Houston Journal of Mathematics
© University of Houston
Volume , No. ,

EXEL'S CROSSED PRODUCT AND CROSSED PRODUCTS BY COMPLETELY POSITIVE MAPS

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Communicated by Kenneth R. Davidson

Abstract. We introduce crossed products of a C^*-algebra A by a completely positive map $\varrho : A \to A$ relative to an ideal in A. When ϱ is multiplicative they generalize various crossed products by endomorphisms. When A is commutative they include C^*-algebras associated to Markov operators by Ionescu, Muhly, Vega, and to topological relations by Brenken, but in general they are not modeled by topological quivers popularized by Muhly and Tomforde.

We show that Exel's crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, generalized to the case where A is not necessarily unital, is the crossed product of A by the transfer operator $\mathcal{L}$ relative to the ideal generated by $\alpha(A)$. We give natural conditions under which $\alpha(A)$ is uniquely determined by $\mathcal{L}$, and hence $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ depends only on $\mathcal{L}$. Moreover, the C^*-algebra $\mathcal{O}(A, \alpha, \mathcal{L})$ associated to $(A, \alpha, \mathcal{L})$ by Exel and Royer always coincides with our unrelative crossed product by $\mathcal{L}$.

As another non-trivial application of our construction we extend a result of Brownlowe, Raeburn and Vittadello, by showing that the C^*-algebra of an arbitrary infinite graph E can be realized as a crossed product of the diagonal algebra $\mathcal{D}_E$ by a 'Perron-Frobenious' operator $\mathcal{L}$. The important difference to the previous result is that in general there is no endomorphism α of $\mathcal{D}_E$ making $(\mathcal{D}_E, \alpha, \mathcal{L})$ an Exel system.

2010 *Mathematics Subject Classification.* 46L05,46L55.

Key words and phrases. Completely positive map, transfer operator, crossed product, Exel's crossed product, graph C^*-algebra, Cuntz-Pimsner algebra.

This research was supported by NCN grant number DEC-2011/01/D/ST1/04112 and by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme; project 'OperaDynaDual' (2014-2016). The author thanks Adam Skalski and the anonymous reviewers for their valuable comments and suggestions that improved the quality of the paper.

1. Introduction

In the present state of the art the theory of crossed products of C^*-algebras by endomorphisms breaks down into two areas that involve two different constructions. The first approach originated in late 1970's in the work of Cuntz [11] and was developed by many authors [44], [51], [1], [41], [2], [30], [29]. Another approach was initiated by Exel [12] in the beginning of the present century and immediately received a lot of attention; in particular, Exel's construction was extended in [10], [35], [14], [7]. By now, both of the approaches have proved to be useful in an innumerable variety of problems and their importance is well-acknowledged. They (or their semigroup versions) serve as tools to construct and analyse the most intensively studied C^*-algebras in recent years. These include: Cuntz algebras [11], Cuntz-Krieger algebras [12], Exel-Laca algebras [14], graph algebras [10], [18], [28], higher-rank graph algebras [7], C^*-algebras arising from semigroups [1], number fields [33], [3], or algebraic dynamical systems [8]. Among the applications one could mention their significant role in classification of C^*-algebras [48], study of phase transitions [32], or short exact sequences and tensor products [34].

In view of what has been said, it is somewhat surprising that the intersection of these two approaches is relatively small: the two constructions coincide for injective corner endomorphisms [12] and more generally for systems called complete in [2], [26], and reversible in [29]. Nowadays, it is known, see, for instance, [9], [30], that the aforementioned crossed-products can be unified in the framework of relative Cuntz-Pimsner algebras $\mathcal{O}(J, X)$ of Muhly and Solel [39]. However, different constructions are associated with different C^*-correspondences and different ideals J.

In fact, the relationship between the two aforesaid lines of research is still shrouded in mystery and calls for clarification. One of the overall aims of the present paper is to cover this demand. We do it by showing that the two areas are different special cases of one natural construction of a crossed product by a completely positive map. In particular, since completely positive maps are ubiquitous in the C^*-theory and in quantum physics, the crossed products we introduce have an ample potential for further study and applications. We hope that the present article will not only clear the decks but also give an impulse for such a development (see, for instance, our remarks concerning crossed products of commutative algebras (subsection 3.5); also the study of ergodic properties of non-commutative Perron-Frobenius operators that we introduce is of interest (see subsection 5.2)).

We note that Schweizer defined in [49, Subsection 3.3] a crossed product by a completely positive map as a particular case of Pimsner's (augmented) C^*-algebra [46]. However, apart from giving a simplicity criterion [50, Theorem 4.6] he didn't study the structure of these algebras. Schweizer's crossed product is covered by our construction (cf. Remark 3.14).

Let us explain our strategy in more detail. We introduce (in Definition 3.5) the relative crossed product $C^*(A,\varrho;J)$ of a C^*-algebra A by a completely positive mapping $\varrho : A \to A$ relative to an ideal J in A. The unrelative crossed product is $C^*(A,\varrho) := C^*(A,\varrho;N_\varrho^\perp)$ where N_ϱ is the largest ideal contained in $\ker\varrho$. When $\alpha := \varrho$ is multiplicative, hence an endomorphism of A, the crossed products $C^*(A,\alpha;J)$ cover the line of research we attributed to Cuntz. More specifically (see Subsection 3.4 below), the C^*-algebras $C^*(A,\alpha;J)$ coincide with crossed products by endomorphisms studied in [30], for unital A, and in [29], for extendible α. In particular, if α is extendible then $C^*(A,\alpha;A)$ is Stacey's crossed product [51], and $C^*(A,\alpha;\{0\})$ is the partial isometric crossed product introduced, in a semigroup context, by Lindiarni and Raeburn [37] (see Proposition 3.26). Accordingly, $C^*(A,\alpha) = C^*(A,\alpha;\ker\alpha^\perp)$ is a good candidate for the (unrelative) crossed product by an arbitrary endomorphism, cf. [30], [29]. In contrast to this multiplicative case, we claim that Exel's crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is a crossed product by the transfer operator $\mathcal{L}$ (which as a rule is not multiplicative). In order to make this statement precise we need to thoroughly re-examine - take 'a new look at' Exel's construction.

We recall that Exel introduced in [12] the crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ of a unital C^*-algebra A by an endomorphism $\alpha : A \to A$ which also depends on the choice of a *transfer operator*, i.e. a positive linear map $\mathcal{L} : A \to A$ such that $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$, for all $a,b \in A$. This construction was generalized to the non-unital case in [10], [35] were authors assumed that both α and $\mathcal{L}$ extend to strictly continuous maps on the multiplier algebra $M(A)$. We show however that extendability of $\mathcal{L}$ is automatic and since extendability of α does not play any role in the definition, in the present paper, we consider crossed products $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ for Exel systems $(A,\alpha,\mathcal{L})$ where A, α and $\mathcal{L}$ are arbitrary. Obviously, in a typical situation there are infinitely many different transfer operators for a fixed α. On the other hand, under natural assumptions, such as faithfulness of $\mathcal{L}$, which usually appear in applications [12], [15], [9], [10], the endomorphism α is uniquely determined by a fixed transfer operator $\mathcal{L}$. Moreover, any transfer operator $\mathcal{L}$ is necessarily a completely positive map and therefore it is suitable to form a crossed product on its own. This provokes the question:

To what extent $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ depends on α?

Before giving an answer, we need to stress that the pioneering Exel's definition of $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, [12, Definition 3.7], was to some degree experimental. In general it requires a modification. Namely, Brownlowe and Raeburn in [9] recognized $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ as a relative Cuntz-Pimsner algebra $\mathcal{O}(K_\alpha, M_\mathcal{L})$ where $M_\mathcal{L}$ is a C^*-correspondence associated to $(A, \alpha, \mathcal{L})$ and $K_\alpha = \overline{A\alpha(A)A} \cap J(M_\mathcal{L})$ is the intersection of the ideal generated by $\alpha(A)$ and the ideal of elements that the left action ϕ of A on $M_\mathcal{L}$ sends to 'compacts'. Then it follows from general results on relative Cuntz-Pimsner algebras, see [38, Proposition 2.21], [23, Proposition 3.3] or [9, Lemma 2.2], that A embeds into $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ if and only if K_α is contained in the ideal $(\ker\phi)^\perp \cap J(M_\mathcal{L})$. But the latter condition is not always satisfied. In particular, the theory of Cuntz-Pimsner algebras, and most notably the work of Katsura [22], [23], indicates that Exel's construction should be improved by replacing in [12, Definition 3.7] the ideal $\overline{A\alpha(A)A}$ with $(\ker\phi)^\perp \cap J(M_\mathcal{L})$. This is done by Exel and Royer in [14], cf. also [7, Proposition 4.5], where they associate to $(A, \alpha, \mathcal{L})$ a C^*-algebra $\mathcal{O}(A, \alpha, \mathcal{L})$ which is isomorphic to Katsura's Cuntz-Pimsner algebra $\mathcal{O}_{M_\mathcal{L}}$ (as a matter of fact, authors of [14] deal with more general Exel systems where α and $\mathcal{L}$ are only 'partially defined').

Turning back to our question, the results of the present paper give the following answer, which consists of three parts:

(1) the modified Exel's crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$ always coincides with our unrelative crossed product $C^*(A, \mathcal{L})$ of A by $\mathcal{L}$ (Theorem 4.7),
(2) original Exel's crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, for regular systems, does not depend on α, if we assume certain conditions assuring that A embeds into $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ (Proposition 4.18, Theorem 4.22),
(3) the three algebras $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$, $\mathcal{O}(A, \alpha, \mathcal{L})$, $C^*(A, \mathcal{L})$ coincide for most of systems appearing in applications (Proposition 4.9, Theorem 4.22, Theorem 5.6 i)).

In connection with point (3) it is interesting to note that, in general, there seems to be no clear relation between the ideals

$$\overline{A\alpha(A)A}, \qquad (\ker\phi)^\perp, \qquad J(M_\mathcal{L}).$$

However, for many natural systems $(A, \alpha, \mathcal{L})$, for instance for all such systems arising from graphs (cf. Lemma 5.9 below), we always have $\overline{A\alpha(A)A} = (\ker\phi)^\perp \cap J(M_\mathcal{L})$ and consequently $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} = \mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$. This shows that (by incorporating the ideal $\overline{A\alpha(A)A}$ into his original construction) Exel exhibited an incredibly good intuition; especially that, in contrast to $\overline{A\alpha(A)A}$, determining

$(\ker\phi)^{\perp}\cap J(M_{\mathcal{L}})$ is very hard in practice. In particular, it is an important task to identify Exel systems $(A,\alpha,\mathcal{L})$ for which $A\rtimes_{\alpha,\mathcal{L}}\mathbb{N}=\mathcal{O}(A,\alpha,\mathcal{L})=C^*(A,\mathcal{L})$. We find a large class of such objects in the present article.

We test the results of our findings on graph C^*-algebras. We recall that the main motivation for introduction of $A\rtimes_{\alpha,\mathcal{L}}\mathbb{N}$ in [12] was to realize Cuntz-Krieger algebras as crossed products associated with one-sided Markov shifts. This result was adapted in [10] to graph C^*-algebras $C^*(E)$ where E is a locally finite graph with no sinks or sources (by [7, Proposition 4.6], it can be generalized to graphs admitting sinks). For such a graph E the space of infinite paths E^∞ is a locally compact Hausdorff space and the one-sided shift $\sigma: E^\infty\to E^\infty$ is a surjective proper local homeomorphism. In particular, the formulas

$$\alpha(a)(\mu)=a(\sigma(\mu)),\qquad \mathcal{L}(a)(\mu)=\frac{1}{|\sigma^{-1}(\mu)|}\sum_{\eta\in\sigma^{-1}(\mu)}a(\eta), \tag{1}$$

$a\in C_0(E^\infty)$, $\mu\in E^\infty$, yield well-defined mappings on $C_0(E^\infty)$. Actually, $(C_0(E^\infty),\alpha,\mathcal{L})$ is an Exel system and $C_0(E^\infty)$ is naturally isomorphic to the diagonal C^*-subalgebra $\mathcal{D}_E$ of $C^*(E)$. By [10, Theorem 5.1], the isomorphism $C_0(E^\infty)\cong\mathcal{D}_E$ extends to the isomorphism $C_0(E^\infty)\rtimes_{\alpha,\mathcal{L}}\mathbb{N}\cong C^*(E)$. In order to generalize that result to arbitrary graphs one is forced to pass to a boundary path space ∂E of E, cf. [53]. Then $C_0(\partial E)\cong\mathcal{D}_E$, but the analogues of maps given by (1) are in general not well defined onto the whole of $C_0(\partial E)$. One possible solution, see [7], is to consider 'partial' Exel systems defined in [14]. In the present paper, we circumvent this problem by studying a more general class of 'Perron-Frobenious operators' of the form:

$$\mathcal{L}_\lambda(a)(\mu)=\sum_{e\in E^1,\, e\mu\in\partial E}\lambda_e\, a(e\mu),\qquad a\in C_0(\partial E), \tag{2}$$

where $\lambda=\{\lambda_e\}_{e\in E^1}$ is a family of strictly positive numbers indexed by the edges of E. We find necessary and sufficient conditions on λ assuring that $\mathcal{L}_\lambda: C_0(\partial E)\to C_0(\partial E)$ is well-defined. For any such λ we get an isomorphism

$$C^*(E)\cong C^*(C_0(\partial E),\mathcal{L}_\lambda).$$

Moreover, the map induced on $\mathcal{D}_E\cong C_0(E^\infty)$ by $\mathcal{L}_\lambda$ extends in a natural way to a completely positive map on $C^*(E)$. The latter deserves a name of *non-commutative Perron-Frobenious operator*. This indicates, at least in the present context, a somewhat superior role of a Perron-Frobenious operator $\mathcal{L}_\lambda$ over the standard non-commutative Markov shift, cf., for instance, [20], which in general is not even well-defined.

Finally, we mention our findings concerning an arbitrary (necessarily completely) positive map ϱ on a commutative C^*-algebra $A = C_0(D)$, where D is a locally compact Hausdorff space. Any such map defines a relation on D:

$$(x, y) \in R \stackrel{def}{\iff} \left(\forall_{a\in A_+}\ \varrho(a)(x) = 0 \implies a(y) = 0\right).$$

If the set $R \subseteq D \times D$ is closed, then ϱ give rise to a topological relation μ in the sense of [5] and a topological quiver $\mathcal{Q}$ in the sense of [40]. Then we prove that for the corresponding C^*-algebras associated to μ and $\mathcal{Q}$, in [5] and [40] respectively, we have $C^*(A, \varrho; A) \cong \mathcal{C}(\mu)$ and $C^*(A, \varrho) \cong C^*(\mathcal{Q})$. In particular, if ϱ is a Markov operator in the sense of [19], the C^*-algebra $C^*(\varrho)$ considered in [19] coincides with $C^*(A, \varrho)$. However, as we explain in detail and show by concrete examples, when R is not closed in $D\times D$, then $C^*(A, \varrho)$ cannot be modeled in any obvious way by the C^*-algebras studied in [40]. In particular, an analysis, similar to that in [19], for general positive maps on commutative C^*-algebras requires a generalization of the theory of topological quivers [40].

The content of the paper is organized as follows.

In Section 2, which serves as preliminaries, we gather certain facts on positive maps, explain in detail what we mean by a universal representation and recall definitions of relative Cuntz-Pimsner algebras. Also, we present a definition of Exel's crossed product $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ for arbitrary Exel systems $(A, \alpha, \mathcal{L})$, and recall a definition of Exel-Royer's crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$ for such systems.

In Section 3 we introduce relative crossed products $C^*(A, \varrho; J)$ for a completely positive map $\varrho : A \to A$. We present three pictures of $C^*(A, \varrho; J)$: as a quotient of a certain Toeplitz algebra (Definition 3.5); as a relative Cuntz-Pimsner algebra associated with a GNS correspondence X_ϱ of (A, ϱ) (Theorem 3.13); and as a universal C^*-algebra generated by suitably defined covariant representations of (A, ϱ) (Proposition 3.17). We finish this section by revealing relationships between construction and various crossed products by endomorphisms (subsection 3.4), and with C^*-algebras associated to topological relations, topological quivers, and Markov operators (subsection 3.5).

In Section 4 we show that the Toeplitz algebra $\mathcal{T}(A, \alpha, \mathcal{L})$ of $(A, \alpha, \mathcal{L})$ coincides with the Toeplitz algebra $\mathcal{T}(A, \mathcal{L})$ of $(A, \mathcal{L})$ (Proposition 4.3), which leads us to identities $\mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$ and $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; \overline{A\alpha(A)A})$ (Theorem 4.7). Using this result we conclude that $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$ for instance when $\mathcal{L}$ faithful and α extendible (Proposition 4.9). In Subsection 4.2 we study Exel systems $(A, \alpha, \mathcal{L})$ with the additional property that $E := \alpha \circ \mathcal{L}$ is a conditional expectation onto $\alpha(A)$. We give a number of characterizations and an intrinsic description of such Exel systems. This leads us to convenient conditions implying that $A\times_{\alpha,\mathcal{L}}\mathbb{N}$ does not depend on α (cf. Proposition 4.18). In particular, if $\alpha(A)$ is a hereditary subalgebra of A we prove that $A\times_{\alpha,\mathcal{L}}\mathbb{N}=C^*(A,\mathcal{L})\cong C^*(A,\alpha)$ (Theorem 4.22).

In the closing Section 5, we analyze the C^*-algebra $C^*(E)=C^*(\{p_v : v\in E^0\}\cup\{s_e : e\in E^1\})$ associated to an arbitrary infinite graph $E=(E^0,E^1,r,s)$. We briefly present Brownlowe's [7] realization of $C^*(E)$ as Exel-Royer's crossed product for a partially defined Exel system $(C_0(\partial E),\alpha,\mathcal{L})$. We find conditions on the numbers $\lambda=\{\lambda_e\}_{e\in E^1}$ assuring that (2) defines a self-map on $C_0(\partial E)$ (Proposition 5.4). For any such choice of λ we prove, using an algebraic picture of the system $(C_0(\partial E),\mathcal{L}_\lambda)$, that $C^*(E)\cong C^*(\mathcal{D}_E,\mathcal{L})$, where $\mathcal{L}(a):=\sum_{e\in E^1}\lambda_e s_e^* a s_e$, $a\in\mathcal{D}_E$ (see Theorem 5.6). If E is locally finite and without sources then the (non-commutative) Markov shift $\alpha(a):=\sum_{e\in E^1}s_e a s_e^*$ is the unique endomorphism of $\mathcal{D}_E$ such that $(\mathcal{D}_E,\alpha,\mathcal{L})$ is an Exel system and $C^*(\mathcal{D}_E,\mathcal{L})=\mathcal{D}_E\times_{\alpha,\mathcal{L}}\mathbb{N}$ (Theorem 5.6 iii)). In general there is no endomorphism making $(\mathcal{D}_E,\alpha,\mathcal{L})$ an Exel system (Theorem 5.6 ii)). One of possible interpretations of these results is that in order to associate a non-commutative shift to an arbitrary infinite graph one is forced to fix a certain measure system and encode the shift in its 'transfer operator', as the 'composition endomorphism' does not exist.

1.1. **Conventions and notation.** All ideals in C^*-algebras (unless stated otherwise) are assumed to be closed and two-sided. If I is an ideal in a C^*-algebra A we denote by $I^\perp=\{a\in A : aI=0\}$ the annihilator of I. We denote by 1 the unit in the multiplier algebra $M(A)$ of A. Any approximate unit in A is assumed to compose of contractive positive elements. All homomorphisms between C^*-algebras are assumed to be $*$-preserving. For actions $\gamma\colon A\times B\to C$ such as multiplications, inner products, etc., we use the notation:

$$\gamma(A,B)=\{\gamma(a,b) : a\in A, b\in B\},\qquad \overline{\gamma(A,B)}=\overline{\operatorname{span}}\{\gamma(a,b) : a\in A, b\in B\}.$$

By the Cohen-Hewitt Factorization Theorem we have $\gamma(A,B)=\overline{\gamma(A,B)}$ whenever γ can be interpreted as a continuous representation of a C^*-algebra A on a Banach space B. We emphasize that we will use this fact without further warning. In particular, a C^*-subalgebra A of a C^*-algebra B is non-degenerate if $AB=B$.

2. Preliminaries

In this section, we present certain facts concerning positive maps. Most of them are known, but usually they are stated in the literature in the unital case. We also present definitions of a universal C^*-algebra and a universal representation, which are well suited for our analysis. We briefly recall definitions of C^*-algebras associated to C^*-correspondences. In the last part of this section, we introduce a definition of Exel crossed product for arbitrary Exel systems, and also recall the definition of crossed products associated to such systems in [14].

2.1. **Positive maps.** Throughout this subsection we fix a positive map $\varrho : A \to B$ between two C^*-algebras A and B. This means that $\varrho : A \to B$ is linear and $\varrho(aa^*) \geq 0$ for every $a \in A$. Such ϱ is automatically $*$-preserving: $\varrho(a^*) = \varrho(a)^*$, $a \in A$; and bounded, see [31, Lemma 5.1]. We have the following formula for the norm of ϱ, which is well known for completely positive maps, cf. [31, Lemma 5.3(i)], and less known for positive maps.[1]

Lemma 2.1. *For any approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A the norm of ϱ is given by the limit $\|\varrho\| = \lim_{\lambda\in\Lambda} \|\varrho(\mu_\lambda)\|$.*

Proof. Recall that the double dual A^{**} of A can be identified with the enveloping von Neumann algebra of A, cf. [52, III, Theorem 2.4]. Similarly for B. Then the double dual $\varrho^{**} : A^{**} \to B^{**}$ of $\varrho : A \to B$ is a σ-weakly continuous extension of ϱ. As positive elements in A are σ-weakly dense in the set of positive elements in A^{**}, ϱ^{**} is positive. Hence Russo-Dye theorem implies, see [42, Corollary 2.9], that $\|\varrho^{**}\| = \|\varrho^{**}(1)\|$. Moreover, since $\{\mu_\lambda\}_{\lambda\in\Lambda}$ converges σ-weakly to 1, $\{\varrho^{**}(\mu_\lambda)\}_{\lambda\in\Lambda}$ converges σ-weakly to $\varrho^{**}(1)$. Since the norm is weakly lower-semicontinuous we get

$$\|\varrho^{**}\| = \|\varrho^{**}(1)\| \leq \liminf_{\lambda\in\Lambda} \|\varrho^{**}(\mu_\lambda)\| = \liminf_{\lambda\in\Lambda} \|\varrho(\mu_\lambda)\|.$$

As clearly we have $\limsup_{\lambda\in\Lambda} \|\varrho(\mu_\lambda)\| \leq \|\varrho\| \leq \|\varrho^{**}\|$, we get the desired equality. □

The formula for the kernel of the classic GNS representation yields also an important ideal for an arbitrary positive map.

Proposition 2.2. *The set*

$$N_\varrho := \{a \in A : \varrho((ab)^* ab)) = 0 \text{ for all } b \in A\} \tag{3}$$

is the largest ideal in A contained in the kernel of the mapping $\varrho : A \to B$.

Proof. Obviously, N_ϱ is a closed right ideal in A. Let $a, b \in A$. Since $b^*a^*ab \leq \|a^*a\| b^*b$ we get $\varrho((ab)^*ab) \leq \|a^*a\|\varrho(b^*b)$. The latter inequality implies that N_ϱ is a left ideal. In particular, if a is a positive element in N_ϱ, then $a^{1/4} \in N_\varrho$ and therefore $\varrho(a) = \varrho((a^{1/4}a^{1/4})^* a^{1/4}a^{1/4}) = 0$. This implies that $N_\varrho \subseteq \ker \varrho$. Clearly, if I is an ideal in A contained in $\ker \varrho$, then $I \subseteq N_\varrho$. □

[1] the author thanks Paul Skoufranis for providing the following short proof.

Definition 2.3. We call the ideal in (3) the *GNS-kernel* of $\varrho : A \to B$.

