_p : \widehat{\langle X_p, X_p\rangle_p} \to \widehat{A}$ coincide.*

Proof. With our identifications, we have

$$\widehat{\alpha_p(A)} = \{[\pi] \in \widehat{A} : \pi(\alpha_p(A)) \neq 0\} = \widehat{\langle X_p, X_p\rangle_p}.$$

Let $\pi : A \to \mathcal{B}(H)$ be an irreducible representation such that $\pi(\alpha_p(A)) \neq 0$. Then $\widehat{\alpha}_p([\pi])$ is the equivalence class of the representation $\pi \circ \alpha_p : A \to \mathcal{B}(\pi(\alpha_p(A))H)$. Since $\pi(\alpha_p(A))H = \pi(\alpha_p(A)A)H$ and

$$\|\sum_i a_i \otimes_\pi h_i\|^2 = \|\sum_{i,j}\langle h_i, \pi(a_i^* a_j)h_j\rangle_p\| = \|\sum_i \pi(a_i)h_i\|^2,$$

$a_i \in X_p = \alpha_p(A)A$, $h_i \in H$, $i = 1, ..., n$, we see that $a \otimes_\pi h \mapsto \pi(a)h$ yields a unitary operator $U : X_p \otimes_\pi H \to \pi(\alpha_p(A))H$. Furthermore, for $a \in A$, $b \in \alpha_p(A)$ and $h \in H$ we have

$$[X_p\text{-Ind}(\pi)(a)U^*]\pi(b)h = X_p\text{-Ind}(\pi)(a)\, b \otimes_\pi h = (\alpha_p(a)b) \otimes_\pi h = [U^*(\pi \circ \alpha_p)(a)]\pi(b)h.$$

Hence U intertwines X_p-Ind and $\pi \circ \alpha_p$. This proves that $\widehat{X}_p = \widehat{\alpha}_p$. □

Before stating our criterion of simplicity for semigroup crossed products, we need to define minimality for semigroup actions.

Definition 6.8. Let α be an action of a semigroup P on a C^*-algebra A. We say that α is *minimal* if for every ideal J in A such that $\alpha_p^{-1}(J) = J$ for all $p \in P$ we have $J = A$ or $J = \{0\}$.

Let us note that if X is the product system associated to a twisted semigroup C^*-dynamical system (A, α, P, ω) then minimality of α in the sense of Definition 6.8 is equivalent to minimality of X in the sense of Definition 5.7.

Proposition 6.9. *Suppose (A, α, P, ω) is a twisted semigroup C^*-dynamical system with P of Ore type. We assume that each endomorphism α_p, $p \in P$, is injective and has hereditary range. As above, we regard $\widehat{\alpha}_p$, $p \in P$, as partial homeomorphisms of $\widehat{A}$. The formulas*

$$D_{[q,p]} := \widehat{\alpha}_q(\widehat{\alpha_p(A)}), \qquad \widehat{\alpha}_{[p,q]}([\pi]) := \widehat{\alpha}_p(\widehat{\alpha}_q^{-1}([\pi])), \qquad [\pi] \in D_{[q,p]}, \; p, q \in P,$$

yield a well defined partial action $(\{D_g\}_{g\in G(P)}, \{\widehat{\alpha}_g\}_{g\in G(P)})$ which coincides with the partial action induced by the Fell bundle $\{B_g\}_{g\in G(P)}$ described in Proposition 6.6 part (ii). Moreover,

$$(\{D_g\}_{g\in G(P)}, \{\widehat{\alpha}_g\}_{g\in G(P)}) \text{ is topologically free } \Longrightarrow \{\widehat{\alpha}_p\}_{p\in P} \text{ is topologically aperiodic},$$

and

i) *if the semigroup* $\{\widehat{\alpha}_p\}_{p\in P}$ *is topologically aperiodic, then for any ideal* I *in* $A\times_{\alpha,\omega}P$ *such that* $I\cap A=\{0\}$ *we have* $I\subseteq I_\lambda$*;*
ii) *if the semigroup* $\{\widehat{\alpha}_p\}_{p\in P}$ *is topologically aperiodic and* α *is minimal, then the reduced twisted crossed product* $A\rtimes^r_{\alpha,\omega}P$ *is simple.*

