Advances in the theory of crossed products by endomorphisms

Bartosz Kosma Kwaśniewski
Univeristy of Southern Denmark, Odense

1. Crossed products by endomorphisms (ideal structure)
2. Crossed products by c.p. maps (Exel's crossed product)
3. Two model examples

Plan

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2. Crossed products by c.p. maps (Exel's crossed product)
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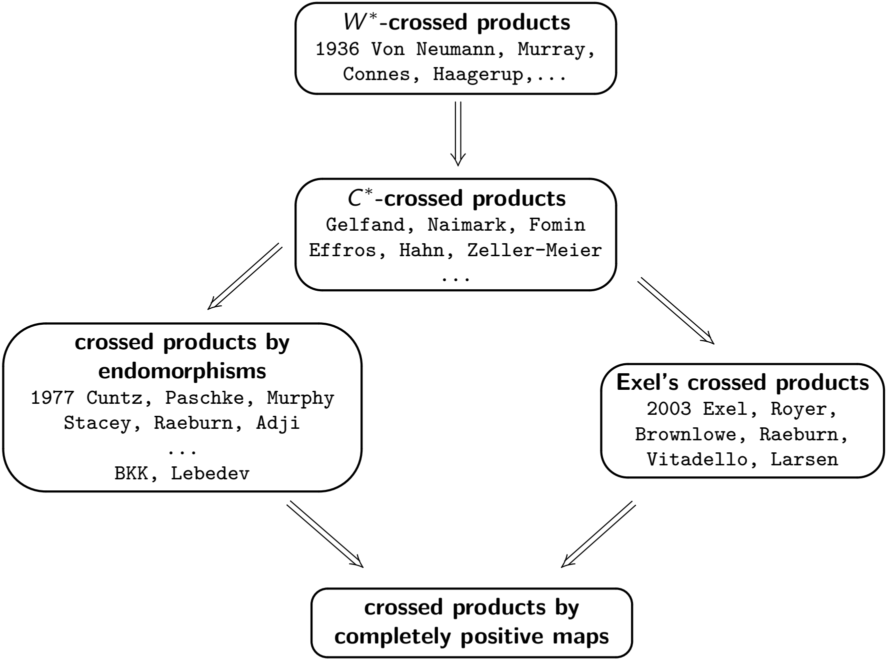

W^*-crossed products
1936 Von Neumann, Murray,
Connes, Haagerup,...
C^*-crossed products
Gelfand, Naimark, Fomin
Effros, Hahn, Zeller-Meier
...
crossed products by endomorphisms
1977 Cuntz, Paschke, Murphy
Stacey, Raeburn, Adji
...
BKK, Lebedev
Exel's crossed products
2003 Exel, Royer,
Brownlowe, Raeburn,
Vitadello, Larsen
crossed products by completely positive maps

What is crossed product by an endomorphism?

Throughout A is unital C^*-algebra.

Crossed product by an automorphism $\alpha : A \to A$ is a universal C^*-algebra $C^*(A, u)$ generated by A and u subject to relations:

$$\alpha(a) = uau^*, \qquad \alpha^{-1}(a) = u^*au, \qquad a \in A$$

Problem If $\alpha : A \to A$ is an endomorphism, then ~~$\alpha^{-1}(a) = u^*au$~~.

What relation should we use instead?

Let $A \subset B$ be C^*-algebras with a common unit 1, $U \in B$.

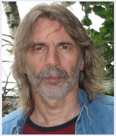
Proposition (the Hint).

Let $\alpha : A \to A$ be a map of the form $\alpha(a) = UaU^*$. Then

$$\alpha \text{ is an endomorphism} \iff U \text{ is a partial isometry, } U^*U \in A'$$
$$\iff U \text{ is a partial isometry and } Ua = \alpha(a)U, \ a \in A.$$

Def. A pair (π, U) is a **representation** of (A, α) in a C^*-algebra B if

$\pi : A \to B$ is a unital homomorphism, $U \in B$ and



$$U\pi(a)U^* = \pi(\alpha(a)), \qquad a \in A.$$

Let $J \triangleleft A$. We say that (π, U) is a J-**covariant representation** if

$$J \subseteq \{a \in A : U^*U\pi(a) = \pi(a)\}.$$

Prop. There is a J-covariant represen. (ι, u) in a C^*-algebra $C^*(A, \alpha; J)$ s.t.:

a) $C^*(A, \alpha; J)$ is generated by $\iota(A)$ and u,

b) for every J-covariant representation (π, U) there is a homomorphism of $\pi \rtimes U$ of $C^*(A, \alpha; J)$ given by $(\pi \rtimes U) \circ \iota = \pi$ and $(\pi \rtimes U)(u) = U$.