The ideal (3) is closely related to the notion of almost faithfulness introduced, in the context of Exel systems, in [9]. Namely, following [9, Definition 4.1], we say that ϱ is *almost faithful* on an ideal I in A if

$$a \in I \text{ and } \varrho((ab)^*ab)) = 0 \text{ for all } b \in A \implies a = 0.$$

The above implication is equivalent to the equality $I \cap N_\varrho = \{0\}$. In other words,

$$\varrho \text{ is almost faithful on } I \iff I \subseteq N_\varrho^\perp.$$

In particular, the annihilator $N_\varrho^\perp$ of the GNS-kernel of ϱ is the largest ideal in A on which ϱ is almost faithful. We recall that ϱ is *faithful* on a C^*-subalgebra $C \subseteq A$ if for any $a \in C$, $\varrho(a^*a) = 0$ implies $a = 0$. The following lemma sheds considerable light on the relationship between the two aforementioned notions.

Lemma 2.4. *Let $C \subseteq A$ be a C^*-subalgebra and consider the following conditions:*

i) *ϱ is faithful on the ideal $\overline{ACA}$,*
ii) *ϱ is faithful on the hereditary C^*-subalgebra CAC,*
iii) *ϱ is almost faithful on the ideal $\overline{ACA}$.*

Then i) $\Rightarrow$ ii) $\Rightarrow$ iii) *and if A is commutative then the above conditions are equivalent.*

Proof. The inclusion $CAC \subseteq \overline{ACA}$ yields the implication i)$\Rightarrow$ ii).
ii) $\Rightarrow$ iii). Let $a \in N_\varrho$ and $c \in C$. Since N_ϱ is an ideal, we have $\varrho((acb)^*acb) = 0$ for all $b \in A$. Taking $b = c^*$ and using faithfulness of ϱ on CAC we infer that $(acc^*)^*acc^* = 0$. This implies that $ac = 0$. Accordingly, $C \subseteq N_\varrho^\perp$ and since $N_\varrho^\perp$ is an ideal in A we get $\overline{ACA} \subseteq N_\varrho^\perp$.
Assume now that A is commutative and $\overline{ACA} \subseteq N_\varrho^\perp$. Consider an element ac of $\overline{ACA} = AC$, $a \in A$, $c \in C$, such that $\varrho((ac)^*ac) = 0$. For all $b \in A$ we have

$$\varrho((acb)^*acb) = \varrho((bac)^*bac) \leq \|b^*b\|\varrho((ac)^*ac) = 0.$$

Thus (by almost faithfulness) $ac = 0$. Hence ϱ is faithful on $\overline{ACA} = CA$. $\square$

There is a natural C^*-subalgebra of A on which ϱ is multiplicative.

Definition 2.5. Let $\varrho : A \to B$ be a positive map. We call the set

$$MD(\varrho) := \{a \in A : \varrho(b)\varrho(a) = \varrho(ba) \text{ and } \varrho(a)\varrho(b) = \varrho(ab) \text{ for every } b \in A\} \tag{4}$$

the *multiplicative domain of ϱ*.

It is immediate that $MD(\varrho)$ is a C^*-subalgebra of A. Hence $\varrho : MD(\varrho) \to B$ is a homomorphism of C^*-algebras. In the literature, see e.g. [42, p. 38], multiplicative domains are considered for contractive completely positive maps, which is due to the fact we express in Proposition 2.6 below. We recall that ϱ is *completely positive* if for every integer $n > 0$ the amplified map $\varrho^{(n)} : M_n(A) \to M_n(B)$ obtained by applying ϱ to each matrix element: $\varrho^{(n)}((a_{ij})) = (\varrho(a_{ij}))$, is positive, see [42, p. 5], [31, p. 39], or [52, IV, Definition 3.3]. It is not hard to show, cf. [43, Remark 5.1], see [52, IV, Corollary 3.4], that a linear map $\varrho : A \to B$ is completely positive if and only if

$$\sum_{i,j=1}^{n} b_i^* \varrho(a_i^* a_j) b_j \geq 0, \qquad \text{for all } a_1, ..., a_n \in A \text{ and } b_1, ..., b_n \in B.$$

The following fact is a generalization of [42, Theorem 3.18] to not necessarily unital completely positive maps on not necessarily unital C^*-algebras.

Proposition 2.6. *Let $\varrho : A \to B$ be a contractive completely positive map between C^*-algebras. Then*

$$MD(\varrho) = \{a \in A : \varrho(a)^*\varrho(a) = \varrho(a^*a) \text{ and } \varrho(a)\varrho(a)^* = \varrho(aa^*)\}. \tag{5}$$

In particular, $MD(\varrho)$ is the largest C^*-subalgebra of A on which ϱ restricts to a homomorphism.

Proof. To show the equality (5) note that the argument of the proof of [42, Theorem 3.18] applies, only modulo the fact that the Schwarz inequality

$$\varrho^{(2)}(a^*)\varrho^{(2)}(a) \leq \varrho^{(2)}(a^*a), \qquad a \in M_2(A),$$

used there holds for arbitrary contractive completely positive maps, see [31, Lemma 5.3 (ii)]. Plainly, (5) implies that for any C^*-subalgebra C of A such that $\varrho : C \to B$ is a homomorphism we have $C \subseteq MD(\varrho)$. □

We recall that any positive map $\varrho : A \to B$ is automatically completely positive whenever A or B is commutative [52, Corollary 3.5, Proposition 3.9]. Of course any homomorphism is a completely positive contraction. Also it is well known, cf., for instance, [52, III, Theorem 3.4, IV, Corollary 3.4], that if B is a C^*-subalgebra of A then for a linear idempotent $E : A \to B$ we have

$$\begin{aligned} E \text{ is contractive } &\Longleftrightarrow E \text{ is positive and } B \subseteq MD(E) \\ &\Longleftrightarrow E \text{ is completely positive and } B \subseteq MD(E) \end{aligned}$$

An idempotent E satisfying the above equivalent conditions is called a *conditional expectation.*

Definition 2.7. Let $\varrho : A \to B$ be a positive map. We say that ϱ is *strict* if $\{\varrho(\mu_\lambda)\}_{\lambda\in\Lambda}$ is strictly convergent in $M(A)$ for some approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A. We say that ϱ is *extendible* if it extends to a strictly continuous mapping $\overline{\varrho} : M(A) \to M(B)$.

Remark 2.8. The positive elements in A are strictly dense in the set of positive elements in $M(A)$. Thus if ϱ is an extendible (completely) positive map, then $\overline{\varrho} : M(A) \to M(B)$ is also (completely) positive. Clearly, every extendible map is strict, and it is well known that for homomorphisms these notions are actually equivalent.

2.2. **Universal C^*-algebras.** Defining a universal C^*-algebra for a given set of generators $\mathcal{G}$ subject to a set of relations $\mathcal{R}$ can be tricky as free C^*-algebras do not exist. A recent and perhaps the most compelling approach is elaborated by Loring in [36], see [36] for references to previous approaches. We propose a slightly more general framework that fits our setting. As in [36] we concentrate on a class of representations of $\mathcal{G}$ that are determined by prescribed relations, rather than on the relations themselves.

Definition 2.9. Let $\mathcal{G}$ be a set and let $\mathcal{R}$ be a certain class of maps from $\mathcal{G}$ to C^*-algebras. We refer to elements of $\mathcal{R}$ as to *representations of* $\mathcal{G}$. We define a preorder relation on $\mathcal{R}$ by writing $\pi \precsim \sigma$ for any $\pi, \sigma \in \mathcal{R}$ such that the map

$$\sigma(a) \longmapsto \pi(a), \qquad a \in \mathcal{G}, \tag{6}$$

extends to a (necessarily unique) homomorphism from $C^*(\sigma(\mathcal{G}))$ onto $C^*(\pi(\mathcal{G}))$. We denote by $\approx$ the equivalence relation on $\mathcal{R}$ induced by this preorder: $\pi \approx \sigma \iff \pi \precsim \sigma$ and $\sigma \precsim \pi$.

Remark 2.10. Let $\mathcal{R}$ be a class of representations of a set $\mathcal{G}$. It is straightforward to see that, if $\pi, \sigma \in \mathcal{R}$ are such that $\pi \approx \sigma$, then the map (6) extends to an isomorphism $C^*(\sigma(\mathcal{G})) \cong C^*(\pi(\mathcal{G}))$. Moreover, if $\pi, \sigma \in \mathcal{R}$ are upper bounds for $(\mathcal{R}, \precsim)$ then $\pi \approx \sigma$.

Definition 2.11. Suppose that a class $\mathcal{R}$ of representations of a set $\mathcal{G}$ admits an upper bound $\iota \in \mathcal{R}$, that is, $\pi \precsim \iota$ for all $\pi \in \mathcal{R}$. By the obvious abuse of language, cf. Remark 2.10, we say that ι *the universal representation* of $\mathcal{G}$ and the C^*-algebra

$$C^*(\mathcal{G}, \mathcal{R}) := C^*(\iota(\mathcal{G}))$$

is *the universal C^*-algebra for $\mathcal{R}$.*

Remark 2.12. By the definition of the universal C^*-algebra $C^*(\mathcal{G},\mathcal{R})$, for any $\pi\in\mathcal{R}$ the map $\iota(a)\mapsto\pi(a)$, $a\in\mathcal{G}$, extends to a (necessarily unique) epimorphism from $C^*(\iota(\mathcal{G}))$ onto $C^*(\pi(\mathcal{G}))$. We will write equality between any two C^*-algebras generated by ranges of two universal representations for the same class of representations of the same set of generators.

Now, we give a condition on $(\mathcal{G},\mathcal{R})$ that imply existence of an upper bound in $\mathcal{R}$. A version of this condition appears in all of the previous approaches starting from Blackadar's [4, Definition 1.1]. It appears also in Loring's characterisation [36, Theorem 3.1.1] of C^*-relations admitting universal C^*-algebras.

For any set of C^*-algebras B_i, $i\in I$, we denote their direct product by $\prod_{i\in I}B_i$, so the elements of $\prod_{i\in I}B_i$ are $\prod_{i\in I}a_i$ where $a_i\in B_i$, $i\in I$, and $\sup_{i\in I}\|a_i\|<\infty$.

Definition 2.13. We say that a class $\mathcal{R}$ of representations of a set $\mathcal{G}$ is *closed under products* if for every set of mappings $\pi_i:\mathcal{G}\to B_i$, $i\in I$, that belong to $\mathcal{R}$ the following two conditions are satisfied:

i) $\prod_{i\in I}\pi_i(a)\in\prod_{i\in I}B_i$ for all $a\in\mathcal{G}$.
ii) there exists an injective homomorphism $\tau:\prod_{i\in I}B_i\to B$ into a C^*-algebra B such that the map $\pi:\mathcal{G}\to B$ given by $\pi(a):=\tau\left(\prod_{i\in I}\pi_i(a)\right)$, $a\in\mathcal{G}$, belongs to $\mathcal{R}$.

Proposition 2.14. *If a class $\mathcal{R}$ of representations of a set $\mathcal{G}$ is closed under products then $\mathcal{R}$ has an upper bound and therefore the universal C^*-algebra $C^*(\mathcal{G},\mathcal{R})$ exists.*

Proof. First we need to show that the collection $\mathcal{R}/\approx$ of equivalence classes for $\approx$ form a set. To this end, we note that there is a Hilbert space H with the property that any C^*-algebra B generated by $|\mathcal{G}|$ generators can be embedded into $B(H)$. Indeed, any GNS representation of B is determined by a function from the set of generators to complex numbers, and the dimension of the resulting Hilbert space cannot exceed the cardinality of the free $*$-algebra $\mathcal{F}(\mathcal{G})$ generated by $\mathcal{G}$. Hence, by GNS construction, there is a faithful representation of B on a Hilbert space with dimension not exceeding $|\{f:\mathcal{G}\to\mathbb{C}\}|\cdot|\mathcal{F}(\mathcal{G})|$. This implies our claim.

Let H be the aforesaid Hilbert space and denote by F the set of all mappings from $\mathcal{G}$ into $B(H)$. For each $\pi\in\mathcal{R}$ we choose an embedding $\phi_\pi:C^*(\pi(\mathcal{G}))\to B(H)$, so that $\phi_\pi\circ\pi\in F$. We define an equivalence relation on F in a similar fashion as we did for $\mathcal{R}$. For $\pi,\sigma\in F$ we write

$$\pi\approx_F\sigma \iff \text{the map (6) extends to an isomorphism } C^*(\sigma(\mathcal{G}))\cong C^*(\pi(\mathcal{G})).$$

It is straightforward to see that, for any $\pi, \sigma \in \mathcal{R}$ we have $\pi \approx \sigma$ if and only if $\phi_\pi \circ \pi \approx_F \phi_\sigma \circ \sigma$. Thus the assignment $\mathcal{R} \ni \pi \longmapsto \phi_\pi \circ \pi \in F$ factors through to a bijective assignment from $\mathcal{R}/\approx$ onto a subset I of the set $F/\approx_F$.

Accordingly, there is a set $\{\pi_i\}_{i\in I} \subseteq \mathcal{R}$ such that for any $\sigma \in \mathcal{R}$ we have $\sigma \approx \pi_i$ for some $i \in I$. Let $\pi \in \mathcal{R}$ be the product of these representations $\{\pi_i\}_{i\in I}$ as described in Definition 2.13, so that, $\pi(a) := \tau\left(\prod_{i\in I}\pi_i(a)\right)$, $a \in \mathcal{G}$, for an injective homomorphism $\tau : \prod_{i\in I} B_i \to B$. To see that π is an upper bound for $\mathcal{R}$, let $\sigma \in \mathcal{R}$. Choose $i_0 \in I$ such that $\sigma \approx \pi_{i_0}$. Let $\Psi : C^*(\pi_{i_0}(\mathcal{G})) \to C^*(\sigma(\mathcal{G}))$ be the isomorphism determined by $\Psi(\pi_{i_0}(a)) = \sigma(a)$, $a \in \mathcal{G}$. Denote by p_{i_0} : $\prod_{i\in I} B_i \to B_{i_0}$ the projection onto B_{i_0} and let τ^{-1} denote the inverse to the isomorphism $\tau : \prod_{i\in I} B_i \to \tau(\prod_{i\in I} B_i)$. Putting $\Phi := \Psi \circ p_{i_0} \circ \tau^{-1}$ we get

$$\Phi(\tau(a)) = (\Psi \circ p_{i_0})\left(\prod_{i\in I}\pi_i(a)\right) = \Psi(\pi_{i_0}(a)) = \sigma(a), \qquad \text{for all } a \in \mathcal{G}.$$

Hence $\Phi : C^*(\pi(\mathcal{G}) \to C^*(\sigma(\mathcal{G}))$ is the homomorphism showing that $\sigma \precsim \pi$. $\square$

In the present paper, we will consider only two types of generators and their representations. One type comes from a C^*-correspondence X over a C^*-algebra A. Then the set of generators is $\mathcal{G} = A \cup X$ and we identify representations σ of $\mathcal{G}$ with pairs (π, π_X) where $\pi = \sigma|_A$ and $\pi_X = \sigma|_X$. Another type comes from a C^*-dynamical system or an Exel system on a C^*-algebra A. Then the set of generators is $\mathcal{G} = A \cup \{s\}$ where s is an abstract element and we identify representations $\sigma \in \mathcal{R}$ with pairs (π, S) where $\pi = \sigma|_A$ and $S = \sigma(s)$. In the latter case we will study the C^*-subalgebra $C^*(\iota(A) \cup \iota(A)\iota(s))$ of the universal C^*-algebra $C^*(\mathcal{G}, \mathcal{R}) = C^*(\iota(\mathcal{G}))$ (which can be also viewed as a universal C^*-algebra but with a different set of generators).

2.3. **Relative Cuntz-Pimsner algebras.** We assume that the reader is familiar with the theory of Hilbert modules (for an introduction see, for instance, [31]). A (right) *C^*-correspondence* over a C^*-algebra A is a right Hilbert A-module X together with a left action of A on X given by a homomorphism ϕ of A into the C^*-algebra $\mathcal{L}(X)$ of all adjointable operators on X: we write $a \cdot x = \phi(a)x$. Sometimes C^*-correspondences are called Hilbert bimodules, see [9], [10]. However, it seems to become a standard to use the term Hilbert bimodule in the sense of [6, Definition 3.1]. Namely, by a *Hilbert bimodule* over A we mean a space X which is at the same time a right and a left C^*-correspondence and the corresponding right $\langle \cdot, \cdot \rangle_A$ and left ${}_A\langle \cdot, \cdot \rangle$ A-valued inner products satisfy $x \cdot$

$\langle y, z\rangle_A = {}_A\langle x, y\rangle \cdot z$, for all $x, y, z \in X$, cf. [22, Definition 3.1] or [27, Definition 1.10] and the remarks below these definitions.

Definition 2.15. A representation (π, π_X) of a C^*-correspondence X consists of a representation $\pi : A \to B(H)$ in a Hilbert space H and a linear map $\pi_X : X \to B(H)$ such that

$$\pi_X(a \cdot x \cdot b) = \pi(a)\pi_X(x)\pi(b), \quad \pi_X(x)^*\pi_X(y) = \pi(\langle x, y\rangle_A), \quad a, b \in A,\, x \in X.$$

The C^*-algebra generated by $\pi(A) \cup \pi_X(X)$ is denoted by $C^*(\pi, \pi_X)$.

Remark 2.16. If (π, π_X) is a representation of a C^*-correspondence X then for each $x \in X$ we have $\|\pi_X(x)\|^2 = \|\pi(\langle x, x\rangle_A)\| \leq \|\langle x, x\rangle_A\| = \|x\|$. Thus the map π_X is automatically contractive (it is isometric if π is faithful), and using Proposition 2.14 one readily sees that a universal representation of X exists.

Definition 2.17 (Pimsner)**.** We denote by (i_A, i_X) the universal representation of a C^*-correspondence X and we call $\mathcal{T}(X) := C^*(i_A, i_X)$ the *Toeplitz algebra* of X.

Remark 2.18. Originally, Pimsner [46] constructed the Toeplitz algebra $\mathcal{T}(X)$ by means of the Fock representation of X, which as he noticed is the universal representation of X.

We recall that the set $\mathcal{K}(X)$ of *generalized compact operators* on X is the closed linear span of the operators $\Theta_{x,y}$ where $\Theta_{x,y}(z) = x\langle y, z\rangle_A$ for $x, y, z \in X$. In particular, $\mathcal{K}(X)$ is an ideal in $\mathcal{L}(X)$. Any representation (π, π_X) of X induces a homomorphism $(\pi, \pi_X)^{(1)} : \mathcal{K}(X) \to B(H)$ which satisfies

$$(\pi, \pi_X)^{(1)}(\Theta_{x,y}) = \pi_X(x)\pi_X(y)^*, \qquad (\pi, \pi_X)^{(1)}(T)\pi_X(x) = \pi_X(Tx)$$

for $x, y \in X$ and $T \in \mathcal{K}(X)$, cf. [46, Page 202] or [21, Proposition 4.6.3]. Let $J(X) := \phi^{-1}(\mathcal{K}(X))$. For any representation (π, π_X) of X the restrictions $(\pi, \pi_X)^{(1)} \circ \phi|_{J(X)}$ and $\pi|_{J(X)}$ yield two representations of $J(X)$. Putting constraints on the set on which these two representations coincide leads us to the following definition.

Definition 2.19 (Muhly and Solel)**.** Let J be an ideal in $J(X) = \phi^{-1}(\mathcal{K}(X))$. We say that a representation (π, π_X) of X is J*-covariant* if

$$(\pi, \pi_X)^{(1)}(\phi(a)) = \pi(a), \qquad \text{for all } a \in J.$$

We denote by (j_A, j_X) the universal J-covariant representation of X and we call the C^*-algebra $\mathcal{O}(J, X) := C^*(j_A, j_X)$ the *relative Cuntz-Pimsner algebra* determined by J.

Remark 2.20. It is clear that the relative Cuntz-Pimsner algebra $\mathcal{O}(J, X)$ is naturally isomorphic to the quotient of the Toeplitz algebra $\mathcal{T}(X)$ by the ideal generated by $\{i_A(a) - (i_A, i_X)^{(1)}(\phi(a)) : a \in J\}$. Actually, Muhly and Solel [38, Definition 2.18] introduced the C^*-algebras $\mathcal{O}(J, X)$ as quotients of $\mathcal{T}(X)$. The C^*-algebra $\mathcal{O}(J(X), X)$, related to the ideal $J(X)$, coincides with the (augmented) C^*-algebra associated to X by Pimsner [46].

Katsura [22], [23] observed that among the relative Cuntz-Pimsner algebras $\mathcal{O}(J, X)$, perhaps, the most natural one is determined by the ideal J equal to

$$J_X := (\ker \phi)^{\perp} \cap J(X). \tag{7}$$

In particular, [38, Proposition 2.21] and [23, Proposition 3.3], see also [9, Lemma 2.3], imply the following proposition.

Proposition 2.21. *Let X be a C^*-correspondence and let J be an ideal in $J(X)$. The universal representation $j_A : A \to \mathcal{O}(J, X)$ is injective if and only if $J \subseteq (\ker \phi)^{\perp}$.*

Definition 2.22 (Katsura [22], Definition 2.6)**.** The (unrelative) *Cuntz-Pimsner algebra* associated to a C^*-correspondence X is $\mathcal{O}_X := \mathcal{O}(J_X, X)$ where J_X is Katsura's ideal (7).

If the C^*-correspondence X is *essential*, that is if $AX = X$, we may restrict our attention to representations (π, π_X) where π is non-degenerate. This is due to the following statement which was proved in [10] for a certain concrete C^*-correspondence X. However, the proof uses only the fact that X is essential.

Lemma 2.23 ([10], Lemma 3.4)**.** *For any representation (π, π_X) of an essential C^*-correspondence X on the Hilbert space H, the subspace $K = \pi(A)H$ is reducing for (π, π_X) and we have $\pi|_{K^{\perp}} = 0$ and $\pi_X|_{K^{\perp}} = 0$.*

Since all the C^*-correspondences considered in the text will be essential,

> *all the representations (π, π_X) of C^*-correspondences will be assumed to be non-degenerate,*

in the sense that π is non-degenerate. It will force our universal homomorphisms to be also non-degenerate. We recall that a homomorphism $h : A \to B$ between two C^*-algebras is *non-degenerate* if $h(A)$ is non-degenerate in B, that is if $h(A)B = B$.

2.4. **Exel's and Exel-Royer's crossed products.** Initially, Exel defined his crossed product for unital C^*-algebras [12], and then it was generalized in [10], [35] to Exel systems that consist of extendible maps. Nevertheless, the definition of the crossed product makes sense for an arbitrary Exel system and can be expressed as follows.

Definition 2.24. Let $\alpha : A \to A$ be an endomorphism of a C^*-algebra A and let $\mathcal{L} : A \to A$ be a positive linear map such that

$$\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b, \qquad \text{for all } a, b \in A. \tag{8}$$

Then $\mathcal{L}$ is called a *transfer operator* for α and the triple $(A, \alpha, \mathcal{L})$ is an *Exel system*.