Proof. With the identification of $A\rtimes_{\alpha,\omega}P$ with $\mathcal{O}_X$, for each $g\in G(P)$ we have the correspondence between $(\mathcal{O}_X)_g$ and B_g. Thus Lemma 6.7 and Proposition 6.4 imply the initial part of the assertion. The remaining claims (i) and (ii) follow from Lemma 6.7 and Theorems 5.6 and 5.10. ☐

Remark 6.10. If $P=G$ is a group, the multiplier $\omega\equiv 1$ is trivial, and all α_p, $p\in P$, are automorphisms, then $A\times_{\alpha,\omega}P=A\times_\alpha G$ is the classical crossed product. Then parts (i) and (ii) of Proposition 6.9 coincide with [4, Theorem 1] and [4, Corollary on p. 122], respectively. More generally, let us suppose that ω is arbitrary, P is left Ore semigroup, and $\alpha:P\to\mathrm{Aut}(A)$ is a semigroup action by automorphisms. By [34, Theorem 2.1.1] both the action α and the multiplier ω extend uniquely to the group $G=PP^{-1}$ in such a way that (A,α,G,ω) is a twisted group C^*-dynamical system and we have a natural isomorphism

$$A\times_{\alpha,\omega}P\cong A\times_{\alpha,\omega}G.$$

Then the partial action of G described in Proposition 6.9 is by homeomorphisms and coincides with the standard action $\widehat{\alpha}$ of G on $\widehat{A}$. Now, when P is both left and right Ore we can infer from Proposition 6.4 part (ii) that

$$\text{semigroup } \{\widehat{\alpha}_p\}_{p\in P} \text{ is topologically aperiodic} \iff \text{group } \{\widehat{\alpha}_g\}_{g\in G} \text{ is topologically free.}$$

It also follows from [4, Theorem 2] and the implication in Proposition 6.9 that the above equivalence holds when P is an arbitrary left Ore semigroup and A is commutative. In fact, in this case both these conditions are equivalent to the intersection property described in Proposition 6.9 part (i).

6.3. **Topological graph algebras.** Let $E=(E^0,E^1,s,r)$ be a *topological graph* as introduced in [27]. This means we assume that vertex set E^0 and edge set E^1 are locally compact Hausdorff spaces, source map $s:E^1\to E^0$ is a local homeomorphism, and range map $r:E^1\to E_0$ is a continuous map.

A *C^*-correspondence X_E of the topological graph E* is defined in the following manner, [27]. The space X_E consists of functions $x\in C_0(E^1)$ for which

$$E^0\ni v\longmapsto \sum_{\{e\in E^1:s(e)=v\}}|x(e)|^2$$

belongs to $A:=C_0(E^0)$. Then X_E is a C^*-correspondence over A with the following structure.

$$\begin{aligned}(x\cdot a)(e)&:=x(e)a(s(e)) \text{ for } e\in E^1,\\ \langle x,y\rangle_A(v)&:=\sum_{\{e\in E^1:s(e)=v\}}\overline{x(e)}y(e) \text{ for } v\in E^0, \text{ and}\\ (a\cdot x)(e)&:=a(r(e))x(e) \text{ for } e\in E^1.\end{aligned}$$

C^*-correspondence X_E generates a product system over $\mathbb{N}$. It follows from [27, Proposition 1.24] that this product system (or simply, this C^*-correspondence X_E) is regular if and only if

$$\overline{r(E^1)}=E^0 \text{ and every } v\in E^0 \text{ has a neighborhood } V \text{ such that } r^{-1}(V) \text{ is compact.} \tag{36}$$