Moreover, ι is injective if and only if $J \subseteq (\ker \alpha)^\perp$.

Def. We call $C^*(A, \alpha; J)$ the **relative crossed product** of (A, α) relative to J.

We define the **crossed product** by putting $C^*(A, \alpha) := C^*(A, \alpha; (\ker \alpha)^\perp)$.

Remark.

1. $C^*(A,\alpha)$ is the (unrelative) **crossed product**

$$u \text{ is an isometry} \iff \alpha \text{ is a monomorphism}$$

2. $C^*(A,\alpha;A)$ is **Stacey's crossed product** (u is always an isometry)

$$A \text{ embeds into } C^*(A,\alpha;A) \iff \alpha \text{ is a monomorphism}$$

3. $C^*(A,\alpha;\{0\})$ **'Toeplitz' crossed product** (u is never an isometry) studied by Raeburn et al.

Remark.

For any (A,α) and J there is a canonical endomorphism $\alpha_J : A_J \to A_J$ s.t.:

- $C^*(A,\alpha;J) \cong C^*(A_J,\alpha_J)$
- $\ker \alpha_J$ is a complemented ideal in A_J

From now on we assume that $\ker\alpha$ is a complemented ideal.

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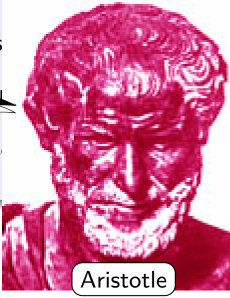
$$\{0\} \subseteq (\ker\alpha)^{\perp} \subseteq A$$



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Ideal structure

Def. We say that $I \triangleleft A$ is **invariant** if $(\ker \alpha)^{\perp} \cap \alpha^{-1}(I) = I \cap (\ker \alpha)^{\perp}$.

If I is invariant then $\alpha(I) \subseteq I$ and we have the restricted $\alpha|_I : I \to I$ and the quotinet $\alpha_I : A/I \to A/I$ endomorphism.

Theorem. Equality $I = A \cap \mathcal{I}$ yields a bijection between

invariant ideals I in A and gauge-invariant ideals $\mathcal{I}$ in $C^*(A, \alpha)$. Moreover:

$$C^*(A, \alpha)/\mathcal{I} \cong C^*(A/I, \alpha_I)$$

and $\mathcal{I}$ is Morita-Rieffel equivalent to $C^*(I, \alpha|_I)$.

Theorem. $C^*(A, \alpha)$ is simple if and only if

there are no non-trvial invariant ideals in A and either

i) α is pointwise quasinilpotent $(\forall_{a \in A} \ \alpha^n(a) \to 0)$ or

ii) α is injective and no power α^n, $n > 0$, is inner.

Endomorphisms of $C(X)$-algebras

Suppose that $X := \mathbf{Prim}(A)$ is Hausdorff and $\alpha(Z(A)) \subseteq Z(A)\alpha(1)$.

Then $Z(A) \cong C(X)$ and we may treat A as a section algebra of the bundle $\bigsqcup_{x\in X} A(x)$, where $A(x) := A/x$, $x \in X$. Then we have

$$\alpha(a)(x) = \begin{cases} \alpha_x(a(\varphi(x)), & x \in \Delta, \\ 0, & x \notin \Delta, \end{cases} \qquad a \in A, \; x \in X.$$

where

1) $\varphi : \Delta \to X$ continuous proper map, $\Delta \subset X$ is open,
2) $\{\alpha_x\}_{x\in\Delta}$ 'continuous' bundle of homomorphisms $\alpha_x : A(\varphi(x)) \to A(x)$.