Definition 2.25. A *representation of an Exel system* $(A, \alpha, \mathcal{L})$ is a pair (π, S) consisting of a non-degenerate representation $\pi : A \to B(H)$ and an operator $S \in B(H)$ such that

$$S\pi(a) = \pi(\alpha(a))S \quad \text{and} \quad S^*\pi(a)S = \pi(\mathcal{L}(a)) \text{ for all } a \in A. \tag{9}$$

A *redundancy* of a representation (π, S) of $(A, \alpha, \mathcal{L})$ is a pair $(\pi(a), k)$ where $a \in A$ and $k \in \overline{\pi(A)SS^*\pi(A)}$ are such that

$$\pi(a)\pi(b)S = k\pi(b)S, \qquad \text{for all } b \in A.$$

The *Toeplitz algebra* $\mathcal{T}(A, \alpha, \mathcal{L})$ of $(A, \alpha, \mathcal{L})$ is the C^*-algebra generated by $i_A(A) \cup i_A(A)t$ for a universal representation (i_A, t) of $(A, \alpha, \mathcal{L})$. *Exel's crossed product* $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ of $(A, \alpha, \mathcal{L})$ is the quotient C^*-algebra of $\mathcal{T}(A, \alpha, \mathcal{L})$ by the ideal generated by the set

$$\{i_A(a) - k : a \in \overline{A\alpha(A)A} \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}.$$

Existence of the universal representation (i_A, t) of an Exel system $(A, \alpha, \mathcal{L})$ can be deduced from Proposition 2.14, cf. the proof of Lemma 3.2 below. It can be also obtained by realizing $\mathcal{T}(A, \alpha, \mathcal{L})$ as a Toeplitz algebra of a C^*-correspondence $M_{\mathcal{L}}$ introduced by Exel in [12].

More specifically, let $(A, \alpha, \mathcal{L})$ be an Exel system. One makes A into a semi-inner product (right) A-module $A_{\mathcal{L}}$ by putting $m \cdot a := m\alpha(a)$, $\langle m, n\rangle_{\mathcal{L}} := \mathcal{L}(m^*n)$, $n, m \in A_{\mathcal{L}}, a \in A$, and defines $M_{\mathcal{L}}$ to be the associated Hilbert A-module:

$$M_{\mathcal{L}} := \overline{A_{\mathcal{L}}/N}, \qquad N := \{m \in A_{\mathcal{L}} : \langle m, m\rangle_{\mathcal{L}} = 0\}.$$

Denoting by $q : A_{\mathcal{L}} \to M_{\mathcal{L}}$ the quotient map one gets, cf. [12], [9], that

$$a \cdot q(m) := q(am), \qquad m \in A_{\mathcal{L}},\ a \in A,$$

yields a well defined left action of A on $M_{\mathcal{L}}$ making $M_{\mathcal{L}}$ into a C^*-correspondence. We note that $A \cdot M_{\mathcal{L}} = M_{\mathcal{L}}$, that is $M_{\mathcal{L}}$ is an essential C^*-correspondence. Moreover, the kernel of the left action of A on $M_{\mathcal{L}}$ coincides with the GNS-kernel $N_{\mathcal{L}}$ of $\mathcal{L}$, cf. Definition 2.3.

The following fact was proved in [10, Lemmas 3.2 and 3.3] for extendible Exel systems. However, the proofs exploit only extendability of a transfer operator. We will show in Proposition 4.2 below, that this is automatic. Another reason for omitting the proof of the next Proposition 2.26 is that it will follow from our more general results, cf. Corollary 4.5. We include it here, for the sake of discussion.

Proposition 2.26. *We have a one-to-one correspondence between representations (π, S) of $(A, \alpha, \mathcal{L})$ and representations $(\pi, \pi_{M_{\mathcal{L}}})$ of the C^*-correspondence $M_{\mathcal{L}}$. In particular, we have an isomorphism $\mathcal{T}(A, \alpha, \mathcal{L}) \cong \mathcal{T}(M_{\mathcal{L}})$.*

Previous versions of the above result were a point of departure in [14]. More specifically, the authors of [14] considered 'partial Exel systems' $(A, \alpha, \mathcal{L})$ where $\mathcal{L}$ is not everywhere defined and α may attain values outside of A. For such triples they defined a crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$, in essence, simply to be $\mathcal{O}_{M_{\mathcal{L}}}$, where $M_{\mathcal{L}}$ is a generalization of the C^*-correspondence defined above to the 'partial case'. In the present paper we will only make use of [14, Definition 1.6] applied to 'global' Exel systems. Thus we adopt the following definition.

Definition 2.27. The *Exel-Royer's crossed product* $\mathcal{O}(A, \alpha, \mathcal{L})$ associated to an Exel system $(A, \alpha, \mathcal{L})$ is the quotient of $\mathcal{T}(A, \alpha, \mathcal{L})$ by the ideal generated by the set

$$\{i_A(a) - k : a \in J_{M_{\mathcal{L}}} \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}$$

where $J_{M_{\mathcal{L}}}$ is Katsura's ideal (7) associated to the C^*-correspondence $M_{\mathcal{L}}$.

Remark 2.28. Since the kernel of the left action of A on $M_{\mathcal{L}}$ is equal to $N_{\mathcal{L}}$, we have $J_{M_{\mathcal{L}}} = N_{\mathcal{L}}^{\perp} \cap J(M_{\mathcal{L}})$. It can be readily deduced from Proposition 2.26 and Remark 2.20, see [7, Proposition 4.5], that we have a natural isomorphism $\mathcal{O}(A, \alpha, \mathcal{L}) \cong \mathcal{O}_{M_{\mathcal{L}}}$.

3. Crossed products by completely positive maps

Throughout this section, we fix a completely positive map $\varrho : A \to A$, and refer to the pair (A, ϱ) as to a *C^*-dynamical system.* We introduce relative crossed products $C^*(A, \varrho; J)$ as quotients of a certain Toeplitz algebra. Then we realize them as relative Cuntz Pimsner algebras and as universal C^*-algebras generated by appropriately defined covariant representations of (A, ϱ). At the end of this section we discuss two important special cases when: 1) ϱ is multiplicative; 2) A is commutative.

3.1. **Crossed products.** Following the original idea of Exel [12], we first define a Toeplitz algebra, and then construct crossed products by 'eliminating redundancies' in the latter.

Definition 3.1. A *representation of* (A, ϱ) is a pair (π, S) consisting of a non-degenerate representation $\pi : A \to B(H)$ and an operator $S \in B(H)$ such that

$$S^*\pi(a)S = \pi(\varrho(a)) \qquad \text{for all } a \in A. \tag{10}$$

If π is faithful we call (π, S) *faithful*. We denote by $C^*(\pi, S)$ the C^*-algebra generated by $\pi(A) \cup \pi(A)S$. We define the *Toeplitz algebra of* (A, ϱ) to be the C^*-algebra $\mathcal{T}(A, \varrho) := C^*(i_A(A), t)$ where (i_A, t) is the universal representation of (A, ϱ). Hence for any representation (π, S) of (A, ϱ) the assignments

$$i_A(a) \longmapsto \pi(a), \qquad i_A(a)t \longmapsto \pi(a)S, \qquad a \in A, \tag{11}$$

define the epimorphism from $\mathcal{T}(A, \varrho)$ onto $C^*(\pi, S)$.

Lemma 3.2. *Any C^*-dynamical system (A, ϱ) admits a universal representation and hence the Toeplitz algebra $\mathcal{T}(A, \varrho)$ exists.*

Proof. Let (π, S) be a representation of (A, ϱ) and let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit in A. Using non-degeneracy of π and relation (10) we get

$$\|S\|^2 = \lim_{\lambda\in\Lambda} \|\pi(\mu_\lambda)S\|^2 = \lim_{\lambda\in\Lambda} \|S^*\pi(\mu_\lambda^2)S\| = \lim_{\lambda\in\Lambda} \|\pi(\varrho(\mu_\lambda^2))\| \le \|\varrho\|.$$

Now let $\{(\pi_i, S_i)\}_{i\in I}$ be a set of representation, where $\pi_i : A \to B(H_i)$ and $S_i \in B(H_i)$. The above inequality implies that the direct product $\prod_{i\in I} S_i$ is an element of $\prod_{i\in I} B(H_i)$. In particular, embedding $\prod_{i\in I} B(H_i)$ in a non-degenerate way into $B(H)$ for some Hilbert space H, we see that $(\prod_{i\in I} \pi_i, \prod_{i\in I} S_i)$ is a representation of (A, ϱ). Thus the assertion follows by Proposition 2.14. □

To study the structure of $C^*(\pi, S) = C^*(\pi(A) \cup \pi(A)S)$ one needs to understand the relationship between the following 'monomials':

$$\pi(a)S\pi(b)S\pi(c), \quad \pi(a)S^*\pi(b)S^*\pi(c), \quad \pi(a)S^*\pi(b)S\pi(c), \quad \pi(a)S\pi(b)S^*\pi(c).$$

As we will see in the course of our analysis, the first two behave like 'simple tensors', and by (10) the third one is in $\pi(A)$. Establishing the relationship with the fourth 'monomial' requires determining additional data which is encoded in an ideal that we are about to introduce. In the context of C^*-correspondences, this ideal is closely related with the one considered in [23, Definition 5.8], and is called the ideal of covariance in [30, Definition 4.5].

Definition 3.3. By a *redundancy* of a representation (π, S) of (A, ϱ) we mean a pair $(\pi(a), k)$ such that $a \in A$, $k \in \overline{\pi(A)S\pi(A)S^*\pi(A)}$ and

$$\pi(a)\pi(b)S = k\pi(b)S \quad \text{for all } b \in A.$$

Let $J_{(\pi,S)}$ be the set of elements $a \in A$ such that $(\pi(a), k)$ is a redundancy of (π, S) with $\pi(a) = k$. Clearly, it is an ideal in A and

$$J_{(\pi,S)} = \{a \in A : \pi(a) \in \overline{\pi(A)S\pi(A)S^*\pi(A)}\}.$$

We call it the *ideal of covariance* for (π, S).

Remark 3.4. Using (10), we see that $\overline{\pi(A)S\pi(A)S^*\pi(A)}$ is a C^*-algebra that acts on the space $\pi(A)S$. Moreover, this action is faithful. Thus, if $(\pi(a), k)$ is a redundancy of (π, S), then k is uniquely determined by a, and we have $\pi(a) = k$ if and only if $a \in J_{(\pi,S)}$.

Let us consider the GNS-kernel N_ϱ of ϱ, see (3), and a representation (π, S) of (A, ϱ). If $a \in N_\varrho$ then the pair $(\pi(a), 0)$ is necessarily a redundancy because $\|\pi(a)\pi(b)S\|^2 \leq \|\varrho((ab)^*ab))\| = 0$, for all $b \in A$. In particular, $a \in J_{(\pi,S)} \cap N_\varrho$ implies $\pi(a) = 0$. Accordingly, if (π, S) is faithful then $J_{(\pi,S)} \subseteq N_\varrho^\perp$. An argument of this sort stands behind Katsura's motivation for introducing the ideal (7). It explains the special role of the ideal $N_\varrho^\perp$ in the following definition.

Definition 3.5. We define the *crossed product* $C^*(A, \varrho)$ of A by ϱ to be the quotient of the Toeplitz C^*-algebra $\mathcal{T}(A, \varrho)$ by the ideal generated by the set

$$\{i_A(a) - k : a \in N_\varrho^\perp \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}.$$

For any ideal J in A we define the *relative crossed product* $C^*(A, \varrho; J)$ to be the quotient of the Toeplitz C^*-algebra $\mathcal{T}(A, \varrho)$ by the ideal generated by the set

$$\{i_A(a) - k : a \in J \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}.$$

We denote by (j_A, s) the representation of (A, ϱ) that generates $C^*(A, \varrho; J)$.

3.2. **Crossed products as relative Cuntz-Pimsner algebras.** A C^*-correspondence associated to a completely positive map was already considered by Paschke [43, section 5] and sometimes is called the GNS or the KSGNS-correspondence (for Kasparov, Stinespring, Gelfand, Naimark, Segal), cf. [31], [19]. Namely, we let X_ϱ to be a Hausdorff completion of the algebraic tensor product $A \odot A$ with respect to the seminorm associated to the A-valued sesquilinear form given by

$$\langle a \odot b, c \odot d \rangle_\varrho := b^* \varrho(a^* c) d, \qquad a, b, c, d \in A. \tag{12}$$

In the sequel we use the symbol $a \otimes b$ to denote the image of the simple tensor $a \odot b$ in X_ϱ. The space X_ϱ becomes a C^*-correspondence over A with the left and right actions determined by: $a \cdot (b \otimes c) = (ab) \otimes c$ and $(b \otimes c) \cdot a = b \otimes (ca)$ where $a, b, c \in A$.

Definition 3.6. We call X_ϱ defined above the *C^*-correspondence of* (A, ϱ).

Remark 3.7. Clearly, the C^*-correspondence X_ϱ is essential. The GNS-kernel (3) of ϱ coincides with the kernel of the left action of A on X_ϱ. Hence the left action of A on X_ϱ is faithful if and only if ϱ is almost faithful on A.

If A is not unital we give a meaning to the symbol $a \otimes 1$, $a \in A$, using the following lemma.

Lemma 3.8. *Let $\{\mu_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit in A. Then, for any $a \in A$, the limit*

$$a \otimes 1 := \lim_{\lambda \in \Lambda} a \otimes \mu_\lambda \tag{13}$$

exists and defines a bounded linear map $A \ni a \mapsto a \otimes 1 \in X_\varrho$ of norm $\|\varrho\|^{\frac{1}{2}}$.

Proof. Since $\|a \otimes (\mu_\lambda - \mu_{\lambda'})\|^2 = \|(\mu_\lambda - \mu_{\lambda'}) \varrho(a^* a)(\mu_\lambda - \mu_{\lambda'})\|$ tends to zero, as λ and λ' tend to 'infinity', the net $\{a \otimes \mu_\lambda\}_{\lambda \in \Lambda}$ is Cauchy and hence convergent. Since

$$\sup_{a \in A, \|a\|=1} \|a \otimes 1\|^2 = \sup_{a \in A, \|a\|=1} \|\varrho(a^* a)\| = \|\varrho\|,$$

we see that $A \ni a \mapsto a \otimes 1 \in X_\varrho$ is a bounded operator of norm $\|\varrho\|^{\frac{1}{2}}$. □

Remark 3.9. Let $\{\mu_\lambda\}_{\lambda \in \Lambda}$ be an approximate unit in A. The limit

$$1 \otimes a := \lim_{\lambda \in \Lambda} \mu_\lambda \otimes a, \qquad a \in A, \tag{14}$$

in general may not exist. However, if ϱ is strict then it does exist and the map $A \ni a \mapsto 1 \otimes a \in X_\varrho$ is linear bounded, again of norm $\|\varrho\|^{\frac{1}{2}}$, see [31, p. 50].

The mapping in the latter remark, which exists when ϱ is strict, plays a key role in the construction of KSGNS-dilation of ϱ, cf. [31, Theorem 5.6]. We adjust this construction to get a description of representations of the C^*-correspondence X_ϱ for arbitrary ϱ.

Proposition 3.10. *Let X_ϱ be the C^*-correspondence of (A,ϱ). We have a one-to-one correspondence between representations (π,S) of (A,ϱ) and representations (π,π_{X_ϱ}) of X_ϱ where*

$$\pi_{X_\varrho}(a\otimes b)=\pi(a)S\pi(b),\qquad a\otimes b\in X_\varrho, \tag{15}$$

$$S^*\pi_{X_\varrho}(a\otimes b)=\pi(\varrho(a)b),\quad a,b\in A,\quad S^*|_{(\pi_{X_\varrho}(X_\varrho)H)^\perp}\equiv 0. \tag{16}$$

For the corresponding representations, we have $C^*(\pi,S)=C^*(\pi(A)\cup\pi_{X_\varrho}(X_\varrho))$ and for any approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A:

$$S=s\text{-}\lim_{\lambda\in\Lambda}\pi_{X_\varrho}(\mu_\lambda\otimes 1) \tag{17}$$

where the limit is taken in the strong operator topology. If ϱ is strict, the limit in (17) is strictly convergent in $M(C^*(\pi,S))$, and the multiplier $S\in M(C^*(\pi,S))$ is determined by the formula $S\pi(a)=\pi_{X_\varrho}(1\otimes a)$, $a\in A$, cf. Remark 3.9.

Proof. Let (π,S) be a representation of (A,ϱ) on H. The following computation

$$\begin{aligned}\Big(\sum_i\pi(a_i)S\pi(b_i)\Big)^*\sum_j\pi(c_j)S\pi(d_j)&=\sum_{i,j}\pi(b_i^*)S^*\pi(a_i^*c_j)S\pi(d_j)\\&=\sum_{i,j}\pi(b_i^*\varrho(a_i^*c_j)d_j)\\&=\pi(\langle\sum_i a_i\otimes b_i,\sum_j c_j\otimes d_j\rangle_\varrho)\end{aligned}$$

implies that the mapping (15) extends to a linear contractive map $\pi_{X_\varrho}:X_\varrho\to B(H)$. It is evident that (π,π_{X_ϱ}) is a representation of X_ϱ. We have $S^*\pi_{X_\varrho}(a\otimes b)=S^*\pi(a)S\pi(b)=\pi(\varrho(a)b)$, for any $a,b\in A$. Since π is non-degenerate, the range of S is contained in $\pi(A)SH=\pi(A)S\pi(A)H\subseteq\pi_{X_\varrho}(X_\varrho)H$. Hence $S^*|_{(\pi_{X_\varrho}(X_\varrho)H)^\perp}\equiv 0$ and (16) holds. Furthermore, note that for any approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A, the net $\{\pi(\mu_\lambda)\}_{\lambda\in\Lambda}$ converges strongly to the identity in $B(H)$. Thus by Lemma 3.8 and (15), we have $\pi_{X_\varrho}(a\otimes 1)=\lim_{\lambda\in\Lambda}\pi(a)S\pi(\mu_\lambda)=\pi(a)S$, for any $a\in A$. Therefore

$$\pi(A)S=\pi_{X_\varrho}(A\otimes 1)\subseteq\pi_{X_\varrho}(X_\varrho)=\pi(A)S\pi(A).$$

Hence $C^*(\pi,S)=C^*(\pi(A)\cup\pi(A)S)=C^*(\pi(A)\cup\pi(A)S\pi(A))=C^*(\pi(A)\cup\pi_{X_\varrho}(X_\varrho))$.

Suppose now that (π,π_{X_ϱ}) is a representation of X_ϱ. We need to show that there exists an operator $S\in B(H)$ such that (π,S) is a representation of (A,ϱ) satisfying (15). Let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit in A and consider the net of bounded operators $S_\lambda:=\pi_{X_\varrho}(\mu_\lambda\otimes 1)$, $\lambda\in\Lambda$. Note that $S_\lambda\pi(a)=\pi_{X_\varrho}(\mu_\lambda\otimes a)$,

$a \in A$. To see that (17) determines a bounded operator it suffices to show that the net $\{S_\lambda\}_{\lambda\in\Lambda}$ is strongly Cauchy. To this end, let $a \in A$, $h \in H$ and $\lambda \le \lambda'$, in the directed set Λ. Then

$$\begin{aligned}\|(S_\lambda - S_{\lambda'})\pi(a)h\|^2 &= \|\pi_{X_\varrho}((\mu_\lambda - \mu_{\lambda'}) \otimes a)h\|^2\\ &= \langle h, \pi(\langle(\mu_\lambda - \mu_{\lambda'}) \otimes a), (\mu_\lambda - \mu_{\lambda'}) \otimes a\rangle_\varrho)h\rangle\\ &= \langle h, \pi(a^*\varrho((\mu_\lambda - \mu_{\lambda'})^2)a)h\rangle\\ &\le \langle h, \pi(a^*\varrho(\mu_\lambda - \mu_{\lambda'})a)h\rangle\\ &= \langle \pi(a)h, \pi(\varrho(\mu_\lambda - \mu_{\lambda'}))\pi(a)h\rangle.\end{aligned}$$

Since the net $\{\pi(\varrho(\mu_\lambda))\}_{\lambda\in\Lambda}$ is strongly convergent the last expression tends to zero. Hence $S := \text{s-}\lim_{\lambda\in\Lambda} S_\lambda$ defines a bounded operator. Let $a, b \in A$. As $\|(a - a\mu_\lambda) \otimes b\|^2 = \|b^*\varrho((a^* - \mu_\lambda a^*)(a - a\mu_\lambda))b\|$ tends to zero, we get that $a \otimes b = \lim_{\lambda\in\Lambda} a\mu_\lambda \otimes b$ in X_ϱ. Thus

$$\pi_{X_\varrho}(a \otimes b) = \lim_{\lambda\in\Lambda} \pi_{X_\varrho}(a\mu_\lambda \otimes b) = \pi(a) \lim_{\lambda\in\Lambda} S_\lambda\pi(b) = \pi(a)S\pi(b),$$

that is (15) holds. Moreover, for any $a, b \in A$ and $h, f \in H$ we have

$$\begin{aligned}\langle \pi_{X_\varrho}(a \otimes b)h, Sf\rangle &= \lim_{\lambda\in\Lambda} \langle \pi_{X_\varrho}(a \otimes b)h, \pi_{X_\varrho}(\mu_\lambda \otimes 1)f\rangle\\ &= \lim_{\lambda\in\Lambda} \langle \pi(\varrho(\mu_\lambda a)b)h, f\rangle = \langle \pi(\varrho(a)b)h, f\rangle.\end{aligned}$$

Hence $S^*\pi_{X_\varrho}(a \otimes b) = \pi(\varrho(a)b)$ and therefore $S^*\pi(a)S\pi(b) = S^*\pi_{X_\varrho}(a \otimes b) = \pi(\varrho(a))\pi(b)$. Since π is non-degenerate this implies (10).
Suppose now that ϱ is strict. Then, in view of Remark 3.9, for any $a \in A$ the following limit exists

$$\lim_{\lambda\in\Lambda} S_\lambda\pi(a) = \lim_{\lambda\in\Lambda} \pi_{X_\varrho}(\mu_\lambda \otimes a) = \pi_{X_\varrho}(1 \otimes a).$$

Similarly, we get $\lim_{\lambda\in\Lambda} \pi(a)S_\lambda = \pi_{X_\varrho}(a\otimes 1)$. Thus, since $\pi(A)$ is non-degenerate in $C^*(\pi, S) = C^*(\pi(A) \cup \pi(A)S\pi(A))$, the limit in (17) is strictly convergent. $\square$

Remark 3.11. Let (j_A, s) be the representation of (A, ϱ) that generates the relative crossed product $C^*(A, \varrho; J)$. By the above proposition s is an element of the enveloping von Neumann algebra $C^*(A, \varrho; J)^{**}$ and if ϱ is strict then actually $s \in M(C^*(A, \varrho; J))$.

The following lemma is a translation of [23, Proposition 3.3] to our setting, cf. also [9, Lemma 3.5]. It implies that when considering the relative crossed products it suffices to restrict attention to ideals J contained in the ideal $J(X_\varrho) = \phi^{-1}(\mathcal{K}(X_\varrho))$. It will also lead us to the main result of this subsection.

Lemma 3.12. *Let (π, S) be a faithful representation of (A, ϱ) and let (π, π_{X_ϱ}) be the representation of X_ϱ with π_{X_ϱ} given by* (15)*. A pair $(\pi(a), k)$ is a redundancy of (π, S) if and only if $a \in J(X_\varrho)$ and $k = (\pi, \pi_{X_\varrho})^{(1)}(\phi(a))$.*

Proof. Note that $(\pi, \pi_{X_\varrho})^{(1)} : \mathcal{K}(X_\varrho) \to \overline{\pi(A)S\pi(A)S^*\pi(A)}$. Thus if $a \in J(X_\varrho)$ and $k = (\pi, \pi_{X_\varrho})^{(1)}(\phi(a))$ then $k \in \overline{\pi(A)S\pi(A)S^*\pi(A)}$. Moreover, for any $b, c \in A$ we have

$$\begin{aligned}\pi(a)\pi(b)S\pi(c) &= \pi(a)\pi_{X_\varrho}(b \otimes c) = \pi_{X_\varrho}(\phi(a)b \otimes c) = (\pi, \pi_{X_\varrho})^{(1)}(\phi(a))\pi_{X_\varrho}(b \otimes c)\\ &= (\pi, \pi_{X_\varrho})^{(1)}(\phi(a))\pi(b)S\pi(c).\end{aligned}$$

Hence by non-degeneracy of π, we see that $(\pi(a), k)$ is a redundancy.
Now let $(\pi(a), k)$ be any redundancy. Then $k = (\pi, \pi_{X_\varrho})^{(1)}(t)$ for a certain $t \in \mathcal{K}(X_\varrho)$, and for any $b, c \in A$ we have

$$\begin{aligned}\pi_{X_\varrho}(\phi(a)b \otimes c) &= \pi(a)\pi_{X_\varrho}(b \otimes c) = \pi(a)\pi(b)S\pi(c) = k\pi(b)S\pi(c)\\ &= (\pi, \pi_{X_\varrho})^{(1)}(t)\pi_{X_\varrho}(b \otimes c) = \pi_{X_\varrho}(t(b \otimes c)).\end{aligned}$$

Since faithfulness of π implies injectivity of π_{X_ϱ}, we get $\phi(a)b \otimes c = tb \otimes c$, for all $b, c \in A$. Consequently, $\phi(a) = t$ as desired. □

Theorem 3.13. *Let X_ϱ be the C^*-correspondence of (A, ϱ). For any ideal J in A we have*

$$C^*(A, \varrho; J) = C^*(A, \varrho; J \cap J(X_\varrho)) \cong \mathcal{O}(J \cap J(X_\varrho), X_\varrho).$$

The universal homomorphism $j_A : A \to C^*(A, \varrho; J)$ is injective if and only if ϱ is almost faithful on $J \cap J(X_\varrho)$, that is if and only if $J \cap J(X_\varrho) \subseteq N_\varrho^\perp$. In particular,

$$C^*(A, \varrho) \cong \mathcal{O}_{X_\varrho}$$

and $j_A : A \to C^*(A, \varrho)$ is injective.