In particular, (36) holds whenever $r : E^1 \to E^0$ is a proper surjection. If both E^0 and E^1 are discrete then E is just a usual directed graph and then (36) says that every vertex in E^0 receives at least one and at most finitely many edges (in other words, graph E is row-finite and without sources). According to [27, Definition 2.10], the C^*-algebra of E is

$$C^*(E) := \mathcal{O}_{X_E}.$$

Let $e = (e_n, ..., e_1)$, $r(e_i) = s(e_{i+1})$, $i = 1, ..., n-1$, be a path in E. Then e is a cycle if $r(e_n) = s(e_1)$, and vertex $s(e_1)$ is called the base point of e. A cycle e is said to be without entries if $r^{-1}(r(e_k)) = e_k$ for all $k = 1, ..., n$. Graph E is *topologically free*, [27, Definition 5.4], if base points of all cycles without entries in E have empty interiors. It is known, see [28, Theorem 6.14], that topological freeness of E is equivalent to the uniqueness property for $C^*(E)$.

In general, topological aperiodicity of X_E is stronger than topological freeness of E. However, when $E = (E^0, E^0, s, id)$ is a graph that comes from a mapping $s : E^0 \to E^0$, these two notions coincide.

Proposition 6.11. *Suppose X_E is a C^*-correspondence of a topological graph E satisfying (36). The dual C^*-correspondence acts on E^0 (identified with the spectrum of $A = C_0(E^0)$) via the formula*

$$\widehat{X}_E(v) = r(s^{-1}(v)). \tag{37}$$

In particular,

i) *X_E is topologically aperiodic if and only if the set of base points for periodic paths in E has empty interior;*
ii) *If r is injective, topological aperiodicity of X_E is equivalent to topological freeness of E;*
iii) *If E is discrete, then X_E is topologically aperiodic if and only if E has no cycles, and this in turn is equivalent (see* [30, Theorem 2.4]*) to $C^*(E)$ being an AF-algebra.*

Proof. We identify $\widehat{A}$ with E^0 by putting $v(a) := a(v)$ for $v \in E^0$, $a \in A = C_0(E^0)$. We fix $v \in E^0$ and an orthonormal basis $\{x_e\}_{e\in s^{-1}(v)}$ in the Hilbert space $\mathbb{C}^{|s^{-1}(v)|}$. Let us consider the representation $\pi_v : A \to \mathcal{B}(\mathbb{C}^{|s^{-1}(v)|})$ given by

$$\pi_v(a) = \sum_{e\in s^{-1}(v)} a(r(e))x_e, \qquad a \in A = C_0(E^0).$$

One readily checks that the mapping

$$X_E \otimes_v \mathbb{C} \ni x \otimes_v \lambda \longmapsto \sum_{e\in s^{-1}(v)} \lambda x(e)x_e \in \mathbb{C}^{|s^{-1}(v)|}$$

gives rise to a unitary which establishes equivalence X_E-Ind$(v) \cong \pi_v$. Furthermore, we have

$$\{w \in E^0 : w \leqslant \pi_v\} = \{w \in E^0 : w = r(e) \text{ for some } e \in s^{-1}(v)\} = r(s^{-1}(v)).$$

This yields (37). Claim (i) follows from (37), part (iii) of Proposition 5.5 and the Baire category theorem. Claims (ii) and (iii) are now straightforward. □

Corollary 6.12. *Keeping the assumptions of Proposition 6.11, let $V \subseteq E^0$ be closed. Then ideal $J = C_0(E^0\backslash V)$ is X_E-invariant if and only if $\widehat{X}_E(V) = V$.*