Proof. By Proposition 3.10 Toeplitz algebras $\mathcal{T}(A, \varrho)$ and $\mathcal{T}(X_\varrho)$ are naturally isomorphic and identifying them explicitly we may assume that the universal representation (i_A, i_{X_ϱ}) of X_ϱ in $\mathcal{T}(X_\varrho)$ satisfies $i_{X_\varrho}(a \otimes b) = i_A(a)t\, i_A(b)$ for $a, b \in A$. Then $\mathcal{T}(A, \varrho) = \mathcal{T}(X_\varrho)$ and by Lemma 3.12

$$\begin{aligned}&\{i_A(a) - k : a \in J \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}\\ &= \{i_A(a) - k : a \in J \cap J(X_\varrho) \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}\\ &= \{i_A(a) - (i_A, i_{X_\varrho})^{(1)}(\phi(a)) : a \in J \cap J(X_\varrho)\}.\end{aligned} \tag{18}$$

Hence the three algebras $C^*(A, \varrho; J)$, $C^*(A, \varrho; J \cap J(X_\varrho))$ and $\mathcal{O}(J \cap J(X_\varrho), X_\varrho)$ arise as quotients of the same algebra by the same ideal, cf. Remark 2.20. Thus they are isomorphic. For the remaining part of the assertion apply Proposition 2.21 and the fact that $(\ker\phi)^\perp = N_\varrho^\perp$. $\square$

Remark 3.14. In [49, Subsection 3.3], a crossed product by a completely positive map ϱ was defined as Pimsner's (augmented) C^*-algebra associated to the C^*-correspondence X_ϱ. Thus Schweizer's crossed product is isomorphic to the relative crossed product $C^*(A,\varrho;A) = C^*(A,\varrho;J(X_\varrho))$.

Remark 3.15. Using the notion of multiplicative domain, an *interaction* $(\mathcal{V},\mathcal{H})$ on a C^*-algebra A introduced by Exel in [13, Definition 3.1] can be defined as a pair of positive maps $\mathcal{V},\mathcal{H}: A\to A$ such that

$$\mathcal{V}\mathcal{H}\mathcal{V}=\mathcal{V},\quad \mathcal{H}\mathcal{V}\mathcal{H}=\mathcal{H},\quad \mathcal{V}(A)\subseteq MD(\mathcal{H}),\quad \mathcal{H}(A)\subseteq MD(\mathcal{V}).$$

Then $\mathcal{V}$ and $\mathcal{H}$ are automatically contractive completely positive maps, see [13, Corollary 3.3]. In [28] the author considered an interaction $(\mathcal{V},\mathcal{H})$ on a unital C^*-algebra A such that the ranges $\mathcal{V}(A)$, $\mathcal{H}(A)$ are corners in A. The crossed product $C^*(A,\mathcal{V},\mathcal{H})$ defined in [28, Definition 2.10] is a universal C^*-algebra generated by a copy of A and an operator s subject to relations

$$\mathcal{V}(a) = sas^* \quad \text{and} \quad \mathcal{H}(a)=s^*as \quad \text{for all } a\in A.$$

By [28, Corollary 2.16], $C^*(A,\mathcal{V},\mathcal{H})$ coincides with the covariance algebra associated to $(\mathcal{V},\mathcal{H})$ in [13]. The C^*-correspondences $X_\mathcal{V}$ and $X_\mathcal{H}$ are (mutually opposite) Hilbert bimodules, see [28, Lemma 2.11]. Hence by [28, Proposition 2.14] and [22, Proposition 3.7], $C^*(A,\mathcal{V},\mathcal{H})$ is isomorphic to both $\mathcal{O}_{X_\mathcal{V}}$ and $\mathcal{O}_{X_\mathcal{H}}$. In particular, by Theorem 3.13, we get

$$C^*(A,\mathcal{V},\mathcal{H})\cong C^*(A,\mathcal{V})\cong C^*(A,\mathcal{H}), \tag{19}$$

where $C^*(A,\mathcal{V})$ and $C^*(A,\mathcal{H})$ are crossed products in the sense of Definition 3.5.

3.3. **Universal description and gauge-invariant uniqueness theorem.** We describe the C^*-algebra $C^*(A,\varrho;J)$ as a universal object in the following way.

Definition 3.16. Let J be an ideal in A. We say that a representation (π,S) of (A,ϱ) is *J-covariant* if $J\cap J(X_\varrho)\subseteq J_{(\pi,S)}$ or just *covariant* if $N_\varrho^\perp\cap J(X_\varrho)\subseteq J_{(\pi,S)}$.

Proposition 3.17. *Let J be an ideal in A. The crossed product $C^*(A,\varrho;J)$ is universal with respect to J-covariant representations of (A,ϱ), that is (j_A,s) is J-covariant and for every J-covariant representation (π,S) of (A,ϱ) the mapping*

$$j_A(a)\longmapsto \pi(a),\qquad j_A(a)s\longmapsto \pi(a)S,\qquad a\in A, \tag{20}$$

extends to a (necessarily unique) epimorphism $\pi\rtimes_J S: C^*(A,\varrho;J)\to C^*(\pi,S)$.

Proof. That (j_A, s) is J-covariant follows from the equality (18) and the definition of $C^*(A,\varrho;J)$. Let (π,S) be a J-covariant representation of (A,ϱ), and let $\pi\rtimes_{\{0\}} S:\mathcal{T}(A,\varrho)\to C^*(\pi,S)$ be the epimorphism determined by (11). Clearly, $\pi\rtimes_{\{0\}} S$ maps redundancies of (i_A, i_{X_ϱ}) onto redundancies of (π,S). Thus in view of (18), using $J\cap J(X_\varrho)\subseteq J_{(\pi,S)}$, we conclude that $\pi\rtimes_{\{0\}} S$ factors through to the desired epimorphism $\pi\rtimes_J S: C^*(A,\varrho;J)\to C^*(\pi,S)$. □

Using Katsura's gauge-invariant uniqueness theorem for Cuntz-Pimsner algebras [23, Theorem 6.4] we get the following version of this standard tool for $C^*(A,\varrho)$. One could formulate a more general result for crossed products $C^*(A,\varrho;J)$, cf. for instance [27, Theorem 9.1], but we will not need it here.

Proposition 3.18. *Let (π,S) be a covariant representation of (A,ϱ). The epimorphism given by (20) is an isomorphism:*

$$C^*(\pi,S)\cong C^*(A,\varrho)$$

if and only if (π,S) is faithful and there exists a strongly continuous action $\beta:\mathbb{T}\to \mathrm{Aut}(C^*(\pi,S))$ such that $\beta_z(\pi(a))=\pi(a)$ and $\beta_z(\pi(a)S)=z\pi(a)S$ for all $a\in A$ and $z\in\mathbb{T}$.

Proof. If (π,S) is a faithful covariant representation of (A,ϱ) then by Lemma 3.12 the corresponding representation (π,π_{X_ϱ}) of X_ϱ is covariant in the sense of [23, Definition 3.4]. Since $C^*(A,\varrho)\cong\mathcal{O}_{X_\varrho}$, by Theorem 3.13, it suffices to apply [23, Theorem 6.4]. □

3.4. **The case when ϱ is multiplicative.** In this section, we assume that ϱ is multiplicative and we denote it by α. In other words, we assume that $\alpha:A\to A$ is an endomorphism. We show that our relative crossed products $C^*(A,\alpha;J)$ coincide with various crossed products by endomorphisms appearing in the literature. The latter are typically studied in the case where α is extendible. We warn the reader that representations of endomorphisms are considered in a different convention than we adopted in Definition 3.1. For the sake of discussion we include the following definition.

Definition 3.19. Let $\alpha:A\to A$ be an extendible endomorphism and let J be an ideal in A. We say that (π,U) is a *representation of the endomorphism* α if (π,S), where $S=U^*$, is a representation of (A,α) in the sense of Definition 3.1. Thus we assume that (π,U) consists of a non-degenerate representation $\pi:A\to B(H)$ and an operator $U\in B(H)$ such that

$$U\pi(a)U^*=\pi(\alpha(a))\qquad\text{for all } a\in A. \tag{21}$$

Further under these assumptions:

i) We say that (π, U) is a *J-covariant representation of* α if

$$J \subseteq \{a \in A : U^*U\pi(a) = \pi(a)\}.$$

We put $C^*_{endo}(A, \alpha; J) := C^*(i_A(A) \cup i_A(A)u)$ where (i_A, u) is the universal J-covariant representation of the endomorphism α. We also put $C^*_{endo}(A, \alpha) := C^*_{endo}(A, \alpha; (\ker \alpha)^\perp)$.

ii) We say that (π, U) is *isometric* if U is an isometry. The *Stacey's crossed product* is $A \rtimes^1_\alpha \mathbb{N} := C^*(\{i_A(a)u^n u^{*m} : a \in A,\ n, m \in \mathbb{N}\})$ where (i_A, u) is the universal isometric representation of α.

iii) We say that (π, U) is *partial-isometric* if

(22) U is a partial isometry and U^*U belongs to the commutant of A.

The *partial-isometric crossed product* is $A \rtimes_\alpha \mathbb{N} := C^*(i_A(A) \cup u i_A(A))$ where (i_A, u) is the universal partial-isometric representation of α.

Remark 3.20. The relation (21) implies that U is necessary a partial isometry (cf. Proposition 3.21 below). All of the authors mentioned below assumed that the universal operator u belongs to the multiplier algebra of the corresponding crossed product. We did not require that explicitly in Definition 3.19 but it follows from the axioms (see the second part of Proposition 3.26). Moreover, we have the following comments:

i). The crossed product $C^*_{endo}(A, \alpha; J)$ was introduced in [30, Definition 1.12] in the case A is unital. It was generalized to the non-unital case in [29, Definition 4.8]. These papers deal only with the ideals J contained in $(\ker \alpha)^\perp$. However, as explained, for instance in [29, Remark 4.3] or [30, Subsection 5.3], if $J \subsetneq (\ker \alpha)^\perp$, then there is a canonical quotient system $(A/R, \alpha_R)$ such that the image $q_R(J)$ of J in A/R is contained in $(\ker \alpha_R)^\perp$ and $C^*_{endo}(A, \alpha; J) \cong C^*_{endo}(A/R, \alpha_R; q_R(J))$.

ii). The crossed product $A \rtimes^1_\alpha \mathbb{N}$ was introduced in [51, Definition 3.1] as a crossed product of multiplicity 1. The author of [51] did not assume explicitly that α is extendible but he uses extendibility of α in his arguments. We note that a representation (π, U) of α is isometric if and only if it is A-covariant.

iii). The crossed product $A \rtimes_\alpha \mathbb{N}$ was defined in [37, p. 73] (in a semigroup context) essentially as Fowler's Toeplitz crossed product [17, p. 344], cf. [17, Proposition 3.4] or [37, Proposition 4.7].

The fact that the following covariance relation (23) is automatic went unnoticed in a few papers preceding [30], cf. [30, Remark 1.3].

Proposition 3.21. *Let $\alpha : A \to A$ be an endomorphism and let (π, S) be a representation of (A, α), in the sense of Definition 3.1. Then we automatically have that S is a partial isometry and*

$$\pi(a)S = S\pi(\alpha(a)), \qquad \text{for all } a \in A. \tag{23}$$

In particular, the projection SS^* belongs to the commutant of $\pi(A)$, and the ideal of covariance for (π, S), in the sense of Definition 3.3, is given by the formula

$$J_{(\pi,S)} = \{a \in A : SS^*\pi(a) = \pi(a)\}. \tag{24}$$

Proof. Let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit in A. Multiplicativity of α implies that the increasing net $\{\pi(\alpha(\mu_\lambda))\}_{\lambda\in\Lambda}$ converges strongly to a projection in $B(H)$. By (10), and non-degeneracy of π, we have $s\text{-}\lim_{\lambda\in\Lambda} \pi(\alpha(\mu_\lambda)) = s\text{-}\lim_{\lambda\in\Lambda} S^*\pi(\mu_\lambda)S = SS^*$. Hence S is a partial isometry. Following the argument in the proof of [30, Lemma 1.2], with $U = S^*$, we get the commutation relation (23).

Now, by (23) and its adjoint version ($S^*\pi(a) = \pi(\alpha(a))S^*$, $a \in A$) one gets $SS^*\pi(a) = S\pi(\alpha(a))S^* = \pi(a)SS^*$, for $a \in A$. Hence SS^* belongs to the commutant of $\pi(A)$.

By (23) we have $\overline{\pi(A)S\pi(A)S^*\pi(A)} = S\pi(\alpha(A)A\alpha(A))S^*$. Since S is a partial isometry, this implies that $J_{(\pi,S)} \subseteq \{a \in A : SS^*\pi(a) = \pi(a)\}$. Conversely, for any $a \in A$ such that $SS^*\pi(a) = \pi(a)$, again by (23), we have

$$\pi(a) = SS^*\pi(a) = S\pi(\alpha(a))S^* \in S\pi(\alpha(A)A\alpha(A))S^* = \overline{\pi(A)S\pi(A)S^*\pi(A)}.$$

This proves (24). □

Remark 3.22. If A is unital and (π, U) is a representation of α (as in Definition 3.19), then the set $I_{(\pi,U)} := \{a \in A : U^*U\pi(a) = \pi(a)\}$ was called in [30, Definition 1.7] the ideal of covariance for (π, U). In view of (24) we have $I_{(\pi,U)} = J_{(\pi,S)}$ where $S = U^*$.

The following lemma implies that in the definition of crossed products by extendible endomorphisms we can put the generating operator either on the left or on the right of the generating algebra (the outcome will be the same).

Lemma 3.23. *If $\alpha : A \to A$ is an extendible endomorphism and (π, S) is representation of (A, α), then*

$$C^*(\pi(A) \cup \pi(A)S) = C^*(\pi(A) \cup S\pi(A)) = C^*(\{\pi(a)S^{*n}S^m : a \in A,\ n, m \in \mathbb{N}\}).$$

Proof. By (23) we have $\pi(A)S = S\pi(\alpha(A)) \subseteq S\pi(A)$. Thus we get

$$C^*(\pi(A) \cup \pi(A)S) \subseteq C^*(\pi(A) \cup S\pi(A)) \subseteq C^*(\{\pi(a)S^{*n}S^m : a \in A,\ n, m \in \mathbb{N}\}).$$

We show that the rightmost algebra is contained in the leftmost one. Recall that for any approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A the net $\{\pi(\alpha(\mu_\lambda))\}_{\lambda\in\Lambda}$ converges strongly to S^*S, cf. the proof Proposition 3.21. Moreover, since α is extendible, for any $a\in A$, the net $\{a\alpha(\mu_\lambda)\}_{\lambda\in\Lambda}$ converges in norm (to $a\overline{\alpha}(1)$). Thus for any $a\in A$, using the adjoint of (23), we get

$$\pi(a)S^* = \pi(a)S^*SS^* = \lim_{\lambda\in\Lambda}\pi(a\alpha(\mu_\lambda))S^* = \lim_{\lambda\in\Lambda}\pi(a)S^*\pi(\mu_\lambda)\in\pi(A)S^*\pi(A)$$

($\pi(A)S^*\pi(A)$ is a closed space by the Cohen-Hewitt Factorization Theorem). Hence $\pi(A)S^*\subseteq\pi(A)S^*\pi(A)$. By (23), we also have $\pi(A)S\subseteq S\pi(A)$. The last two inclusions, used inductively, give us that

$$\pi(A)S^{*n}S^m\subseteq\underbrace{(\pi(A)S^*)...(\pi(A)S^*)}_{n \text{ times}}\underbrace{(\pi(A)S)...(\pi(A)S)}_{m\text{ times}}.$$

Hence $C^*(\{\pi(a)S^{*n}S^m : a\in A,\ n,m\in\mathbb{N}\})\subseteq C^*(\pi(A)\cup\pi(A)S)$. □

Corollary 3.24. *For any extendible endomorphism $\alpha : A\to A$ we have that $A\rtimes^1_\alpha\mathbb{N} = C^*_{endo}(A,\alpha;A)$ and $A\rtimes_\alpha\mathbb{N} = C^*_{endo}(A,\alpha;\{0\})$.*

Proof. Since a representation (π,U) of α is isometric if and only if it is A-covariant, we may identify the corresponding universal representations. Then applying Lemma 3.23 to (i_A,u^*) we get

$$\begin{aligned}A\rtimes^1_\alpha\mathbb{N} &= C^*(\{i_A(a)u^nu^{*m} : a\in A,\ n,m\in\mathbb{N}\}) = C^*(i_A(A)\cup i_A(A)u)\\ &= C^*_{endo}(A,\alpha;A).\end{aligned}$$

Similarly, using Proposition 3.21 we see that partial-isometric representations of α coincide with $\{0\}$-covariant representations of α (which are simply representations of α). Hence identifying the corresponding universal representations and applying Lemma 3.23 to (i_A,u^*) we get $A\rtimes_\alpha\mathbb{N} = C^*(i_A(A)\cup ui_A(A)) = C^*(i_A(A)\cup i_A(A)u) = C^*_{endo}(A,\alpha;\{0\})$. □

The crossed products $C^*_{endo}(A,\alpha;J)$ can be realized as relative Cuntz-Pimsner algebras associated to a certain C^*-correspondence E_α associated to α (this fact is extensively discussed in [30] in the case when A is unital). The C^*-correspondence in question was already considered by Pimsner [46] and it is defined by the formulas

$$E_\alpha := \alpha(A)A,\quad \langle x,y\rangle_A := x^*y,\quad a\cdot x\cdot b := \alpha(a)xb,\quad x,y\in\alpha(A)A,\ a,b\in A.$$

Pimsner's C^*-correspondence E_α and KSGNS C^*-correspondence X_α are naturally isomorphic.

Lemma 3.25. *If $\alpha : A \to A$ is an endomorphism, then the map $X_\alpha \ni a \otimes b \to \alpha(a)b \in E_\alpha$ determines an isomorphism of C^*-correspondences $X_\alpha \cong E_\alpha$. In particular, we have $J(X_\alpha) = J(E_\alpha) = A$.*

Proof. We leave it to the reader, as an easy exercise, to check that the prescribed map determines the desired isomorphism. For every $a \in A$ and $x \in E$ we can write $a = a_1a_2$, where $a_1, a_2 \in A$, and then $a \cdot x = \Theta_{\alpha(a_1),\alpha(a_2^*)}x$. Thus $J(E_\alpha) = A$. □

Now we are ready to state the main result of this subsection.

Proposition 3.26. *Let $\alpha : A \to A$ be an endomorphism and let J be an ideal in A. A representation (π, S) of (A, α) is J-covariant (in the sense of Definition 3.16) if and only if*

$$J \subseteq \{a \in A : SS^*\pi(a) = \pi(a)\}. \tag{25}$$

Thus the C^*-algebra $C^*(A, \alpha; J)$ is generated by $j_A(A) \cup j_A(A)s$ where (j_A, s) is a universal representation for the representations of (A, α) satisfying (25).

If α is extendible, then $s \in M(C^*(A, \alpha; J))$, $C^*(A, \alpha; J)$ is generated by $j_A(A)\cup sj_A(A)$, and the assignments $j_A(a) \to i_A(a)$, $sj_A(a) \to u^*i_A(a)$, $a \in A$, establish the isomorphisms

$$C^*(A, \alpha; J) \cong C^*_{endo}(A, \alpha; J), \qquad C^*(A, \alpha) \cong C^*_{endo}(A, \alpha),$$
$$C^*(A, \alpha; A) \cong A \rtimes^1_\alpha \mathbb{N}, \qquad C^*(A, \alpha; \{0\}) \cong A \rtimes_\alpha \mathbb{N}.$$

Proof. By the formula (24) and the second part of Lemma 3.25 we get that a representation (π, S) of (A, α) is J-covariant if and only if (25) holds. Then the universal picture of $C^*(A, \alpha; J)$ follows by Proposition 3.17.

Assume now that α is extendible. Then $s \in M(C^*(A, \alpha; J))$ by the last part of Proposition 3.10, and $C^*(A, \alpha; J) = C^*(j_A(A) \cup sj_A(A))$ by Lemma 3.23. Thus the universal descriptions immediately give us the isomorphism $C^*(A, \alpha; J) \cong C^*_{endo}(A, \alpha; J)$. Taking $J = (N_\alpha)^\perp = (\ker \alpha)^\perp$, we get $C^*(A, \alpha) \cong C^*_{endo}(A, \alpha)$. By Corollary 3.24 we get $C^*(A, \alpha; A) \cong A \rtimes^1_\alpha \mathbb{N}$ and $C^*(A, \alpha; \{0\}) \cong A \rtimes_\alpha \mathbb{N}$. □

3.5. **The case when A is commutative.** In this subsection, we assume that $A = C_0(D)$ is the algebra of continuous, vanishing at infinity functions on a locally compact Hausdorff space D. We let $\mathrm{Mes}(D)$ be the space of Radon positive measures on D and treat $\mathrm{Mes}(D)$ as the subset of the dual space A^* equipped with the w^*-topology. Let us start with a few simple observations.

Lemma 3.27. *We have a one-to-one correspondence given by the relation*

$$\varrho(a)(x) = \int_D a(y)d\mu_x(y), \qquad x \in D, a \in A,$$

between positive maps ϱ on A and continuous, uniformly bounded maps

$$D \ni x \overset{\mu}{\longmapsto} \mu_x \in \mathrm{Mes}(D) \tag{26}$$

that vanish at infinity in the w^*-sense, that is for every $a \in A$ and every $\varepsilon > 0$ the set $\{x : |\mu_x(a)| \geq \varepsilon\}$ is compact in D. Under this correspondence $\|\varrho\| = \sup_{x\in D} \|\mu_x\|$.

Proof. The assertion readily follows from Riesz theorem, cf., for instance, [5, Section 1]. In particular, using Lemma 2.1 we get $\|\varrho\| = \sup_{x\in D} \|\mu_x\|$. □

We denote by Closed(D) the set of all closed subsets of D. A mapping $\Phi : D \to \mathrm{Closed}(D)$ is *lower-semicontinuous* if for every open $V \subseteq D$ the set $\{x \in D : V \cap \Phi(x) \neq \emptyset\}$ is open. For any such mapping the set $\mathrm{Dom}(\Phi) := \{x \in D : \Phi(x) \neq \emptyset\}$ is open in D.

Lemma 3.28. *Any continuous mapping* (26) *induces a lower-semicontinuous mapping*

$$D \ni x \overset{\Phi}{\longmapsto} \mathit{supp}\, \mu_x \in \mathrm{Closed}(D). \tag{27}$$

Proof. Assume that $V \cap \Phi(x_0) \neq \emptyset$ where $x_0 \in D$ and $V \subseteq D$ is open. Then there is a positive function $a \in C_c(D)$ that vanishes outside V and such that $\mu_{x_0}(a) > 0$. Continuity of μ implies that $U := \{x \in D : \mu_x(a) > 0\}$ is open. Since $x_0 \in U \subseteq \{x \in D : V \cap \Phi(x) \neq \emptyset\}$, this proves the assertion. □

Let us fix a positive map ϱ and the corresponding maps μ and Φ given by (26) and (27). We associate to Φ the following relation on D:

$$R_\Phi := \bigcup_{x\in \mathrm{Dom}(\Phi)} \{x\} \times \Phi(x). \tag{28}$$

If R_Φ is closed in $D \times D$, then μ is called a *topological relation* in [5]. In this case, as we show below, μ can also be viewed as a topological quiver. In general, the relationship with topological quivers is subtle. We recall the relevant definition, see [39, Example 5.4] or [40, Definition 3.1]. We adopt the convention concerning the roles of the maps r, s as presented in [39] (it differs from the one in [40]). It fits to conventions we adopt in Section 5 for graph C^*-algebras. It is also consistent with the notation used for topological graphs by Katsura [24]. We stress that we use the term topological graph in a broader sense than [24, Definition 2.1]. Namely, we do not assume that the source map is a local homeomorphism. Also we do not assume that the topological spaces underlying a topological quiver are second countable, as it is done in [40, Definition 3.1].

Definition 3.29. A *topological graph* is a quadruple $E = (E^0, E^1, r, s)$ consisting of locally compact Hausdorff spaces E^0, E^1 and continuous maps $r, s : E^1 \to E^0$, where s is additionally assumed to be open. An *s-system* on the graph E is a family of Radon measures $\lambda = \{\lambda_v\}_{v\in E^0}$ on E^1 such that

(Q1) $\operatorname{supp} \lambda_v = s^{-1}(v)$ for all $v \in E^0$,
(Q2) $v \to \int_{E^1} a(e) d\lambda_v(e)$ is an element of $C_c(E^0)$ for all $a \in C_c(E^1)$.

The quintuple $(E^0, E^1, r, s, \lambda)$ where $E := (E^0, E^1, r, s)$ is a topological graph and λ is an s-system on E is called a *topological quiver*.

The following lemma and proposition should be compared with [19, Proposition 2.2] stated (essentially without a proof) in the context of Markov operators, see discussion below.

Lemma 3.30. *Consider the quintuple*

$$\mathcal{Q}_\varrho := (D, \overline{R_\Phi}, r, s, \lambda)$$

where R_Φ is given by (28), $s(x, y) := x$, $r(x, y) := y$ and λ_x is a measure supported on $\{x\} \times \Phi(x)$ given by $\lambda_x(\{x\} \times U) = \mu_x(U)$. Then $\mathcal{Q}_\varrho$ satisfies all axioms of topological quiver, except that openness of the source map and axiom (Q1) hold for the restriction $s|_{R_\Phi}$ to R_Φ, rather than for $s : \overline{R_\Phi} \to D$ itself.

Proof. Openness of $s : R_\Phi \to D$ is equivalent to lower-semicontinuity of Φ and thus follows from Lemma 3.28. To show the axiom (Q2) define a map $\Psi : C_c(D) \odot C_c(D) \to C_0(D)$ by the formula

$$\Psi(\sum_i a_i \odot b_i)(x) := \int \sum_i a_i \odot b_i \, d\lambda_x = \sum_i a_i(x) \int b_i \, d\mu_x = \left(\sum_i a_i \varrho(b_i) \right)(x).$$

It is well defined because $\sum_i a_i \varrho(b_i) \in C_0(D)$ and it is linear because it is given by the integral. It is bounded with $\|\Psi\| \le \sup_{x\in D} \|\lambda_x\| = \sup_{x\in D} \|\mu_x\| = \|\varrho\|$. Thus, since $C_c(D) \odot C_c(D)$ is uniformly dense in $C_0(D \times D)$ we deduce that the formula $\Psi(a)(x) = \int a d\lambda_x$ defines a bounded linear map $\Psi : C_0(D\times D) \to C_0(D)$. Concluding, for any $a \in C_c(\overline{R_\Phi})$ we see that $x \to \int_{E^1} a(x, y) d\lambda_x(y)$ defines a continuous function on D which vanishes outside the compact set $s(\operatorname{supp}(a))$. This proves (Q2). The rest is clear by construction. $\square$

Remark 3.31. By the above lemma the quintuple $(D, R_\Phi, r, s, \lambda)$ is a topological quiver whenever R_Φ is locally compact. However, if R_Φ is not closed in $D \times D$ then the mapping (29) below, with R_Φ in place of $\overline{R_\Phi}$, is not well defined.

Proposition 3.32. *The quintuple* $\mathcal{Q}_\varrho = (D, \overline{R_\Phi}, r, s, \lambda)$ *from Lemma 3.30 gives rise to a* C^**-correspondence* $X_{\mathcal{Q}_\varrho}$ *which is the Hausdorff completion of the semi-inner* C^**-correspondence defined on* $C_c(\overline{R_\Phi})$ *via*

$$(a \cdot f \cdot b)(x, y) = \big((a \circ s) f (b \circ r)\big)(x, y) = a(x) f(x, y) b(y),$$

and

$$\langle f, g\rangle_{\mathcal{Q}_\varrho}(x) = \int_{R_\Phi} \overline{f} g d\lambda_x = \int_{\Phi(x)} \overline{f(x, y)} g(x, y) d\mu_x(y),$$

$f, g \in C_c(\overline{R_\Phi})$, $a, b \in C_0(D) = C_0(D)$. *Moreover, putting* $W(a \odot b)(x, y) := a(x)b(y)$, *for* $(x, y) \in \overline{R_\Phi}$ *the mapping determined by*

$$C_c(D) \odot C_c(D) \ni a \odot b \to W(a \odot b) \in C_c(\overline{R_\Phi}), \tag{29}$$

factors through and extends to an isomorphism of C^**-correspondences* $X_\varrho \cong X_{\mathcal{Q}_\varrho}$.

Proof. It is a routine exercise to check that $C_c(\overline{R_\Phi})$ is a semi-inner product (right) A-module, equipped with a left A-module action by adjointable operators, [31, p. 3]. Thus we get the C^*-correspondence $X_{\mathcal{Q}_\varrho}$. Furthermore, one readily checks that (29) determines a well-defined map $W : C_c(D) \odot C_c(D) \to C_c(\overline{R_\Phi})$ satisfying

$$aW(f \odot g)b = W(af \odot gb), \qquad \langle W(f \odot g), W(f \odot g)\rangle_{\mathcal{Q}} = \langle f \odot g, f \odot g\rangle_\varrho,$$

for all $a, b \in C_0(D), f, g \in C_c(D)$. Since, the image of $C_c(D) \odot C_c(D)$ is dense in X_ϱ it follows that W factors through and extends to an isometric homomorphism of C^*-correspondences $\mathcal{W} : X_\varrho \to X_{\mathcal{Q}_\varrho}$. To see it is surjective, note that by the Stone-Weierstrass theorem for any $f \in C_c(\overline{R_\Phi})$ we can find a sequence $\{f_n\} \subseteq C_c(D) \odot C_c(D)$ such that $W(f_n)$ converges uniformly to f, and thus all the more in the semi-norm induced by $\langle \cdot, \cdot\rangle_{\mathcal{Q}}$. Hence $\mathcal{W}$ is the desired isomorphism. □

By [40, Definition 3.17], the *C^*-algebra associated to a topological quiver $\mathcal{Q}$* is the Cuntz-Pimsner algebra $\mathcal{O}_{X_\mathcal{Q}}$ of a certain C^*-correspondence $X_\mathcal{Q}$. If the quintuple $\mathcal{Q}_\varrho = (D, \overline{R_\Phi}, r, s, \lambda)$ defined in Lemma 3.30 is a topological quiver, then the C^*-correspondence $X_{\mathcal{Q}_\varrho}$ constructed in Proposition 3.32 coincides with the C^*-correspondence associated to $\mathcal{Q}_\varrho$ in [40]. Thus Proposition 3.32 and Theorem 3.13 give the following proposition.

Proposition 3.33. *Suppose that the quintuple* $\mathcal{Q}_\varrho = (D, \overline{R_\Phi}, r, s, \lambda)$ *defined in Lemma 3.30 is a topological quiver (it is automatic when* R_Φ *is closed). Then the crossed product* $C^*(A, \varrho)$ *is naturally isomorphic to the* C^**-algebra* $C^*(\mathcal{Q}_\varrho)$ *associated to* $\mathcal{Q}_\varrho$.

If the set R_Φ is closed, that is, if μ is a topological relation, then the C^*-correspondence $X_{\mathcal{Q}_\varrho}$ constructed in Proposition 3.32 coincides with the one associated to μ in [5]. In particular, Brenken defines a *C^*-algebra $\mathcal{C}(\mu)$ associated to the topological relation μ* as Pimsner's (augmented) C^*-algebra $\mathcal{O}(J(X_{\mathcal{Q}_\varrho}), X_{\mathcal{Q}_\varrho})$. Hence by Proposition 3.32 and Theorem 3.13 we have the following proposition.

Proposition 3.34. *If $\varrho : C_0(D) \to C_0(D)$ is such that the map μ given by (26) is a topological relation then the C^*-algebra $\mathcal{C}(\mu)$ associated to μ is naturally isomorphic to the relative crossed product $C^*(A, \varrho; A)$.*

Suppose now that D is compact and ϱ is unital (equivalently, every measure μ_x, $x \in D$, is a probability distribution). Then ϱ is called a *Markov operator* in [19, Definition 1]. The closure $\overline{R_\Phi}$ of the set (28) coincides with the support of ϱ defined in [19, Definition 4]. It seems that the authors of [19] tacitly assumed that the corresponding set R_Φ is closed, since they model their algebras via C^*-algebras associated to topological quivers. Namely, they define [19, Definition 7] the C^*-algebra $C^*(\varrho)$ of the Markov operator ϱ to be the C^*-algebra of the quintuple $(D, \overline{R_\Phi}, r, s, \lambda)$ described in Lemma 3.30. But in general neither $(D, R_\Phi, r, s, \lambda)$ nor $(D, \overline{R_\Phi}, r, s, \lambda)$, satisfies all of the axioms of a topological quiver.

Example 3.35. Consider the following Markov operators ϱ_1, ϱ_2 on $C([0,1])$:

$$\varrho_1(a)(x) = \begin{cases} \int_0^1 a(t) f_x(t)\, dt & x \neq 0 \\ a(1), & x = 0 \end{cases}, \qquad f_x(t) = \frac{t^{1/x}}{\int_0^1 s^{1/x}\, ds};$$

$$\varrho_2(a)(x) = f(x)a(0) + f(1-x)a(1), \;\; f(x) = \chi_{[0,1/3)}(x) + (2-3x)\chi_{[1/3,2/3)}(x).$$

Then the corresponding relations on $[0,1]$ are

$$R_1 = \Big((0,1] \times [0,1]\Big) \cup \{(0,1)\}, \qquad R_2 = \Big([0,2/3) \times \{0\}\Big) \cup \Big((1/3,1] \times \{1\}\Big).$$

Plainly, R_1 is not locally compact in $(0,1)$, while the source map on $\overline{R_2}$ is not open.

The above example shows that the assertion in [19, Proposition 2.2] is false. Proposition 3.32 could be considered a correct version of this statement. It suggests that in general the C^*-algebra $C^*(\varrho)$ should be defined as the Cuntz-Pimsner algebra $\mathcal{O}_{X_{\mathcal{Q}_\varrho}}$ of the C^*-correspondence $X_{\mathcal{Q}_\varrho}$ described in Proposition 3.32. Then Theorem 3.13 gives us the following proposition.

Proposition 3.36. *The C^*-algebra $C^*(\varrho)$ of a Markov operator $\varrho : C(D) \to C(D)$ is naturally isomorphic to the crossed product $C^*(A, \varrho)$.*

4. A new look at Exel systems and their crossed products

Throughout this section, $(A, \alpha, \mathcal{L})$ denotes an Exel system. We show that Exel-Royer's crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$ is the crossed product $C^*(A, \mathcal{L})$ and Exel's crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is the relative crossed product $C^*(A, \mathcal{L}; \overline{A\alpha(A)A})$. We study in detail the structure of Exel systems with the property that $\alpha \circ \mathcal{L}$ is a conditional expectation, and discuss cases when we have $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$.

4.1. Exel's crossed products as crossed products by transfer operators.

Let us start with the following simple but fundamental observation.

Lemma 4.1. *Any transfer operator is a completely positive map.*

Proof. Using (8) and its symmetrized version: $\mathcal{L}(\alpha(b)a) = b\mathcal{L}(a)$, $a, b \in A$, for any a_i, $b_i \in A$, $i = 1, ..., n$, we get

$$\sum_{i,j=1}^{n} b_i^* \mathcal{L}(a_i^* a_j) b_j = \mathcal{L}\Big(\big(\sum_{i=1}^{n} a_i \alpha(b_i) \big)^* \big(\sum_{j=1}^{n} a_j \alpha(b_j) \big) \Big) \geq 0. \tag{30}$$

Hence $\mathcal{L}$ is a completely positive map. □

Authors of [10] and [35] considered Exel systems $(A, \alpha, \mathcal{L})$ under the additional assumption that both α and $\mathcal{L}$ are extendible. It turns out that extendibility of $\mathcal{L}$ is automatic.

Proposition 4.2. *Any transfer operator $\mathcal{L}$ for α is extendible. Its strictly continuous extension $\overline{\mathcal{L}} : M(A) \to M(A)$ is determined by the formula*

$$\overline{\mathcal{L}}(m)a = \mathcal{L}(m\alpha(a)), \qquad a \in A, \ m \in M(A). \tag{31}$$

In particular, $\overline{\mathcal{L}}(1)$ is a positive central element in $M(A)$, and if α is extendible then the triple $(M(A), \overline{\alpha}, \overline{\mathcal{L}})$ is an Exel system.

Proof. Fix $m \in M(A)$. We claim that (31) defines a multiplier, that is an adjointable mapping $\overline{\mathcal{L}}(m) : A \to A$ where we view A as the standard Hilbert A-module. Indeed, for any $a, b \in A$ we have

$$(\overline{\mathcal{L}}(m)a)^* b = \mathcal{L}(m\alpha(a))^* b = \mathcal{L}(\alpha(a^*)m^*)b = \mathcal{L}(\alpha(a^*)m\alpha(b)) = a^*(\overline{\mathcal{L}}(m^*)b).$$

Hence $\overline{\mathcal{L}}(m) \in M(A)$ and $\overline{\mathcal{L}}(m)^* = \overline{\mathcal{L}}(m^*)$. Accordingly, (31) defines a $*$-preserving mapping $\overline{\mathcal{L}} : M(A) \to M(A)$. It follows directly from (31) that $\overline{\mathcal{L}}$ is a strictly continuous extension of $\mathcal{L} : A \to A$. Moreover, for $a \in A$, we have $\overline{\mathcal{L}}(1)a = \mathcal{L}(1\alpha(a)) = \mathcal{L}(\alpha(a)1) = a\overline{\mathcal{L}}(1)$. Thus $\overline{\mathcal{L}}(1)$ belongs to the commutant of A in $M(A)$. This commutant coincides with the center of $M(A)$. If additionally

α is extendible then $\overline{\mathcal{L}}$ is a transfer operator for $\overline{\alpha}$ because (8) is preserved when passing to strict limits. $\square$

Another somehow unexpected fact is that the first relation in (9) is superfluous.

Proposition 4.3. *Suppose that $(A, \alpha, \mathcal{L})$ is an Exel system. For any representation (π, S) of $(A, \mathcal{L})$ we automatically have*

$$S\pi(a) = \pi(\alpha(a))S, \quad a \in A.$$

Thus the classes of representations of $(A, \mathcal{L})$ and $(A, \alpha, \mathcal{L})$ coincide and

$$\mathcal{T}(A, \mathcal{L}) = \mathcal{T}(A, \alpha, \mathcal{L}).$$

Moreover, the notions of redundancy for (π, S) as a representation of $(A, \mathcal{L})$ and as a representation of $(A, \alpha, \mathcal{L})$ coincide.

Proof. Let (π, S) be a representation of $(A, \mathcal{L})$, $\overline{\pi} : M(A) \to B(H)$ the extension of π, and $\overline{\mathcal{L}}$ the strictly continuous extension of $\mathcal{L}$, which exists by Proposition 4.2. It readily follows that $(\overline{\pi}, S)$ is a representation of $(M(A), \overline{\mathcal{L}})$. In particular, $S^*S = \overline{\pi}(\overline{\mathcal{L}}(1))$. Using this and (31), one sees that each of the expressions

$$S^*\pi(\alpha(a^*))S\pi(a), \quad S^*\pi(\alpha(a^*))\pi(\alpha(a))S, \quad \pi(a^*)S^*S\pi(a), \quad \pi(a^*)S^*\pi(\alpha(a))S$$

is equal to $\pi\big(\mathcal{L}(\alpha(a^*a))\big)$, for any $a \in A$. Hence we get

$$\|S\pi(a) - \pi(\alpha(a))S\|^2 = \|\big(S^*\pi(\alpha(a^*)) - \pi(a^*)S^*\big)\big(S\pi(a) - \pi(\alpha(a))S\big)\| = 0.$$

This finishes the proof of the first part of the assertion. For the second part it suffices to show that $\overline{\pi(A)S\pi(A)S^*\pi(A)} = \overline{\pi(A)SS^*\pi(A)}$. In view of what we have just shown we have

$$\overline{\pi(A)S\pi(A)S^*\pi(A)} = \overline{\pi(A)\pi(\alpha(A))SS^*\pi(A)} \subseteq \overline{\pi(A)SS^*\pi(A)}.$$

Moreover, the last inclusion is the equality because $\pi(a)S = \lim_{\lambda\in\Lambda} \pi(a\alpha(\mu_\lambda))S$ for any approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A. Indeed,

$$\|\pi(a)S - \pi(a\alpha(\mu_\lambda))S\|^2 = \|\pi\big(\mathcal{L}(a^*a) - \mathcal{L}(a^*a)\mu_\lambda - \mu_\lambda\mathcal{L}(a^*a) + \mu_\lambda\mathcal{L}(a^*a)\mu_\lambda\big)\|$$

clearly tends to 0. $\square$

The above coincidence can be explained at the level of C^*-correspondences.

Lemma 4.4. *The C^*-correspondence $M_{\mathcal{L}}$ associated to $(A, \alpha, \mathcal{L})$ and the C^*-correspondence $X_{\mathcal{L}}$ associated to $(A, \mathcal{L})$ are isomorphic, via the mapping determined by $a \otimes b \longmapsto q(a\alpha(b))$, $a, b \in A$.*

Proof. That $a \otimes b \longmapsto q(a\alpha(b))$ yields a well defined isometry follows from the equality in (30). Clearly, it is a C^*-correspondence map. It is onto because $q(a\alpha(\mu_\lambda))$ converges in $M_{\mathcal{L}}$ to $q(a)$ for any $a \in A$ and any approximate unit $\{\mu_\lambda\}$ in A, cf. [25, Lemma 3.6]. $\square$

Corollary 4.5. *For every Exel system* $(A, \alpha, \mathcal{L})$ *we have*

$$\mathcal{T}(M_{\mathcal{L}}) \cong \mathcal{T}(X_{\mathcal{L}}) \cong \mathcal{T}(A, \mathcal{L}) = \mathcal{T}(A, \alpha, \mathcal{L}).$$

Proof. We have $\mathcal{T}(M_{\mathcal{L}}) \cong \mathcal{T}(X_{\mathcal{L}})$ by Lemma 4.4. Proposition 3.10 implies that $\mathcal{T}(X_{\mathcal{L}}) \cong \mathcal{T}(A, \mathcal{L})$, and we have $\mathcal{T}(A, \mathcal{L}) = \mathcal{T}(A, \alpha, \mathcal{L})$ by Proposition 4.3. $\square$

Remark 4.6. The isomorphism $\mathcal{T}(M_{\mathcal{L}}) \cong \mathcal{T}(A, \alpha, \mathcal{L})$ was proved in [9, Corollary 3.3], cf. [25, Theorem 3.7], for A unital, and independently in [10, Proposition 3.1] and [35, Proposition 4.1], for extendible Exel systems.

Now we are in a position to show the main result of this subsection.

Theorem 4.7. *For any Exel system* $(A, \alpha, \mathcal{L})$ *we have*

$$C^*(A, \mathcal{L}) = \mathcal{O}(A, \alpha, \mathcal{L}) \cong \mathcal{O}_{M_{\mathcal{L}}}, \qquad A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; J),$$

where $J := \overline{A\alpha(A)A}$. *In particular, we have*

$$A \times_{\alpha, \mathcal{L}} \mathbb{N} \cong \mathcal{O}(X_{\mathcal{L}}, J \cap J(X_{\mathcal{L}})) \cong \mathcal{O}(M_{\mathcal{L}}, J \cap J(M_{\mathcal{L}})),$$

and the universal homomorphism $j_A : A \to A \times_{\alpha, \mathcal{L}} \mathbb{N}$ *is injective if and only if* $\mathcal{L}$ *is almost faithful on* $\overline{A\alpha(A)A} \cap J(M_{\mathcal{L}})$.

Proof. Equality $A \times_{\alpha, \mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; J)$ follows from Proposition 4.3. To get $\mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \mathcal{L})$ combine Proposition 4.3, Lemma 4.4 and the first part of Theorem 3.13. The isomorphism $C^*(A, \mathcal{L}) \cong \mathcal{O}_{M_{\mathcal{L}}}$ follows now either from Remark 2.28 or from Lemma 4.4 and Theorem 3.13. The second part of the assertion follows now from Lemma 4.4 and Theorem 3.13. $\square$

Remark 4.8. The second part of the assertion in the above theorem generalizes [9, Proposition 3.10 and Theorem 4.2] proved in the unital case, and [10, Theorems 4.1 and 4.3] where authors assumed extendibility of the Exel system.

Brownlowe, Raeburn and Vitadello proved in [10, Corollary 4.2] that Exel's crossed products for Exel systems $(C_0(T), \alpha, \mathcal{L})$ induced by classical dynamical systems (T, τ) are naturally isomorphic to $\mathcal{O}_{M_{\mathcal{L}}}$. For these systems $\mathcal{L}$ is faithful and α is extendible. It turns our that the latter properties suffice to get the corresponding isomorphism.

Proposition 4.9. *Let $(A, \alpha, \mathcal{L})$ be an Exel system and suppose that one of the following conditions hold:*

i) *$\mathcal{L}$ is almost faithful on A and α is a non-degenerate homomorphism.*
ii) *$\mathcal{L}$ is faithful and α is extendible.*

Then $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}) \cong \mathcal{O}_{M_{\mathcal{L}}}$.