Proof. It is known, see for instance [28, Section 2], that ideal $J = C_0(E^0\backslash V)$ is X_E-invariant if and only if V satisfies the following two conditions

$$1)\ (\forall e \in E^1)\ s(e) \in V \Longrightarrow r(e) \in V, \qquad \text{and} \qquad 2)\ v \in V \Longrightarrow (\exists e \in r^{-1}(v))\ s(e) \in V.$$

In view of (37), conditions (1) and (2) are respectively equivalent to the inclusions $\widehat{X}_E(V) \subseteq V$ and $V \subseteq \widehat{X}_E(V)$. □

Example 6.13 (Exel's crossed product for a proper local homeomorphism)**.** Let $A = C_0(M)$ for a locally compact Hausdorff space M and let $\alpha : A \to A$ be the operator of composition with a proper surjective local homeomorphism $\sigma : M \to M$. Then α is an extendible monomorphism possessing a natural left inverse transfer operator $L : A \to A$, defined by

$$L(a)(t) = \frac{1}{|\sigma^{-1}(t)|} \sum_{s\in\sigma^{-1}(t)} a(s),$$

see [6, Subsection 2.1]. Let X_L be the C^*-correspondence with coefficients in A, constructed as follows. X_L is the completion of A with respect to the norm given by the inner-product below, and with the following structure:

$$x \cdot a = x\alpha(a), \qquad \langle x, y\rangle = L(x^*y), \qquad a \cdot x = ax,$$

where $a \in A$, $x, y \in X_L$. Clearly, the left action of A on X_L is injective. One can also show that it is by compacts, see the argument preceding [6, Corollary 4.2]. Hence X_L is a regular C^*-correspondence. It is known that is naturally isomorphic to a C^*-correspondence associated to the topological graph $E = (M, M, \sigma, id)$, [6, Section 6]. Thus, by Proposition 6.11, the dual C^*-correspondence to X_L acts on M, identified with the spectrum of $A = C_0(M)$, via the formula

$$\widehat{X}_L(t) = \sigma^{-1}(t). \tag{38}$$

It is observed in [6] that

$$C_0(M) \rtimes_{\alpha,L} \mathbb{N} := \mathcal{O}_{X_L}$$

is a natural candidate for Exel's crossed product when $A = C_0(M)$ is non-unital. When M is compact, $C(M) \rtimes_{\alpha,L} \mathbb{N}$ coincides with the crossed product introduced in [17] and can be effectively described in terms of generators and relations, [20, Theorem 9.2].

Now, combining Proposition 6.11, [6, Lemma 6.2] and [28, Theorem 6.14], we see that the following conditions are equivalent.

i) X_L is topologically aperiodic;
ii) the set of periodic points of σ has empty interior;
iii) σ is topologically free in the sense of Exel and Vershik [20, Definition 10.1], [6];
iv) every non-trivial ideal in $C_0(M) \rtimes_{\alpha,L} \mathbb{N}$ intersects $C_0(M)$ non-trivially.

Consequently, in view of Corollary 6.12, the crossed product $C_0(M) \rtimes_{\alpha,L} \mathbb{N}$ is simple if and only if in addition to the above equivalent conditions there is no nontrivial closed subset Y of M such that $\sigma^{-1}(Y) = Y$, cf. [6, Theorem 6.4], [20, Theorem 11.2], see also [9] and [45].

6.4. **C^*-algebras of topological P-graphs.** In this subsection, we introduce topological P-graphs which generalize both topological k-graphs [47] and (discrete) P-graphs [40], [7]. Within the framework of a general approach to product systems proposed in [23], the reasoning in [23, Example 1.5 (4)] shows that a topological P-graph defined below is simply a product system over P with values in a groupoid of topological graphs, see [23, Definition 1.1]. In the sequel P is a semigroup of Ore type. We treat elements of P as morphisms in a category with single object e.