PROOF. i). The assumptions imply that $N_{\mathcal{L}}^{\perp} = A$ and $\overline{A\alpha(A)A} = A$. By Theorem 4.7, we get $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; A) = C^*(A, \mathcal{L}; N_{\mathcal{L}}^{\perp}) = C^*(A, \mathcal{L}) \cong \mathcal{O}_{M_{\mathcal{L}}}$.

ii). By item i) it suffices to show that α is non-degenerate. For any $a \in A$ the element $\mathcal{L}((a - a\overline{\alpha}(1))^*(a - a\overline{\alpha}(1)))$ is equal to

$$\mathcal{L}(a^*a) - \mathcal{L}(\overline{\alpha}(1)a^*a) - \mathcal{L}(a^*a\overline{\alpha}(1)) + \mathcal{L}(\overline{\alpha}(1)a^*a\overline{\alpha}(1)) = 0.$$

Thus by faithfulness of $\mathcal{L}$ we have $a = \overline{\alpha}(1)a$. This implies that $A = \overline{\alpha}(1)A = \alpha(A)A$. □

Remark 4.10. Kakariadis and Peters [25, Remark 3.19] raised a question whether Exel's crossed product $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ is always isomorphic to the Cuntz-Pimsner algebra $\mathcal{O}_{M_{\mathcal{L}}}$ (by Theorem 4.7, the latter is always isomorphic to $C^*(A, \mathcal{L})$). The answer to this question, stated as it is, is no. The reason is that A always embeds into $\mathcal{O}_{M_{\mathcal{L}}} \cong C^*(A, \mathcal{L})$ while for Exel's crossed product in general this fails, see for instance [9, Example 4.7]. Thus, taking into account Theorem 4.7, we propose the following modified version of this question:

> *Let $(A, \alpha, \mathcal{L})$ be an Exel system such that $\overline{A\alpha(A)A} \cap J(M_{\mathcal{L}}) \subseteq N_{\mathcal{L}}^{\perp}$. Do the crossed products $C^*(A, \mathcal{L})$ and $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ coincide?*

Since $C^*(A, \mathcal{L}; J) = C^*(A, \mathcal{L}; J \cap J(M_{\mathcal{L}}))$, the answer to the above question, for systems under consideration, is positive if and only if

$$N_{\mathcal{L}}^{\perp} \cap J(M_{\mathcal{L}}) \subseteq \overline{A\alpha(A)A}. \tag{32}$$

The most problematic part in establishing (32) is determining $J(M_{\mathcal{L}})$. For instance, when α is extendible and $\mathcal{L}$ is faithful then $\overline{A\alpha(A)A} = A = N_{\mathcal{L}}^{\perp}$ and hence (32) holds independently of $J(M_{\mathcal{L}})$. Interestingly enough, if additionally $E = \mathcal{L} \circ \alpha : A \to \alpha(A) \subseteq A$ is a conditional expectation of finite-type then $J(M_{\mathcal{L}}) = A$, see [15]. However, in general we have $\overline{A\alpha(A)A} \neq N_{\mathcal{L}}^{\perp}$ and $J(M_{\mathcal{L}}) \cap N_{\mathcal{L}}^{\perp} \neq N_{\mathcal{L}}^{\perp}$. This may happen already when $\mathcal{L}$ is faithful but α is not extendible, cf. Lemmas 5.8 and 5.9. Surprisingly, when $\alpha(A)$ is a corner in A, see Theorem 4.22 below, and also for all Exel systems considered in Section 5 we have $J(M_{\mathcal{L}}) \cap N_{\mathcal{L}}^{\perp} = \overline{A\alpha(A)A}$.

4.2. **Regular transfer operators.** Most of natural Exel systems appearing in applications, see [12], [15], [9], [10], [26], have the property that $\alpha \circ \mathcal{L}$ is a conditional expectation onto $\alpha(A)$. In [12] Exel called transfer operators with that property *non-degenerate*. However, we have reasons to change this name. Firstly, the term 'non-degenerate' when referred to a positive operator is sometimes used to mean a faithful map [13, page 60] or a strict map [49, subsection 3.3]. Secondly, there are historical reasons. Namely, transfer operators on unital commutative C^*-algebras, under the name averaging operators, were studied at least from 1950's see [45], cf. [26]. The averaging operators are called *regular* exactly when the corresponding composition $\alpha \circ \mathcal{L}$ is a conditional expectation, cf. [26, Proposition 2.1.i)]. Therefore we adopt the following definition.

Definition 4.11. Let $(A, \alpha, \mathcal{L})$ be an Exel system. We say that both the transfer operator $\mathcal{L}$ and the Exel system $(A, \alpha, \mathcal{L})$ are *regular*, if $E := \alpha \circ \mathcal{L}$ is a conditional expectation onto $\alpha(A)$.

Let us start with a simple fact.

Lemma 4.12. *Let $(A, \alpha, \mathcal{L})$ be an Exel system. The range $\mathcal{L}(A)$ of the transfer operator $\mathcal{L}$ is a self-adjoint two-sided (not necessarily closed) ideal in A such that* $\ker \alpha \subseteq \mathcal{L}(A)^{\perp}$.

Proof. Since $\mathcal{L}$ is linear and $*$-preserving, $\mathcal{L}(A)$ is a self-adjoint linear space. The space $\mathcal{L}(A)$ is a two-sided ideal in A by (8). For $a \in \ker \alpha$ we have $a\mathcal{L}(A) = \mathcal{L}(\alpha(a)A) = \mathcal{L}(0) = 0$. Hence $\ker \alpha \subseteq \mathcal{L}(A)^{\perp}$. $\square$

Suppose that the central positive element $\overline{\mathcal{L}}(1) \in M(A)$, described in Proposition 4.2, is a projection. Then by the above lemma the multiplier $\overline{\mathcal{L}}(1)$ projects A onto an ideal contained in $(\ker \alpha)^{\perp}$. It turns out that $\mathcal{L}$ is regular exactly when $\overline{\mathcal{L}}(1)$ projects A onto $(\ker \alpha)^{\perp}$.

Proposition 4.13. *Let $(A, \alpha, \mathcal{L})$ be an Exel system and let $\{\mu_\lambda\}$ be an approximate unit in A. The following conditions are equivalent:*

i) *$\mathcal{L}$ is regular, that is $E = \alpha \circ \mathcal{L} : A \to \alpha(A)$ is a conditional expectation,*
ii) *$\{\alpha(\mathcal{L}(\mu_\lambda))\}$ is an approximate unit in $\alpha(A)$,*
iii) *$\alpha \circ \mathcal{L} \circ \alpha = \alpha$,*
iv) *$\{\mathcal{L}(\mu_\lambda)\}$ converges strictly to a projection $\overline{\mathcal{L}}(1) \in M(A)$ onto* $(\ker \alpha)^{\perp}$,
v) *$(\alpha, \mathcal{L})$ is an interaction in the sense of* [13, Definition 3.1]*, see Remark 3.15.*

In particular, if the above equivalent conditions hold, then $\ker\alpha$ *is a complemented ideal in* A*,* $\overline{\mathcal{L}}(1)A = \mathcal{L}(A) = (\ker\alpha)^{\perp}$*,* $(1-\overline{\mathcal{L}}(1))A = \ker\alpha$*, and*

$$\mathcal{L}(a) = \alpha^{-1}(E(a)), \qquad \alpha(a) = \mathcal{L}^{-1}(\overline{\mathcal{L}}(1)a), \qquad a \in A, \tag{33}$$

where α^{-1} *is the inverse to the isomorphism* $\alpha : (\ker\alpha)^{\perp} \to \alpha(A)$*, and* $\mathcal{L}^{-1}$ *is the inverse to the isomorphism* $\mathcal{L} : \alpha(A) \to \mathcal{L}(A)$*.*

Proof. i)$\Rightarrow$ii). For any $a \in \alpha(A)$ we have $\lim_{\lambda\in\Lambda} \alpha(\mathcal{L}(\mu_\lambda))a = \lim_{\lambda\in\Lambda} E(\mu_\lambda a) = E(a) = a$.

ii)$\Rightarrow$iii). For any $a \in A$ we have

$$\alpha(a) = \lim_{\lambda\in\Lambda} \alpha(\mathcal{L}(\mu_\lambda))\alpha(a) = \lim_{\lambda\in\Lambda} \alpha(\mathcal{L}(\mu_\lambda)a) = \lim_{\lambda\in\Lambda} \alpha(\mathcal{L}(\mu_\lambda\alpha(a))) = \alpha(\mathcal{L}(\alpha(a))).$$

iii)$\Rightarrow$iv). By Proposition 4.2, $\{\mathcal{L}(\mu_\lambda)\}$ converges strictly to a central element $\overline{\mathcal{L}}(1)$ in $M(A)$. In particular, using (31), for $a \in A$, we get

$$\overline{\mathcal{L}}(1)^2 a = \overline{\mathcal{L}}(1)\mathcal{L}(\alpha(a)) = \mathcal{L}(\alpha(\mathcal{L}(\alpha(a)))) = \mathcal{L}(\alpha(a)) = \overline{\mathcal{L}}(1)a.$$

Hence $\overline{\mathcal{L}}(1)$ is a projection. On one hand $(1-\overline{\mathcal{L}}(1))A \subseteq \ker\alpha$ because

$$\alpha((1-\overline{\mathcal{L}}(1))a) = \alpha(a) - \alpha(\mathcal{L}(\alpha(a))) = \alpha(a) - \alpha(a) = 0,$$

for any $a \in A$. On the other hand, $\ker\alpha \subseteq (1-\overline{\mathcal{L}}(1))A$ because if $a \in \ker\alpha$ then

$$(1-\overline{\mathcal{L}}(1))a = a - \mathcal{L}(\alpha(a)) = a.$$

Accordingly, $\ker\alpha = (1-\overline{\mathcal{L}}(1))A$ and $(\ker\alpha)^{\perp} = \overline{\mathcal{L}}(1)A$.

iv)$\Rightarrow$v). If $\overline{\mathcal{L}}(1)$ is a projection onto $(\ker\alpha)^{\perp}$ then in view of (31) for $a \in A$ we have

$$\alpha(a) = \alpha(\overline{\mathcal{L}}(1)a) = \alpha(\mathcal{L}(\alpha(a))),$$

that is $\alpha = \alpha \circ \mathcal{L} \circ \alpha$. By Lemma 4.12, $\ker\alpha \subseteq \mathcal{L}(A)^{\perp}$. This implies that $\mathcal{L}(A) \subseteq (\mathcal{L}(A)^{\perp})^{\perp} \subseteq (\ker\alpha)^{\perp} = \overline{\mathcal{L}}(1)A$. Consequently,

$$\mathcal{L}(a) = \overline{\mathcal{L}}(1)\mathcal{L}(a) = \mathcal{L}(\alpha(\mathcal{L}(a))),$$

that is $\mathcal{L} = \mathcal{L} \circ \alpha \circ \mathcal{L}$. We note that as $\overline{\mathcal{L}}(1)A = \mathcal{L}(\alpha(A)) \subseteq \mathcal{L}(A)$ we actually get $\mathcal{L}(A) = \overline{\mathcal{L}}(1)A = (\ker\alpha)^{\perp}$. Clearly, $\mathcal{L}(A) \subseteq MD(\alpha) = A$. We have $\alpha(A) \subseteq MD(\mathcal{L})$ because

$$\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b) = \overline{\mathcal{L}}(1)a\mathcal{L}(b) = \mathcal{L}(\alpha(a))\mathcal{L}(b),$$

and similarly $\mathcal{L}(b\alpha(a)) = \mathcal{L}(b)\mathcal{L}(\alpha(a))$, $a, b \in A$.

v)$\Rightarrow$i). $E = \alpha \circ \mathcal{L}$ is a conditional expectation by [13, Corollary 3.3]. $\square$

Remark 4.14. If A is unital the equivalence of conditions i), ii), iii) above (with units in place of approximate units) was proved by Exel [12, Proposition 2.3], and in [26, Proposition 1.5] it was noticed that they imply that $\mathcal{L}(1)$ is a central projection with $\mathcal{L}(1)A = \mathcal{L}(A) = (\ker\alpha)^{\perp}$.

The following classification of regular transfer operators generalizes [26, Theorem 1.6] to the non-unital case.

Proposition 4.15. *Fix an endomorphism $\alpha : A \to A$. If α admits a regular transfer operator then its kernel is a complemented ideal in A and the formulas*

$$\mathcal{L} = \alpha^{-1} \circ E, \qquad E = \alpha \circ \mathcal{L}$$

where α^{-1} is the inverse to $\alpha : (\ker\alpha)^{\perp} \to \alpha(A)$, establish a one-to-one correspondence between conditional expectations E from A onto $\alpha(A)$ and regular transfer operators $\mathcal{L}$ for α.

In particular, if the range of α is a hereditary subalgebra of A, then α admits at most one regular transfer operator.

Proof. The first part follows immediately from Proposition 4.13. For the second part notice that every conditional expectation $E : A \to B \subseteq A$ is determined by its restriction to the hereditary C^*-subalgebra BAB of A generated by B. $\square$

Now, we reverse the situation and parametrize all regular Exel systems for a fixed transfer operator. To this end, we recall, cf. [49, subsection 1.3], that a completely positive contraction $\varrho : A \to B$ is called a *retraction* if there exists a homomorphism $\theta : B \to A$ such that $\varrho \circ \theta = id_B$; then θ is called a *section* of ϱ.

Proposition 4.16. *A completely positive mapping $\mathcal{L} : A \to A$ is a regular transfer operator for a certain endomorphism if and only if $\mathcal{L}(A)$ is a complemented ideal in A and $\mathcal{L} : A \to \mathcal{L}(A)$ is a retraction.*

If the above conditions hold, we have bijective correspondences between the following objects:

- i) *endomorphisms $\alpha : A \to A$ making $(A, \alpha, \mathcal{L})$ into a regular Exel system,*
- ii) *sections $\theta : \mathcal{L}(A) \to A$ of $\mathcal{L} : A \to \mathcal{L}(A)$,*
- iii) *C^*-subalgebras $B \subseteq MD(\mathcal{L})$ such that $\mathcal{L} : B \to \mathcal{L}(A)$ is a bijection.*

These correspondences are given by the relations

$$\alpha(a) = \theta(\overline{\mathcal{L}}(1)a), \qquad a \in A, \qquad B = \alpha(A) = \theta(\mathcal{L}(A)), \tag{34}$$

where $\overline{\mathcal{L}}(1)$ is the projection onto $\mathcal{L}(A)$ and $\theta : \mathcal{L}(A) \to B$ is the inverse to $\mathcal{L} : B \to \mathcal{L}(A)$.

Proof. By virtue of Proposition 4.13, if $\mathcal{L}$ is a regular transfer operator for α then $\mathcal{L}(A)$ is a complemented ideal in A and α is given by the first formula in (34). Conversely, if $\mathcal{L}(A)$ is a complemented ideal in A, then the projection $\overline{\mathcal{L}}(1) \in M(A)$ onto $\mathcal{L}(A)$ commutes with elements of A. Hence for any section $\theta : \mathcal{L}(A) \to A$, the first formula in (34) defines a homomorphism such that $(A, \alpha, \mathcal{L})$ is a regular Exel system. Thus we have a bijection between objects in items i) and ii). If θ is a section of $\mathcal{L} : A \to \mathcal{L}(A)$ and α is the corresponding endomorphism then $B := \alpha(A) = \theta(\mathcal{L}(A))$ is a C^*-subalgebra of $MD(\mathcal{L})$ by Proposition 4.13v). By the same proposition $\mathcal{L} : B \to \mathcal{L}(A)$ is a bijection. Conversely, if B is a C^*-algebra such that $B \subseteq MD(\mathcal{L})$ and $\mathcal{L} : B \to \mathcal{L}(A)$ is a bijection, then the inverse to $\mathcal{L} : B \to \mathcal{L}(A)$ is a section of $\mathcal{L} : A \to \mathcal{L}(A)$. This shows the correspondence between objects in ii) and iii). $\square$

Example 4.17 (cf. Example 4.7 in [9]). Let $\mathcal{L} : C([0,2]) \to C([0,2])$ be given by $\mathcal{L}(a)(x) = a(x/2)$. Regular Exel systems $(C([0,2]), \alpha, \mathcal{L})$ are parametrized by continuous extensions of the mapping $[0,1] \ni x \to 2x \in [0,2]$; for any such system we have

$$\alpha(a)(x) = \begin{cases} a(2x), & \text{if } x \in [0,1] \\ a(\gamma(x)), & \text{if } x \in [1,2] \end{cases}, \qquad a \in C([0,2]),$$

where $\gamma : [1,2] \to [0,2]$ is continuous and $\gamma(1) = 2$. In other words, the algebras B in Proposition 4.16 correspond to continuous mappings on $[0,2]$ whose restriction to $[0,1]$ is $2x$.

4.3. **Exel's crossed products for regular Exel systems.** By Theorem 4.7, A embeds into $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ if and only if $\overline{A\alpha(A)A} \cap J(M_{\mathcal{L}}) \subseteq N_{\mathcal{L}}^{\perp}$. In this subsection, we consider regular Exel systems satisfying stronger, but easier to check in practice, condition: $\overline{A\alpha(A)A} \subseteq N_{\mathcal{L}}^{\perp}$. The latter inclusion holds, for instance, for systems with faithful transfer operators or with corner endomorphisms. These are the cases when Exel's crossed product boasts its greatest successes, see [12], [15], [9], [10]. We show that for such systems Exel crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ can be defined without a use of α.

We recall that any positive map $\mathcal{L} : A \to A$ restricts to the homomorphism $\mathcal{L} : MD(\mathcal{L}) \to \mathcal{L}(A)$. The kernel $(\ker \mathcal{L}|_{MD(\mathcal{L})})$ of this homomorphism is an ideal in $MD(\mathcal{L})$ and we may consider its annihilator $(\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp}$ in $MD(\mathcal{L})$. Thus $(\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp}$ is a C^*-subalgebra of A.

Proposition 4.18. *Suppose that $(A, \alpha, \mathcal{L})$ is a regular Exel system such that $\mathcal{L}$ is faithful on $\overline{A\alpha(A)A}$. Then $\alpha(A) = (\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp}$. Hence α is uniquely* *determined by* $\mathcal{L}$ *and*

$$A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; \overline{A(\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp} A}).$$

In particular, if $\overline{A(\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp} A} = N_{\mathcal{L}}^{\perp}$*, then* $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$.

Proof. On one hand, by Proposition 4.16, $\alpha(A) \subseteq \overline{A\alpha(A)A} \cap MD(\mathcal{L})$ and $\mathcal{L} : \alpha(A) \to \mathcal{L}(A)$ is an isomorphism. On the other hand, since $\mathcal{L}$ is faithful on $\overline{A\alpha(A)A}$, the map $\mathcal{L} : \overline{A\alpha(A)A} \cap MD(\mathcal{L}) \to \mathcal{L}(A)$ is an injective homomorphism. This implies that $\alpha(A) = \overline{A\alpha(A)A} \cap MD(\mathcal{L})$. Since $\alpha(A) = \overline{A\alpha(A)A} \cap MD(\mathcal{L})$ is an ideal in $MD(\mathcal{L})$ and $\mathcal{L} : \alpha(A) \to \mathcal{L}(A)$ is an isomorphism we actually get $\alpha(A) = (\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp}$. Thus, by Proposition 4.16, α is uniquely determined by $\mathcal{L}$. By Theorem 4.7 we get $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; \overline{A(\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp} A})$, and if $\overline{A(\ker \mathcal{L}|_{MD(\mathcal{L})})^{\perp} A} = N_{\mathcal{L}}^{\perp}$, then we actually have $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; N_{\mathcal{L}}^{\perp}) = C^*(A, \mathcal{L}; N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}})) = C^*(A, \mathcal{L})$. $\square$

Now we consider Exel systems $(A, \alpha, \mathcal{L})$ where α and $\mathcal{L}$ have somehow equal rights. Algebras arising from such systems were studied for instance in [43], [12], [2], [26], [28], [29].

Definition 4.19. We say that a regular Exel system $(A, \alpha, \mathcal{L})$ is a *corner system* if $\alpha(A)$ is a hereditary subalgebra of A.

The above terminology is justified by Lemma 4.20 and Remark 4.23 below. We note that corner systems $(A, \alpha, \mathcal{L})$ satisfy condition $\overline{A\alpha(A)A} \subseteq N_{\mathcal{L}}^{\perp}$. Indeed, since $\alpha(A)$ is hereditary and $\mathcal{L}$ is faithful on $\alpha(A)$, Lemma 2.4 implies that $\mathcal{L}$ is almost faithful on $\overline{A\alpha(A)A}$.

Lemma 4.20. *Let* $(A, \alpha, \mathcal{L})$ *be an Exel system. The following statements are equivalent:*

i) $(A, \alpha, \mathcal{L})$ *is a corner system.*
ii) α *is extendible and*

$$\alpha(\mathcal{L}(a)) = \overline{\alpha}(1) a \overline{\alpha}(1) \qquad \textit{for all } a \in A. \tag{35}$$

iii) α *has a complemented kernel and a corner range;* $\mathcal{L}$ *is a unique regular transfer operator for* α *and it is given by the formula*

$$\mathcal{L}(a) = \alpha^{-1}(pap), \qquad a \in A, \tag{36}$$

where $p \in M(A)$ *is a projection such that* $\alpha(A) = pAp$*, and* α^{-1} *is the inverse to the isomorphism* $\alpha : (\ker \alpha)^{\perp} \to pAp$.

iv) $\mathcal{L}(A)$ *is a complemented ideal in* A *and* $(\ker\mathcal{L}|_{MD(\mathcal{L})})^{\perp}$ *is a hereditary subalgebra of* A *mapped by* $\mathcal{L}$ *onto* $\mathcal{L}(A)$*;* α *is given by the formulas:*

$$\alpha|_{\mathcal{L}(A)^{\perp}} \equiv 0 \quad \text{and} \quad \alpha(a) = \mathcal{L}^{-1}(a), \quad \text{for } a \in \mathcal{L}(A), \tag{37}$$

where $\mathcal{L}^{-1}$ *is the inverse to the isomorphism* $\mathcal{L} : (\ker\mathcal{L}|_{MD(\mathcal{L})})^{\perp} \to \mathcal{L}(A)$.

If the above equivalent statements hold then $\alpha(A) = \overline{\alpha}(1)A\overline{\alpha}(1) = (\ker\mathcal{L}|_{MD(\mathcal{L})})^{\perp}$.

Proof. i)⇒ii). Let $\{\mu_\lambda\}_{\lambda\in\Lambda}$ be an approximate unit in A. Since $\mathcal{L}$ is isometric on $\alpha(A) = \alpha(A)A\alpha(A)$, for any $a \in A$, we have

$$\begin{aligned}\|\big(\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'})\big)a\|^2 &= \|\big(\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'})\big)aa^*\big(\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'})\big)\| \\ &= \|\mathcal{L}\Big(\big(\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'})\big)aa^*\big(\alpha(\mu_\lambda) - \alpha(\mu_{\lambda'})\big)\Big)\| \\ &\le 2\|\mathcal{L}(aa^*)(\mu_\lambda - \mu_{\lambda'})\|.\end{aligned}$$

The last term is arbitrarily small for sufficiently large λ and λ'. Accordingly, $\{\alpha(\mu_\lambda)\}_{\lambda\in\Lambda}$ is strictly Cauchy and thereby strictly convergent. Hence α is extendible and we have $\alpha(A) = \overline{\alpha}(1)A\overline{\alpha}(1)$. Since $E(a) = \overline{\alpha}(1)a\overline{\alpha}(1)$ is the unique conditional expectation onto $\alpha(A)$ we conclude, using Proposition 4.15, that (35) holds.

ii)⇒iii). Note that (35) implies that $\alpha(A) = \overline{\alpha}(1)A\overline{\alpha}(1)$ is a corner in A. In particular, $(A, \alpha, \mathcal{L})$ is regular because $E(a) = (\alpha\circ\mathcal{L})(a) = \overline{\alpha}(1)a\overline{\alpha}(1)$ is a conditional expectation onto $\alpha(A)$. Thus by Proposition 4.15, $\ker\alpha$ is complemented and $\mathcal{L}(a) = \alpha^{-1}(\overline{\alpha}(1)a\overline{\alpha}(1))$ $a \in A$.

iii)⇒iv). Decomposing A into parts pAp, $(1-p)Ap$, $pA(1-p)$, $(1-p)A(1-p)$, the map $\mathcal{L}$ assumes the form

$$\mathcal{L}\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{pmatrix} = \alpha^{-1}(a_{11}).$$

By Proposition 2.6, it is immediate that $pAp \oplus (1-p)A(1-p) \subseteq MD(\mathcal{L})$. Moreover, if $a = a_{12} + a_{21} \in MD(\mathcal{L})$, where $a_{12} \in (1-p)Ap$ and $a_{21} \in pA(1-p)$, then using (5) we get

$$\alpha^{-1}(a_{12}^*a_{12}) = 0, \qquad \alpha^{-1}(a_{21}a_{21}^*) = 0.$$

Since $a_{12}^*a_{12}$ and $a_{21}a_{21}^*$ belong to pAp, it follows that $a = a_{12} + a_{21} = 0$. Hence $MD(\mathcal{L}) = pAp \oplus (1-p)A(1-p)$. Consequently, $(\ker\mathcal{L}|_{MD(\mathcal{L})})^{\perp} = pAp$. Now the formula (37) is immediate.

iv)⇒i). It follows from Proposition 4.16. □

Remark 4.21. Transfer operators satisfying (35) are called complete transfer operators in [2], [26], [28], [29]. The pair (A, α) where α is an endomorphism satisfying condition iii) of Lemma 4.20, is called a reversible C^*-dynamical system in [29].