Definition 6.14. By a *topological P-graph* we mean a pair (Λ, d) consisting of:

(1) a small category Λ endowed with a second countable locally compact Hausdorff topology under which the composition map is continuous and open, the range map r is continuous and the source map s is a local homeomorphism;

(2) a continuous functor $d\colon \Lambda \to P$, called degree map, satisfying the factorization property: if $d(\lambda) = pq$ then there exist unique μ, ν with $d(\mu) = p$, $d(\nu) = q$ and $\lambda = \mu\nu$.

Elements (morphisms) of Λ are called paths. $\Lambda^p := d^{-1}(p)$ stands for the set of paths of degree $p \in P$. Paths of degree e are called vertices.

We associate to a topological P-graph (Λ, d) a product system in the same manner as it is done for topological k-rank graphs in [8]. That is, for each $p \in P$ we let $X_p = X_{E_p}$ be the standard C^*-correspondence associated to the topological graph

$$E_p = (\Lambda^e, \Lambda^p, s|_{\Lambda^p}, r|_{\Lambda^p}),$$

so that $A := C_0(\Lambda^e)$ and X_p is the completion of the pre-Hilbert A-module $C_c(\Lambda^p)$ with the structure

$$\langle f, g\rangle_p(v) = \sum_{\eta\in\Lambda^p(v)} \overline{f(\eta)}g(\eta) \quad \text{and} \quad (a\cdot f\cdot b)(\lambda) = a(r(\lambda))f(\lambda)b(s(\lambda)).$$

The proof of [8, Proposition 5.9] works in our more general setting and shows that the formula

$$(fg)(\lambda) := f(\lambda(e,p))g(\lambda(p,pq))$$

defines a product $X_p \times X_q \ni (f,g) \to fg \in X_{pq}$ that makes $X = \bigsqcup_{p\in P} X_p$ into a product system. In view of (36), we see that the product system X is regular if and only if for every $p \in P$ we have

$$\overline{r(\Lambda^p)} = \Lambda^0, \text{ and}$$
$$\text{every } v \in E^0 \text{ has a neighborhood } V \text{ such that } r^{-1}(V) \cap \Lambda^p \text{ is compact in } \Lambda^p,$$

If the above condition holds, we say that the *topological P-graph (Λ, d) is regular*. It follows from [8, Theorem 5.20] that if (Λ, d) is a regular topological k-rank graph (that is, if $P = \mathbb{N}^k$), then the Cuntz-Krieger algebra of (Λ, d) defined in [47] coincides with $\mathcal{O}_X$. Hence it is natural to coin the following definitions, see also Remark 6.16 below.

Definition 6.15. Suppose (Λ, d) is a regular topological P-graph, where P is a semigroup of Ore type. We define a *C^*-algebra $C^*(\Lambda, d)$* and a *reduced C^*-algebra $C^*_r(\Lambda, d)$* of (Λ, d) to be respectively the Cuntz-Pimsner algebra $\mathcal{O}_X$ and the reduced Cuntz-Pimsner algebra $\mathcal{O}^r_X$, where X is the regular product system defined above.

Remark 6.16. If Λ is a discrete space then $C^*(\Lambda, d)$ is a universal C^*-algebra generated by partial isometries $\{s_\lambda : \lambda \in \Lambda\}$ subject to a natural version of Cuntz-Krieger relations, see [40, Theorem 4.2]. If we additionally assume (G, P) is a quasi-lattice ordered group then $C^*_r(\Lambda, d)$ coincides with the co-universal C^*-algebra $C^*_{min}(\Lambda)$ associated to (Λ, d) in [7]. To see the latter combine [40, Proposition 6.4], [7, Theorem 5.3], [8, Theorem 4.1] and [44, Corollary 5.2].

As an application of our main results — Theorems 3.8, 5.6, 5.10, we obtain the following.

Proposition 6.17. *Suppose (Λ, d) is a regular topological P-graph. The C^*-algebras $C^*(\Lambda, d)$ and $C^*_r(\Lambda, d)$ are non-degenerate in the sense that they are generated by the images of injective Cuntz-Pimsner representations of $X = \bigsqcup_{p\in P} X_p$. Moreover,*

i) *X is topologically aperiodic if and only if for every nonempty open set $U \subseteq \Lambda^e$, each finite set $F \subseteq P$ and an element $q \in P$ with $q \not\sim_R p$ for all $p \in F$, there is an* *enumeration* $\{p_1, ..., p_n\}$ *of elements of* F *and there are elements* $s_1, ..., s_n \in P$ *such that* $q \leqslant s_1 \leqslant ... \leqslant s_n$, $p_i \leqslant s_i$, *for* $i = 1, ..., n$, *and the union*

$$\bigcup_{i=1}^{n}\{v \in \Lambda^e : \mu \in \Lambda^{p_i^{-1}s_i}, \nu \in \Lambda^{q^{-1}s_i},\ s(\mu) = s(\nu) \text{ and } r(\mu) = r(\nu) = v\} \tag{39}$$

does not contain U.

ii) X *is minimal if and only if there is no nontrivial closed set* $V \subseteq \Lambda^e$ *such that*

$$r(\Lambda^p \cap s^{-1}(V)) = V \quad \text{for all } p \in P. \tag{40}$$

In particular, if the equivalent conditions in (i) hold, then any non-zero ideal in $C_r^*(\Lambda, d)$ *has non-zero intersection with* $C_0(\Lambda^e)$. *If the conditions described in (i) and (ii) hold, then* $C_r^*(\Lambda, d)$ *is simple.*

Proof. The initial claim of this proposition follows from Theorem 3.8 above. To see that the equivalence in part (i) holds, it suffices to apply formula (37) to the C^*-correspondences $X_p = X_{E_p}$, $p \in P$. Similarly, using (37) and Corollary 6.12, we see that X-invariant ideals in $C_0(\Lambda^e)$ are in one-to-one correspondence with closed sets V satisfying (40). This proves part (ii). The final claim of the proposition now follows from Theorems 5.6 and 5.10 above. □

Remark 6.18. Until now, there has been several different aperiodicity conditions introduced that imply uniqueness theorems for topological (or discrete) higher-rank graphs, that is when $P = \mathbb{N}^k$, cf. [39], [47], [46]. To our knowledge there are no such theorems known for more general semigroups P. We also point out that our topological aperiodicity has an advantage of being *local* – it involves only finite paths in Λ, which is of importance, cf. [39, discussion on page 94].

6.5. **The Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$.** In [12], Cuntz introduced $\mathcal{Q}_{\mathbb{N}}$, the universal C^*-algebra generated by a unitary u and isometries s_n, $n \in \mathbb{N}^\times$, subject to the relations

(Q1) $s_m s_n = s_{mn}$,
(Q2) $s_m u = u^m s_m$, and
(Q3) $\sum_{k=0}^{m-1} u^k s_m s_m^* u^{-k} = 1$,

for all $m, n \in \mathbb{N}^\times$. Cuntz proved that $\mathcal{Q}_{\mathbb{N}}$ is simple and purely infinite. Now we deduce the simplicity of $\mathcal{Q}_{\mathbb{N}}$ from our general result – Theorem 5.10 above, see also Remark 6.19 below.

It was shown in [46] that $\mathcal{Q}_{\mathbb{N}}$ may be viewed as the Cuntz-Pimsner algebra of a certain product system. We recall an explicit description of that product system given in [24].