In the case when A is unital, Exel systems satisfying (35) were considered in [2] and [25]. In particular, it was shown in [2, Theorem 4.16] that $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ is isomorphic to $C^*(A,\alpha)$, and in [25, Theorem 3.14] that $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ is isomorphic to $\mathcal{O}_{M_\mathcal{L}}$. By Theorem 4.7, we know that $\mathcal{O}_{M_\mathcal{L}} \cong \mathcal{O}(A,\alpha,\mathcal{L}) = C^*(A,\mathcal{L})$. Hence combining these results we get the following isomorphisms $A \times_{\alpha,\mathcal{L}} \mathbb{N} \cong C^*(A,\mathcal{L}) \cong C^*(A,\alpha)$. We now generalize this fact to the non-unital case.

Theorem 4.22. *Suppose $(A,\alpha,\mathcal{L})$ is a corner Exel system. Then α and $\mathcal{L}$ determine each other uniquely and we have*

$$A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A,\mathcal{L}) \cong C^*(A,\alpha).$$

In particular, $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ can be viewed as a C^*-algebra generated by $j_A(A) \cup j_A(A)s$ where $j_A : A \to A \times_{\alpha,\mathcal{L}} \mathbb{N}$ is a non-degenerate homomorphism, $s \in M(A \times_{\alpha,\mathcal{L}} \mathbb{N})$,

$$sj_A(a)s^* = j_A(\alpha(a)), \qquad s^*j_A(a)s = j_A(\mathcal{L}(a)), \qquad a \in A, \tag{38}$$

and the pair (j_A, s) is universal for the pairs with the above properties.

Proof. Lemma 4.20 implies that α and $\mathcal{L}$ determine each other uniquely. By Theorem 4.7, to prove the equality $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A,\mathcal{L})$ it suffices to show that $A\alpha(A)A = N_\mathcal{L}^\perp \cap J(X_\mathcal{L})$. To this end, we use the isomorphism $X_\mathcal{L} \cong M_\mathcal{L}$ from Lemma 4.4. For $x \in A$ we have $q(x) = q(x\overline{\alpha}(1)) \in M_\mathcal{L}$. For any x, y, $z \in A$ we get

$$\Theta_{q(x),q(y)}q(z) = q(x)\mathcal{L}(y^*z) = q(x\alpha(\mathcal{L}(y^*z))) = q(x\overline{\alpha}(1)y^*z\overline{\alpha}(1)) = (x\overline{\alpha}(1)y)q(z).$$

Thus $\phi(x\overline{\alpha}(1)y) = \Theta_{q(x),q(y)}$. It follows that ϕ sends $\overline{A\alpha(A)A} = \overline{A\overline{\alpha}(1)A} \subseteq N_\mathcal{L}^\perp = (\ker\phi)^\perp$ isometrically onto $\mathcal{K}(X_\mathcal{L}) \cong \mathcal{K}(M_\mathcal{L})$. Hence $\overline{A\alpha(A)A} = J(X_\mathcal{L}) \cap N_\mathcal{L}^\perp$ and we have $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A,\mathcal{L})$.

In order to show that the first relation in (38) holds in $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A,\mathcal{L})$ (the second holds trivially) it suffices to check that $(i_A(\alpha(a)), ti_A(a)t^*)$, for $a \in A$, is a redundancy of the Toeplitz representation (i_A, t) (note that $\alpha(A) \subseteq N_\mathcal{L}^\perp$). Invoking Proposition 4.3 we have $ti_A(a) = i_A(\alpha(a))t$. Thus, using (35), for any $b, c \in A$ we get

$$\begin{aligned}(ti_A(a)t^*)\, i_A(b)t\, i_A(c) &= ti_A(a\mathcal{L}(b)c) = i_A(\alpha(a\mathcal{L}(b)c))t = i_A(\alpha(a)b\alpha(c))t \\ &= i_A(\alpha(a))\, i_A(b)t\, i_A(c).\end{aligned}$$

Since i_A is non-degenerate this shows that $(i_A(\alpha(a)), ti_A(a)t^*)$ is a redundancy and thus (38) holds. Moreover, since the ideal $J(X_\mathcal{L}) \cap N_\mathcal{L}^\perp = \overline{A\alpha(A)A}$ is generated by $\alpha(A)$ we see that the kernel of the quotient map $\mathcal{T}(A,\mathcal{L}) \to C^*(A,\mathcal{L})$ is the ideal generated by differences $i_A(\alpha(a)) - ti_A(a)t^*$, $a \in A$. Hence $(C^*(A,\mathcal{L}), j_A, s)$

is universal with respect to relations (38), cf. Proposition 3.17. By [29, Proposition 4.6], cf. Proposition 3.26, $C^*(A,\alpha)$ is universal with respect to the same relations and thus $C^*(A,\alpha)\cong C^*(A,\mathcal{L})$. □

Remark 4.23. If A is unital then an Exel system $(A,\alpha,\mathcal{L})$ is a corner system if and only if $(\alpha,\mathcal{L})$ is a corner interaction studied in [28], see Proposition 4.13v). In particular, the isomorphism $C^*(A,\mathcal{L})\cong C^*(A,\alpha)$ is an instance of the isomorphism (19). An examination of the argument leading to (19) shows that it holds also in the non-unital case if one defines a corner interaction as an interaction $(\mathcal{V},\mathcal{H})$ over A where both $\mathcal{V}$ and $\mathcal{H}$ are extendible and have corner ranges. Thus corner interactions give a symmetrized framework for corner Exel systems, and one could think of them as partial automorphism of A whose domain and range are corners in A.

5. Graph C^*-algebras as crossed products by completely positive maps

In this section, we test Exel's construction and the results of the present paper against the original idea standing behind [12] that Cuntz-Krieger algebras (or more generally graph C^*-algebras) could be viewed as crossed products associated to topological Markov shifts. We recall Brownlowe's [7] realization of graph C^*-algebras $C^*(E)$ as Exel-Royer's crossed product for partially defined Exel system $(\mathcal{D}_E,\alpha,\mathcal{L})$ and discuss when the maps α and $\mathcal{L}$ can be extended to the whole of diagonal algebra $\mathcal{D}_E$. This leads us to a complete description of Perron-Frobenious operators on $\mathcal{D}_E$ associated to quivers on E. We prove that the crossed product of $\mathcal{D}_E$ by any such operator is isomorphic to $C^*(E)$.

5.1. **Graph C^*-algebras as Exel-Royer's crossed products.** For graphs and their C^*-algebras we use the notation and conventions of [47], [10], [18]. Throughout this section, we fix an arbitrary countable directed graph $E=(E^0,E^1,r,s)$. Hence E^0 and E^1 are countable sets and $r,s:E^1\to E^0$ are arbitrary maps. We denote by E^n, $n>0$, the set of finite paths $\mu=\mu_1...\mu_n$ satisfying $s(\mu_i)=r(\mu_{i+1})$, for all $i=1,...,n$. Then $|\mu|=n$ stands for the length of μ and $E^*=\bigcup_{n=0}^{\infty}E^n$ is the set of all finite paths (vertices are treated as paths of length zero). We put E^∞ to be the set of infinite paths. The maps r,s extend naturally to E^* and r extends also to E^∞.

The *graph C^*-algebra* $C^*(E)$ is generated by a universal Cuntz-Krieger E-family consisting of partial isometries $\{s_e: e\in E^1\}$ and mutually orthogonal projections $\{p_v: v\in E^1\}$ such that $s_e^*s_e=p_{s(e)}$, $s_es_e^*\leq p_{r(e)}$ and $p_v=\sum_{r(e)=v}s_es_e^*$

whenever the sum is finite (i.e. v is a finite receiver). It follows that $C^*(E) = \overline{\operatorname{span}}\{s_\mu s_\nu^* : \mu, \nu \in E^*\}$ where $s_\mu := s_{\mu_1} s_{\mu_2} .... s_{\mu_n}$ for $\mu = \mu_1 ... \mu_n \in E^n$, $n > 0$, and $s_\mu = p_\mu$ for $\mu \in E^0$. We denote by $\mathcal{D}_E := \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^*\}$ the *diagonal C^*-subalgebra* of $C^*(E)$.

It is attributed to folklore, see [16] or [53] for an extended discussion, that the Gelfand spectrum of $\mathcal{D}_E$ can be identified with the boundary space of E. To be more specific, we define $E^*_{inf} := \{\mu \in E^* : |r^{-1}(s(\mu))| = \infty\}$ and $E^*_s := \{\mu \in E^* : r^{-1}(s(\mu)) = \emptyset\}$, so E^*_{inf} is the set of paths that start in infinite receivers, and E^*_s is the set of paths that start in sources. For any $\eta \in E^* \setminus E^0$ let $\eta E^{\leq \infty} := \{\mu = \mu_1 ... \in E^* \cup E^\infty : \mu_1 ... \mu_{|\eta|} = \eta\}$ and for $v \in E^0$ put $vE^{\leq\infty} := \{\mu \in E^* \cup E^\infty : r(\mu) = v\}$. The *boundary space* of E, cf. [53, Section 2] or [7, Subsection 4.1], is the set

$$\partial E := E^\infty \cup E^*_{inf} \cup E^*_s$$

equipped with the topology generated by the 'cylinders' $D_\eta := \partial E \cap \eta E^{\leq\infty}$, $\eta \in E^*$, and their complements. In fact, the sets $D_\eta \setminus \bigcap_{\mu \in F} D_\mu$, where $\eta \in E^*$ and $F \subseteq \eta E^{\leq\infty} \cap E^*$ is finite, form a basis of compact and open sets for Hausdorff topology on ∂E, [53, Section 2] or [7, Section 2]. Passing to a dual description of the assertion in [53, Theorem 3.7] we get the following proposition.

Proposition 5.1. *We have an isomorphism $\mathcal{D}_E \cong C_0(\partial E)$ determined by the formula*

$$s_\mu s_\mu^* \longmapsto \chi_{D_\mu}, \qquad \mu \in E^*. \tag{39}$$

The one-sided *topological Markov shift* associated to E is the map $\sigma : \partial E \setminus E^0 \to \partial E$ defined, for $\mu = \mu_1 \mu_2 ... \in \partial E \setminus E^0$, by the formulas

$$\sigma(\mu) := \mu_2 \mu_3 ... \ \text{ if } \ \mu \notin E^1, \quad \text{ and } \quad \sigma(\mu) := s(\mu_1) \ \text{ if } \ \mu = \mu_1 \in E^1.$$

By [7, Proposition 2.1] the shift σ is a local homeomorphism. Furthermore, results of [7, Propositions 2.1 and 4.4] imply the following proposition (we adopt the convention that a sum over the empty set is zero).

Proposition 5.2 (Brownlowe)**.** *The formulas*

$$\alpha(a)(\mu) = a(\sigma(\mu)), \qquad \mathcal{L}(a)(\mu) = \sum_{\nu \in \sigma^{-1}(\mu)} a(\nu) \tag{40}$$

define respectively a homomorphism $\alpha : C_0(\partial E) \to M(C_0(\partial E \setminus E^0))$ and a linear map $\mathcal{L} : C_c(\partial E \setminus E^0) \to C_c(\partial E)$. Moreover, the triple $(C_0(\partial E), \alpha, \mathcal{L})$ forms a C^*-dynamical system in the sense of [9, Definition 1.2]*, and we have an isomorphism*

$$\mathcal{O}(C_0(\partial E), \alpha, \mathcal{L}) \cong C^*(E).$$

The above mappings (40) have the following important algebraic description. The isomorphism $\mathcal{D}_E \cong C_0(\partial E)$ from Proposition 5.1 gives rise to $*$-isomorphisms $M(\overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus \{0\}\}) \cong M(C_0(\partial E \setminus E_0))$ and $\mathrm{span}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\} \cong C_c(\partial E)$. Using these isomorphisms the mappings in (40) are intertwined respectively with a homomorphism $\Phi : \mathcal{D}_E \to M(\overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\})$ and a linear map $\Phi_* : \mathrm{span}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\} \to \mathrm{span}\{s_\mu s_\mu^* : \mu \in E^*\}$ which are given by the formulas

$$\Phi(a) = \sum_{e\in E^1} s_e a s_e^*, \qquad \Phi_*(a) = \sum_{e\in E^1} s_e^* a s_e, \tag{41}$$

where the first sum is strictly convergent and the second is finite. (It suffices to check it on the spanning elements $s_\mu s_\mu^*$ and χ_{D_μ}, $\mu \in E^*$, which we leave to the reader.) When E has no infinite emitters (see Proposition 5.3 below) the formula $\Phi(a) = \sum_{e\in E^1} s_e a s_e^*$ defines a self-map on the whole of the graph C^*-algebra $C^*(E)$. In the literature, this mapping, usually considered when E is locally finite (i.e. r and s are finite-to-one), is called a *non-commutative Markov shift* and its ergodic properties are well studied, cf., for instance, [20].

5.2. **Non-commutative Perron-Frobenius operators arising from quivers.** The mappings α and $\mathcal{L}$ considered in Proposition 5.2 are viewed as partial mappings on $C_0(\partial E)$, cf. [14], [7]. Now we discuss the problem of when the formulas (40), or their analogues, define honest mappings on $C_0(\partial E)$.

Proposition 5.3. *The following conditions are equivalent:*

i) *the first of formula in* (40) *defines an endomorphism* $\alpha : C_0(\partial E) \to C_0(\partial E)$*,*
ii) $\sigma : \partial E \setminus E^0 \to \partial E$ *is a proper map (preimage of a compact set is compact),*
iii) σ *is a finite-to-one mapping,*
iv) *there are no infinite emitters in* E*,*
v) *the sum* $\sum_{e\in E^1} s_e a s_e^*$ *converges in norm for every* $a \in C^*(E)$*,*
vi) *the range of the homomorphism* Φ *is contained in* $\overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\} \subseteq \mathcal{D}_E$*, and hence* $\Phi : \mathcal{D}_E \to \mathcal{D}_E$ *is an endomorphism.*

In particular, if the above equivalent conditions hold, then the first formula in (41) *defines a completely positive map* $\Phi : C^*(E) \to C^*(E)$ *which restricts to an endomorphism* $\Phi : \mathcal{D}_E \to \mathcal{D}_E$*.*

Proof. i)$\Leftrightarrow$ii). It is a well known general fact that a continuous mapping $\tau : X \to Y$ between locally compact Hausdorff spaces X, Y, gives rise to the composition operator from $C_0(Y)$ to $C_0(X)$ (rather than to $C_b(X) = M(C_0(X))$) if and only if τ is proper.

ii)⇒iii). If σ is a proper local homeomorphism then $\sigma^{-1}(\mu)$, $\mu\in\partial E$, is compact and cannot have a cluster point. Hence σ is finite-to-one.

iii)⇒iv). It follows readily from the definition of σ.

iv)⇒v). Consider a net, indexed by finite sets $F\subseteq E^1$ ordered by inclusion, consisting of mappings $\alpha_F : C^*(E)\to C^*(E)$ given by $\alpha_F(a):=\sum_{e\in F} s_e a s_e^*$. Since the projections $s_e s_e^*$, $e\in F$, are mutually orthogonal we get $\|\alpha_F(a)\| = \max_{e\in F}\|s_e a s_e^*\|\leq \|a\|$, and thus α_F is a contraction. Let $a\in C^*(E)$. For any $\varepsilon>0$ there is a finite linear combination $b=\sum_{\mu,\nu\in K} c_{\mu,\nu} s_\mu s_\nu^*$ such that $\|a-b\|\leq\varepsilon$ ($K\subseteq E^*$ is finite set). Since E has no infinite emitters the set

$$F=\{e\in E^1 : s(e)=r(\mu) \text{ for some } \mu\in K\}=\bigcup_{v\in r(K)} s^{-1}(v)$$

is finite. Clearly, for any finite set $F'\subseteq E^*$ containing F we have $\alpha_{F'}(b)=\alpha_F(b)$. Thus

$$\|\alpha_{F'}(a)-\alpha_F(a)\|\leq\|\alpha_{F'}(a)-\alpha_{F'}(b)\|+\|\alpha_F(b)-\alpha_F(a)\|\leq 2\varepsilon.$$

Hence the net $\{\alpha_F(a)\}_F$ is Cauchy and the sum $\sum_{e\in E^1} s_e a s_e^*$ converges in norm.

v)⇒vi). It is straightforward.

vi)⇒i). Note that the isomorphism $\mathcal{D}_E\cong C_0(\partial E)$ given by (39) intertwines Φ and α. □

One can check that the second formula in (40) defines a mapping $\mathcal{L}: C_c(\partial E)\to C_c(\partial E)$ if and only if E has no infinite receivers. But even if the graph E is locally finite, this mapping might be unbounded. On the other hand, if E is locally finite, we can adjust the formula for $\mathcal{L}$ by adding averaging as in (1), and then $\mathcal{L}$ has norm one, so in particular it extends to a self-map of $C_0(\partial E)$. This motivates us to consider slightly more general averagings, which will allow us to get a bounded positive operator on $C_0(\partial E)$ for arbitrary graphs. Accordingly, we wish to consider strictly positive numbers $\lambda=\{\lambda_e\}_{e\in E^1}$ such that the formula

$$\mathcal{L}_\lambda(a)(\mu)=\sum_{e\in E^1,\, e\mu\in\partial E}\lambda_e\, a(e\mu) \tag{42}$$

defines a mapping on $C_0(\partial E)$. We note that fixing the family $\{\lambda_e\}_{e\in E^1}$ is equivalent to fixing a system of measures $\{\lambda_v\}_{v\in E^0}$ on E^1 making the graph E into a (topological) quiver. Indeed, the relation $\lambda_e=\lambda_{s(e)}(\{e\})$ establishes a one-to-one correspondence between families $\{\lambda_e\}_{e\in E^1}$ of strictly positive numbers and s-systems of measures $\{\lambda_v\}_{v\in E^0}$ on E, cf. Definition 3.29. In particular, if E

has no infinite emitters one can put $\lambda_e := |s^{-1}(s(e))|^{-1}$, $e \in E^1$, which corresponds to the situation where all the measures $\{\lambda_v\}_{v\in E^0}$ are uniform probability distributions. In this case one recovers from (42) the second formula in (1).

Proposition 5.4. *Let $\lambda = \{\lambda_e\}_{e\in E^1}$ be a family of strictly positive numbers. The following conditions are equivalent:*

i) *the formula (42) defines a bounded operator $\mathcal{L}_\lambda : C_0(\partial E) \to C_0(\partial E)$,*

ii) *the following conditions are satisfied:*

$$\left\{ \sum_{e\in s^{-1}(v)} \lambda_e \right\}_{v\in s(E^1)} \in \ell_\infty\left(s(E^1)\right), \tag{43}$$

$$\left\{ \sum_{e\in r^{-1}(v)\cap s^{-1}(w)} \lambda_e \right\}_{w\in s(r^{-1}(v))} \in c_0\left(s(r^{-1}(v))\right) \quad \text{for all } v \in r(E^1), \tag{44}$$

iii) *the sum*

$$u_\lambda := \sum_{e\in E^1} \sqrt{\lambda_e}\, s_e \tag{45}$$

converges strictly in $M(C^*(E))$,

iv) *the sum $\sum_{e,f\in E^1} \sqrt{\lambda_e\lambda_f} s_e^* a s_f$ converges in norm for every $a \in C^*(E)$ and*

$$\Phi_{*,\lambda}(a) := \sum_{e,f\in E^1} \sqrt{\lambda_e\lambda_f} s_e^* a s_f, \qquad a \in C^*(E), \tag{46}$$

defines a completely positive map $\Phi_{*,\lambda} : C^*(E) \to C^*(E)$.

If the above equivalent conditions hold then $\Phi_{*,\lambda}(a) = u_\lambda^* a u_\lambda$, $a \in C^*(E)$, and the isomorphism $\mathcal{D}_E \cong C_0(\partial E)$ from Proposition 5.1 intertwines $\Phi_{*,\lambda}|_{\mathcal{D}_E}$ and $\mathcal{L}_\lambda$.

Proof. i)⇒ii). One readily sees that

$$\mathcal{L}_\lambda(\chi_{D_\eta}) = \lambda_{\eta_1}\chi_{D_{\sigma(\eta)}}, \qquad \text{for any } \eta = \eta_1 ... \in \partial E \setminus E^0. \tag{47}$$

Hence for any $v \in s(E^1)$ and any finite set $F \subseteq s^{-1}(v)$ we get

$$\sum_{e\in F} \lambda_e = \| \left(\sum_{e\in F} \lambda_e \right) \chi_{D_v} \| = \|\mathcal{L}_\lambda \left(\sum_{e\in F} \chi_{D_e} \right) \| \le \|\mathcal{L}_\lambda\|,$$

which implies condition (43). Now let $v \in r(E^1)$ and note that, for any $\mu \in \partial E$,

$$\begin{aligned}\mathcal{L}_\lambda(\chi_{D_v})(\mu) &= \sum_{e\in r^{-1}(v),\, e\in s^{-1}(r(\mu))} \lambda_e\, \chi_{D_v}(e\mu) = \sum_{e\in r^{-1}(v)} \lambda_e\, \chi_{D_{s(e)}}(\mu)\\ &= \sum_{w\in s(r^{-1}(v))} \sum_{e\in r^{-1}(v)\cap s^{-1}(w)} \lambda_e\, \chi_{D_w}(\mu).\end{aligned}$$

Since the sets D_w are disjoint and open, $\mathcal{L}_\lambda(\chi_{D_v}) \in C_0(\partial E)$ implies condition (44). For future reference, note that in view of the above calculation we have (we treat empty sums as zero)

$$\mathcal{L}_\lambda(\chi_{D_v}) = \sum_{e\in r^{-1}(v)} \lambda_e\, \chi_{D_{s(e)}}, \qquad v \in E^0. \tag{48}$$

ii)$\Rightarrow$iii). Let $v \in s(E^1)$. For a finite set $F \subseteq s^{-1}(v)$ we have $\| \sum_{e\in F} \sqrt{\lambda_e} s_e\|^2 = \| \sum_{e\in F} \lambda_e p_v\| = \sum_{e\in F} \lambda_e$. Since $\sum_{e\in s^{-1}(v)} \lambda_e < \infty$, by (43), it follows that the sum $u_v := \sum_{e\in s^{-1}(v)} \sqrt{\lambda_e} s_e$ converges in norm. Thus for any finite set $F \subseteq s(E^1)$ we have

$$u_F := \sum_{v\in F} u_v = \sum_{v\in F} \sum_{e\in s^{-1}(v)} \sqrt{\lambda_e}\, s_e = \sum_{e\in s^{-1}(F)} \sqrt{\lambda_e}\, s_e \in C^*(E). \tag{49}$$

By (43), $M := \sup_{v\in s(E^1)} \sum_{e\in s^{-1}(v)} \lambda_e$ is finite. The set of elements u_F is bounded:

$$\|u_F\|^2 = \|u_F^* u_F\| = \| \sum_{v\in F} \sum_{e\in s^{-1}(v)} \lambda_e p_v\| = \max_{v\in F} \sum_{e\in s^{-1}(v)} \lambda_e \leq M. \tag{50}$$

Condition (44) implies that for any $v \in E^0$ the sum $\sum_{e\in r^{-1}(v)} \sqrt{\lambda_e} s_e$ converges in norm. Indeed, for any finite set $F \subseteq r^{-1}(v)$ we have

$$\| \sum_{e\in F} \sqrt{\lambda_e} s_e\|^2 = \| \sum_{e\in F} \lambda_e p_{s(e)}\| = \max_{w\in s(r^{-1}(v))} \sum_{e\in r^{-1}(v)\cap s^{-1}(w)\cap F} \lambda_e,$$

which by (44) can be made arbitrarily small by choosing F lying outside a sufficiently large finite subset of $r^{-1}(v)$.