The product system X is over the semigroup $\mathbb{N}^\times$ and its coefficient algebra is $A = C(S^1)$. We denote by Z the standard unitary generator of A. Each fiber X_m, $m \in \mathbb{N}^\times$, is a C^*-correspondence over A associated to the classical covering map $S^1 \ni z \to z^m \in S^1$, as constructed in Example 6.13. Each X_m as left A-module is free with rank 1, and we denote the basis element by 1_m. Hence, each element of X_m may be uniquely written as $\xi 1_m$ with $\xi \in A$. We have

$$\begin{aligned}(\xi 1_m) \cdot a &= \xi \alpha_m(a) 1_m,\\ \langle \xi 1_m, \eta 1_m \rangle_m &= L_m(\xi^* \eta),\\ a \cdot \xi 1_m &= (a\xi) 1_m,\end{aligned}$$

for $\xi, a \in A$. Then

$$X := \bigsqcup_{m \in \mathbb{N}^\times} X_m$$

becomes a product system with multiplication $X_m \times X_r \to X_{mr}$ given by

$$(\xi 1_m)(\eta 1_r) := (\xi \alpha_m(\eta)) 1_{mr}$$

for $m, r \in \mathbb{N}^\times$. By [24, Proposition 3.13] (cf. [44, Corollary 5.2]) we have

$$\mathcal{O}_X \cong \mathcal{Q}_\mathbb{N}.$$

Now, let $E_{i,j}$, $i,j = 0,1,\dots,m-1$, be a system of matrix units in $M_m(\mathbb{C})$. There is an isomorphism

$$C(S^1) \otimes M_m(\mathbb{C}) \cong \mathcal{K}(X_m)$$

such that

$$f \otimes E_{i,j} \leftrightarrow \Theta_{Z^i \alpha_m(f) 1_m, Z^j 1_m}.$$

Thus $\widehat{\mathcal{K}(X_m)}$ may be identified with the circle S^1. With these identifications, we have

$$\phi_m(Z) = Z \otimes E_{0,m-1} + \sum_{j=0}^{m-2} 1 \otimes E_{j+1,j},$$

and hence the multivalued map $\widehat{\phi_m} : S^1 \to S^1$ is such that

$$\widehat{\phi_m}(z) = \{w \in S^1 \mid w^m = z\}.$$

Furthermore, $[X_m\text{-Ind}]$ is identified with the identity map on S^1, and consequently the multivalued map $\widehat{X_m} = \widehat{\phi_m} \circ [X_m\text{-Ind}] : S^1 \to S^1$ is

$$\widehat{X_m}(z) = \{w \in S^1 \mid w^m = z\}.$$

For $m \neq n$ the set $\{z \in S^1 \mid z \in \widehat{X_m}(\widehat{X_n}^{-1}(z))\}$ is finite, while every nonempty open subset of S^1 is infinite. It follows that the product system X is topologically aperiodic.

Now, we see that A does not contain any non-trivial invariant ideals. Indeed, suppose J is an X-invariant ideal in A. Then $L_m(J) \subseteq J$ for all $m \in \mathbb{N}^\times$. There exists an open subset U of S^1 and a function $f \in J$ such that $f \geqslant 0$ and $f(t) \neq 0$ for all $t \in U$. If m is sufficiently large then for each $z \in S^1$ there is a $w \in U$ such that $w^m = z$. Then $L_m(f)$ is strictly positive on S^1 and hence invertible. Since $L_m(f) \in J$, we conclude that $J = A$.

Remark 6.19. We recall, cf. [6, Section 2] and Example 6.13, that for each $m \in \mathbb{N}^\times$ the mapping

$$C(S^1) \ni a \mapsto \sqrt{m} a 1_m \in X_m$$

establishes isomorphism between the fiber X_m and the C^*-correspondence associated to the topological graph (S^1, S^1, α_m, id). Using these isomorphisms, one may recover the product system associated to the topological $\mathbb{N}^\times$-graph (Λ, d) constructed in terms of generators in [46, Proposition 5.1]. In particular, simplicity of $\mathcal{Q}_\mathbb{N}$ could be also deduced from Proposition 6.17 applied to (Λ, d). Moreover, as the range map in each fiber of (Λ, d) is injective, part (ii) of Proposition 6.11 and Example 6.13 indicate that our simplicity criterion in this case might be not only sufficient but also necessary.

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