Now fix a (nonzero) finite linear combination $a = \sum_{\mu,\nu\in K} \lambda_{\mu,\nu} s_\mu s_\nu^*$, where $K \subseteq E^*$ is finite. Since we know, by (50), that $\| \sum_{e\in F} \sqrt{\lambda_e}\, s_e\| \leq \sqrt{M}$ for every finite $F \subseteq E^1$, to prove the strict convergence of the sum in (45) it suffices to check the convergence in norm of the two series $\sum_{e\in E^1} \sqrt{\lambda_e}\, s_e a$ and $\sum_{e\in E^1} \sqrt{\lambda_e}\, s_e^* a$.

Firstly, note that for $v \in E^0$ we have $u_v a = 0$ unless $v \in r(K)$. Hence for any finite set $F \subseteq s(E^1)$ containing $r(K)$ we get $u_F a = u_{r(K)} a$. Recall, see (49), that $u_{r(K)} = \sum_{e\in s^{-1}(r(K))} \sqrt{\lambda_e} s_e$ converges in norm. Therefore, for any $\varepsilon > 0$ there is a finite set $F_0 \subseteq s^{-1}(r(K)) \cap E^1$ such that for any finite $F \subseteq E^1$ disjoint with F_0 we have

$$\|\sum_{e\in F} \sqrt{\lambda_e} s_e a\| = \|\sum_{e\in F\cap s^{-1}(r(K))} \sqrt{\lambda_e} s_e a\| \leq \varepsilon.$$

This means that the sum $\sum_{e\in E^1} \sqrt{\lambda_e}\, s_e a$ converges in $C^*(E)$.

Secondly, note that for $e \in E^1$ we have $s_e^* a = 0$ unless $e\mu \in K$ for some $\mu \in E^*$, or $r(e) \in K \cap E^0$. Recall that the sum $\sum_{e\in r^{-1}(v)} \sqrt{\lambda_e} s_e^*$ is norm convergent for all $v \in E^0$. Thus for a fixed $\varepsilon > 0$ we can find a finite set $F_1 \subseteq E^1$ such that for any F disjoint with F_1 we have

$$\|\sum_{e\in r^{-1}(v)\cap F} \sqrt{\lambda_e} s_e^*\| \leq \frac{\varepsilon}{|K\cap E^0|\cdot \|a\|} \qquad \text{for all } v \in K\cap E^0.$$

Then for any finite $F \subseteq E^1$ lying outside the finite set $F_0 := \{e \in E^1 : e\mu \in K, \mu \in E^*\} \cup F_1$ we get

$$\begin{aligned}\|\sum_{e\in F} \sqrt{\lambda_e} s_e^* a\| &= \|\sum_{v\in K\cap E^0}\sum_{e\in r^{-1}(v)\cap F} \sqrt{\lambda_e} s_e^* a\| \\ &\leq \sum_{v\in K\cap E^0} \|\sum_{e\in r^{-1}(v)\cap F} \sqrt{\lambda_e} s_e^*\|\cdot\|a\| \leq \varepsilon.\end{aligned}$$

Thus $\sum_{e\in E^1} \sqrt{\lambda_e}\, s_e^* a$ converges in $C^*(E)$. This shows that $u_\lambda = \sum_{e\in E^1} \sqrt{\lambda_e}\, s_e$ converges in strict topology in $M(C^*(E))$.

iii)⇒iv). Plainly, as the sum $u_\lambda = \sum_{e\in E^1} \sqrt{\lambda_e}\, s_e$ is strictly convergent the sum $u_\lambda^* a u_\lambda = \sum_{e,f\in E^1} \sqrt{\lambda_e\lambda_f} s_e^* a s_f$ converges in norm for every $a \in C^*(E)$.

iv)⇒i). Using relations (47), (48) one readily verifies that the isomorphism given by (39) intertwines the restriction $\Phi_{*,\lambda}|_{\mathcal{D}_E}$ of $\Phi_{*,\lambda}$ to $\mathcal{D}_E$ with a mapping $\mathcal{L}_\lambda : C_0(\partial E) \to C_0(\partial E)$ given by (42). $\square$

Remark 5.5. For u_λ given by (45) we have $u_\lambda^* u_\lambda = \sum_{v\in E^0}(\sum_{e\in s^{-1}(v)} \lambda_e) p_v$. Hence u_λ is a partial isometry if and only if the measures $\{\lambda_v\}_{v\in E^0}$ arising from $\lambda = \{\lambda_e\}_{e\in E^1}$ are normalized, that is if and only if

$$\sum_{e\in s^{-1}(v)} \lambda_e = 1, \quad \text{for all} \quad v \in E^0. \tag{51}$$

Clearly, (51) implies (43) and if no vertex in E receives edges from infinitely many vertices then (44) is trivial. So in this case u_λ can be chosen to be a partial isometry. Nevertheless, in general there might be no systems satisfying (51) for which the sum (45) is strictly convergent (e.g. consider the infinite countable graph with a vertex receiving one edge from each of the remaining ones). If E is locally finite, one can let λ_v, $v \in E^0$, to be uniform probability distributions by putting $\lambda_e := |s^{-1}(s(e))|^{-1}$, $e \in E^1$. In the latter case and under the assumption that E has no sinks or sources it was noted implicitly in [10, Theorem 5.1] and explicitly in [18, Section 5] that the formula (45) defines an isometry in $M(C^*(E))$. A detailed discussion of history and analysis of operators (45), (46) associated to systems of uniform probability measures for arbitrary finite graphs can be found in [28].

Let us note that $\Phi_{*,\lambda}$, given by (46), restricted to $\mathcal{D}_E$ assumes the form

$$\Phi_{*,\lambda}(a) = \sum_{e \in E^1} \lambda_e s_e^* a s_e, \qquad a \in \mathcal{D}_E. \tag{52}$$

In particular, in view of the last part of Proposition 5.4, it is natural to call $\Phi_{*,\lambda} : C^*(E) \to C^*(E)$ the *non-commutative Perron-Frobenius operator* associated to the quiver $(E^1, E^0, r, s, \lambda)$.

5.3. **Graph C^*-algebras as crossed products $C^*(\mathcal{D}_E, \mathcal{L})$.** Now we are ready to state and prove the main result of this section. In previous subsections we have shown that for positive numbers $\lambda = \{\lambda_e\}_{e \in E^1}$ satisfying (43), (44) we have two mappings $\mathcal{L}_\lambda : C_0(\partial E) \to C_0(\partial E)$ and $\Phi_{*,\lambda} : \mathcal{D}_E \to \mathcal{D}_E$, given respectively by (42) and (52). These mappings are intertwined by the isomorphism $C_0(\partial E) \cong \mathcal{D}_E$ determined by (39). Thus one could express the following statement equally well in terms of $(C_0(\partial E), \mathcal{L}_\lambda)$ or $(\mathcal{D}_E, \Phi_{*,\lambda})$. We choose the second system, as it is more convenient for our proofs. In order to shorten the notation we denote $\Phi_{*,\lambda}$ simply by $\mathcal{L}$.

Theorem 5.6. *Suppose $E = (E^0, E^1, s, r)$ is an arbitrary directed graph and choose the numbers $\lambda_e > 0$, $e \in E^1$, such that the conditions* (43), (44) *hold. Then the sum*

$$\mathcal{L}(a) := \sum_{e \in E^1} \lambda_e s_e^* a s_e, \qquad a \in \mathcal{D}_E, \tag{53}$$

is convergent in norm and defines a (completely) positive map $\mathcal{L} : \mathcal{D}_E \to \mathcal{D}_E$ such that

$$C^*(E) \cong C^*(\mathcal{D}_E, \mathcal{L}),$$

with the isomorphism determined by $a \mapsto j_{\mathcal{D}_E}(a)$, $au_\lambda \mapsto j_{\mathcal{D}_E}(a)s$, $a \in \mathcal{D}_E$, where u_λ is given by the strictly convergent sum (45)*. Further under these assumptions:*

i) *If E has no infinite emitters, then the following sum is convergent in norm:*

$$\alpha(a) := \sum_{e \in E^1} s_e a s_e^*, \qquad a \in \mathcal{D}_E. \tag{54}$$

It defines an endomorphism such that $(\mathcal{D}_E, \alpha, \mathcal{L})$ is an Exel system and

$$C^*(\mathcal{D}_E, \mathcal{L}) = \mathcal{D}_E \rtimes_{\alpha,\mathcal{L}} \mathbb{N}.$$

Moreover, $(\mathcal{D}_E, \alpha, \mathcal{L})$ is a regular Exel system if and only if (51) *holds.*

ii) *If E has no infinite receivers then $\mathcal{L}$ is a transfer operator for a certain endomorphism α if and only if E is locally finite. In this event α given by* (54) *is a unique endomorphism such that $(\mathcal{D}_E, \alpha, \mathcal{L})$ is an Exel system and $\mathcal{L}$ is faithful on $\alpha(\mathcal{D}_E)\mathcal{D}_E$.*

iii) *If E is locally finite and without sources then $\mathcal{L}$ is faithful and α given by* (54) *is a unique endomorphism such that $(\mathcal{D}_E, \alpha, \mathcal{L})$ is an Exel system.*

Remark 5.7. We comment on the corresponding items in the above theorem:

i). Recall that (51) holds if and only if the operator u_λ is a partial isometry. In particular, the general question for which graphs E the numbers $\lambda_e > 0$, $e \in E^1$, can be chosen so that the Exel system $(\mathcal{D}_E, \alpha, \mathcal{L})$ is regular, seems to be a complex problem.

ii). One could conjecture that in general $\mathcal{L}$ is a transfer operator for a certain endomorphism if and only if E has no infinite emitters, and then this endomorphism is the (non-commutative) Markov shift given by (54).

iii). If E is locally finite and without sources then $\partial E = E^\infty$ and we can put $\lambda_e := |s^{-1}(s(e))|^{-1}$, $e \in E^1$. In this case, identifying $\mathcal{D}_E$ with $C_0(E^\infty)$, the mappings (54) and (53) coincide with those given by (1). In particular, Theorem 5.6 yields an isomorphism

$$C^*(E) \cong C_0(E^\infty) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$$

proved by Brownlowe in [7, Proposition 4.6], and when E has no sinks by Brownlowe, Raeburn and Vitadello in [10, Theorem 5.1].

The proof of Theorem 5.6 will rely on the following two lemmas. We fix the notation from the assertion of Theorem 5.6 and note that the map $\mathcal{L} : \mathcal{D}_E \to \mathcal{D}_E$ is well defined by Proposition 5.4. We denote by $E^0_s := \{v \in E^0 : r^{-1}(v) = \emptyset\}$ and $E_{inf} := \{v \in E^0 : |r^{-1}(v)| = \infty\}$ the set of sources and the set of infinite receivers, respectively.

Lemma 5.8. *Let $X_{\mathcal{L}}$ be the C^*-correspondence of $(\mathcal{D}_E, \mathcal{L})$. We have $N_{\mathcal{L}}^{\perp} = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E_s^0\}$ and $J(X_{\mathcal{L}}) = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E_{inf}^0\}$. Hence*

$$N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}}) = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\}.$$

Proof. Note that $\mathcal{D}_E$ is a direct sum of two complemented ideals $\overline{\operatorname{span}}\{p_v : v \in E_s^0\}$ and $\overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E_s^0\}$. One readily sees that $\mathcal{L}$ vanishes on the first one and is faithful on the second one. Hence $N_{\mathcal{L}} = \overline{\operatorname{span}}\{p_v : v \in E_s^0\}$ and $N_{\mathcal{L}}^{\perp} = \overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E_s^0\}$.

Let $\mu \in E^* \setminus E^0$ and put $K := \lambda_{\mu_1}^{-1}\Theta_{(s_\mu s_\mu^* \otimes 1),(s_\mu s_\mu^* \otimes 1)}$ where $\mu_1 \in E^1$ is such that $\mu_1 \overline{\mu} = \mu$ for $\overline{\mu} \in E^*$ (we recall that $a \otimes 1 \in X_{\mathcal{L}}$, for $a \in \mathcal{D}_E$, is given by (13)). We claim that $\phi(s_\mu s_\mu^*) = K$. Indeed, for any $a, b \in \mathcal{D}_E$ we have

$$K(a \otimes b) = K(a \otimes 1)b = \lambda_{\mu_1}^{-1}\left(s_\mu s_\mu^* \otimes \mathcal{L}(s_\mu s_\mu^* a)\right) b = \left(s_\mu s_\mu^* \otimes s_{\overline{\mu}} s_\mu^* a s_{\mu_1}\right) b. \tag{55}$$

Moreover, for any $x, y \in \mathcal{D}_E$ we have

$$\begin{aligned}
\langle s_\mu s_\mu^* \otimes s_{\overline{\mu}} s_\mu^* a s_{\mu_1}, x \otimes y\rangle_{\mathcal{L}} &= s_{\overline{\mu}} s_\mu^* a s_{\mu_1} \mathcal{L}(s_\mu s_\mu^* x) y \\
&= \lambda_{\mu_1} s_{\overline{\mu}} s_\mu^* a s_\mu s_\mu^* x s_{\mu_1} y \\
&= \mathcal{L}(s_\mu s_\mu^* a x) y \\
&= \langle s_\mu s_\mu^*(a \otimes 1), x \otimes y\rangle_{\mathcal{L}}.
\end{aligned}$$

Hence $s_\mu s_\mu^* \otimes s_{\overline{\mu}} s_\mu^* a s_{\mu_1} = s_\mu s_\mu^*(a \otimes 1)$. Thus in view of (55) we get $K(a \otimes b) = (s_\mu s_\mu^* a \otimes 1)b = \phi(s_\mu s_\mu^*)(a \otimes b)$, which proves our claim. If $v \in E^0 \setminus E_{inf}^0$, then using what we have just shown we get

$$\phi(p_v) = \phi(\sum_{f \in r^{-1}(v)} s_f s_f^*) = \sum_{f \in r^{-1}(v)} \lambda_f^{-1} \Theta_{(s_f s_f^* \otimes 1),(s_f s_f^* \otimes 1)} \in \mathcal{K}(X_{\mathcal{L}}).$$

This shows that $\overline{\operatorname{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E_{inf}^0\} \subseteq J(X_{\mathcal{L}})$. Suppose, on the contrary, that this inclusion is proper. Then there exists an element in $J(X_{\mathcal{L}})$ of the form $a = \sum_{\mu \in E^* \setminus E_{inf}^0} c_\mu s_\mu s_\mu^* + \sum_{v \in E_{inf}^0} c_v p_v$ where c_μ, c_v are complex numbers and there is $v_0 \in E_{inf}^0$ such that $c_{v_0} \neq 0$. Then $\sum_{v \in E_{inf}^0} c_v p_v = a - \sum_{\mu \in E^* \setminus E_{inf}^0} c_\mu s_\mu s_\mu^*$ is in $J(X_{\mathcal{L}})$. Hence $p_{v_0} = c_{v_0}^{-1} p_{v_0} \sum_{v \in E_{inf}^0} c_v p_v$ is in $J(X_{\mathcal{L}})$. We show that the latter is impossible. Indeed, any operator in $\mathcal{K}(X_{\mathcal{L}})$ can be approximated by $K \in \mathcal{K}(X_{\mathcal{L}})$ given by a finite linear combination of the form

$$K = \sum_{\mu,\nu,\eta,\tau \in F} \lambda_{\mu,\nu,\eta,\tau} \Theta_{(s_\mu s_\mu^* \otimes s_\nu s_\nu^*),(s_\eta s_\eta^* \otimes s_\tau s_\tau^*)}$$

where $F \subseteq E^*$ is a finite set. For any such combination we can find an edge $g \in r^{-1}(v_0)$ such that the projection $p_{s(g)}$ is orthogonal to every projection $s_\mu s_\mu^*$,

$\mu \in F$. Then for $\tau \in F$, and any $\eta \in E^*$, we have

$$\langle s_\eta s_\eta^* \otimes s_\tau s_\tau^*, s_g s_g^* \otimes 1\rangle_{\mathcal{L}} = s_\tau s_\tau^* \mathcal{L}(s_\eta s_\eta^* s_g s_g^*) = s_\tau s_\tau^* p_{s(g)}(\lambda_g s_g^* s_\eta s_\eta^* s_g) = 0.$$

This implies that $K(s_g s_g^* \otimes 1) = 0$. Thus, as $\lambda_g^{-1}\|s_g s_g^* \otimes 1\| = 1$, we get

$$\|\phi(p_{v_0}) - K\| \geq \lambda_g^{-1}\|\phi(p_v)(s_g s_g^* \otimes 1) - K(s_g s_g^* \otimes 1)\| = \lambda_g^{-1}\|s_g s_g^* \otimes 1\| = 1.$$

Accordingly, $\phi(p_{v_0}) \notin \mathcal{K}(X_{\mathcal{L}})$ which is a contradiction. Thus $\overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0_{inf}\} = J(X_{\mathcal{L}})$. □

By Proposition 5.3, if E has no infinite emitters then (54) defines an endomorphism $\alpha : \mathcal{D}_E \to \mathcal{D}_E$.

Lemma 5.9. *Suppose E has no infinite emitters and α is given by* (54)*. Then*

$$\alpha(\mathcal{D}_E)\mathcal{D}_E = N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}}).$$

Proof. Let $= \mu_1\overline{\mu} \in E^* \setminus E^0$ where $\mu_1 \in E^1$. Since

$$s_\mu s_\mu^* = s_{\mu_1\overline{\mu}} s_{\mu_1\overline{\mu}}^* = s_{\mu_1} s_{\mu_1}^* \alpha(s_{\overline{\mu}} s_{\overline{\mu}}^*) \in \alpha(\mathcal{D}_E)\mathcal{D}_E,$$

it follows from Lemma 5.8 that $N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}}) \subseteq \alpha(\mathcal{D}_E)\mathcal{D}_E$. For the reverse inclusion it suffices to show that for any $a \in \mathcal{D}_E$ we have $\alpha(a) \in N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}})$. To this end, consider a net $\mu_F := \sum_{e\in F} s_e s_e^* \in N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}}) = \overline{\mathrm{span}}\{s_\mu s_\mu^* : \mu \in E^* \setminus E^0\}$ indexed by finite sets $F \subseteq E^1$ ordered by inclusion. Clearly, $\mu_F\alpha(a)$ converges to $\alpha(a)$. Hence $\alpha(a) \in N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}})$. □

Proof of Theorem 5.6:. By Proposition 5.4 the sum (53) converges in norm and the operator $u_\lambda = \sum\limits_{e\in E^1} \sqrt{\lambda_e} s_e$ converges strictly in $M(C^*(E))$. Plainly, $\mathcal{L}(a) = u_\lambda a u_\lambda^*$ for $a \in \mathcal{D}_E$. Let us treat $M(C^*(E))$ as a non-degenerate subalgebra of $B(H)$. Then the pair (id, u_λ) is a faithful representation of $(\mathcal{D}_E, \mathcal{L})$ in $B(H)$. We claim that it is covariant, in the sense of Definition 3.16, i.e. $N_{\mathcal{L}}^{\perp} \cap J(X_{\mathcal{L}}) \subseteq \overline{\mathcal{D}_E u_\lambda \mathcal{D}_E u_\lambda^*}$. Indeed, taking $s_\mu s_\mu^*$ where $\mu \in E^* \setminus E^0$, and writing $\mu = \mu_1\overline{\mu}$ where $\mu_1 \in E^1$ and $\overline{\mu} \in E^*$ we get

$$s_\mu s_\mu^* = s_{\mu_1\overline{\mu}} s_{\mu_1\overline{\mu}}^* = \lambda_{\mu_1}^{-1} s_{\mu_1} s_{\mu_1}^* \, u_\lambda(s_{\overline{\mu}} s_{\overline{\mu}}^*) u_\lambda^* \, s_{\mu_1} s_{\mu_1}^* \in \overline{\mathcal{D}_E u_\lambda \mathcal{D}_E u_\lambda^*}.$$

By virtue of Lemma 5.8 this proves our claim. Hence by Proposition 3.17 the mapping $j_A(a) \mapsto a$, $j_A(a)s \mapsto a u_\lambda$, $a \in \mathcal{D}_E$, gives rise to a homomorphism from $C^*(\mathcal{D}_E, \mathcal{L})$ into $C^*(E)$. Let us denote it by $id \rtimes u_\lambda$ and note that it is actually an epimorphism because we have

$$s_e = (\sqrt{\lambda_e})^{-1}(s_e s_e^*) u_\lambda p_{s(e)}, \qquad \text{for all } e \in E^1.$$

Moreover, for the canonical gauge circle action γ on $C^*(E)$ we have

$$\gamma_z(a) = a, \qquad \gamma_z(au_\lambda) = zau_\lambda, \qquad \text{for all } a \in \mathcal{D}_E,\ z \in \mathbb{T}.$$

Thus applying Proposition 3.18 we see that $id \rtimes u_\lambda$ is an isomorphism. This proves the main part of the assertion.

i). Suppose now that E has no infinite emitters. Then (54) converges in norm by Proposition 5.3. Since

$$\mathcal{L}(\alpha(a)b) = \sum_{e,f\in E^1} \lambda_f s_f^* s_e a s_e^* b s_f = \sum_{e\in E^1} \lambda_e p_{s(e)} a s_e^* b s_e = a \sum_{e\in E^1} \lambda_e s_e^* b s_e = a\mathcal{L}(b),$$

for all a, $b \in \mathcal{D}_E$, the triple $(\mathcal{D}_E, \alpha, \mathcal{L})$ is an Exel system. Similar calculations show that

$$\alpha(\mathcal{L}(\alpha(a))) = \sum_{e\in E^1} \left(\sum_{f\in s^{-1}(s(e))} \lambda_f \right) s_e a s_e^*.$$

Hence, in view of Proposition 4.13, $\mathcal{L}$ is a regular transfer operator for α if and only if (51) holds. The crossed products $C^*(\mathcal{D}_E, \mathcal{L})$ and $\mathcal{D}_E \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ coincide by Lemma 5.9 and Theorem 4.7.

ii). Suppose α is an endomorphism such that $(\mathcal{D}_E, \alpha, \mathcal{L})$ is an Exel system. Putting $b = s_e s_e^*$, $e \in E^1$, in the equation $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$ we get $s_e^*\alpha(a)s_e = a s_e^* s_e$. This in turn implies that

$$\alpha(a) s_e s_e^* = s_e a s_e^*, \qquad e \in E^1.$$

Lack of infinite receivers in E implies that the projections $s_e s_e^*$ sum up strictly to a projection in $M(C^*(E))$. Let us denote it by p. It follows that $\alpha(a)p = \sum_{e\in E^1} s_e a s_e^*$ is in $\mathcal{D}_E$ for any $a \in \mathcal{D}_E$. If there would be an infinite emitter $v \in E^0$, then $\alpha(p_v)p = \sum_{e\in s^{-1}(v)} s_e s_e^*$ would not be an element of $\mathcal{D}_E$ (otherwise it would correspond via the isomorphism $\mathcal{D}_E \cong C_0(\partial E)$ to a characteristic function of a non-compact set). Thus E must be locally finite. Furthermore, in view of Lemma 5.8, we have $p\mathcal{D}_E = N_{\mathcal{L}}^{\perp}$. Therefore if $\alpha(\mathcal{D}_E)\mathcal{D}_E \subseteq N_{\mathcal{L}}^{\perp}$ then α has to be given by (54).

Item iii) follows from item ii) because for a locally finite graph without sources we have $N_{\mathcal{L}}^{\perp} = \mathcal{D}_E$ by Lemma 5.8. $\square$

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