Theorem.

- If φ is topologically free, for every covariant representation (π, U), with π injective, the representation $\pi \rtimes U$ of $C^*(A, \alpha)$ is faithful,
- If φ is free, then we have a bijective correspondence

$$A \cap \mathcal{I} = \{a \in A : a(x) = 0 \text{ for all } x \in V\}$$

  between ideals $\mathcal{I} \triangleleft C^*(A, \alpha)$ and closed sets V s.t. $\varphi(\Delta \cap V) = \varphi(\Delta) \cap V$.

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Exel's crossed product

Def. A **transfer operator** for an endomorphism $\alpha : A \to A$ is a positive linear map $\mathcal{L} : A \to A$ such that $\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b$, $a, b \in A$.

Ex. $\sigma : M \to M$ a local homeomorphism on a compact Hausdorff M. The standard transfer operator for $\alpha(a) = a \circ \sigma$, $a \in A := C(M)$, is

$$\mathcal{L}(a)(x) = \frac{1}{|\sigma^{-1}(x)|} \sum_{y \in \sigma^{-1}(x)} a(y).$$

Any transfer operator is of the form $\mathcal{L}_\rho(a)(x) = \sum_{y \in \sigma^{-1}(x)} \rho(y)a(y)$ where $\rho : M \to [0, \infty)$ is continuous.

Exel's crossed product

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Def. (π, S) is a representation of an Exel system $(A, \alpha, \mathcal{L})$ in B if

$\pi : A \to B$ is a unital homomorphism, $S \in B$ and

~~$S\pi(a) = \pi(\alpha(a))S$~~, $\qquad S^*\pi(a)S = \pi(\mathcal{L}(a)) \qquad$ for all $a \in A$.

Let $\mathcal{T}(A, \cancel{\alpha}, \mathcal{L}) := C^*(\iota(A) \cup \{s\})$ where (ι, s) is the universal representation of $(A, \cancel{\alpha}, \mathcal{L})$. **Exel's crossed product** $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ is the quotient of $\mathcal{T}(A, \cancel{\alpha}, \mathcal{L})$ by the ideal generated by

$$\{\iota(a) - k : a \in \overline{A\alpha(A)A} \text{ and } (\iota(a), k) \text{ is a redundancy}\}.$$

Rem. A transfer operator is a c.p. map; for $a_i, b_i \in A$ we have

$$\sum_{i,j=1}^{n} b_i^* \mathcal{L}(a_i^* a_j) b_j = \mathcal{L}\Big(\big(\sum_{i=1}^{n} a_i\alpha(b_i)\big)^* \big(\sum_{j=1}^{n} a_j\alpha(b_j)\big)\Big) \geq 0.$$

Let $\varrho : A \to A$ be a linear completely positive map (c.p. map)

Def. Representation of (A, ϱ) in B is (π, S) where

$\pi : A \to B$ is a unital homomorphism, $S \in B$ and

$$S^*\pi(a)S = \pi(\varrho(a)) \qquad \text{for all } a \in A.$$

Toeplitz algebra of (A, ϱ) is C^*-algebra $\mathcal{T}(A, \varrho) := C^*(\iota(A) \cup s)$ generated by the universal representation (ι, s) of (A, ϱ).

Def. Redundancy is a pair $(\iota(a), k)$ where

$a \in A$, $k \in \overline{\iota(A)s\iota(A)s^*\iota(A)}$ and $\iota(a)\iota(b)s = k\iota(b)s$ for all $b \in A$.

Def. GNS-kernel of ϱ is $N_\varrho := \{a \in A : \varrho((ab)^*ab) = 0 \text{ for all } b \in A\}$

Rem. If $\varrho = \alpha$ is multiplicative, then $N_\varrho = \ker \alpha$ and

$$(\iota(a), k) \text{ is a redundancy } \iff k = ss^*\iota(a)$$

Def. The **crossed product** $C^*(A, \varrho)$ is the quotient of $\mathcal{T}(A, \varrho)$ by the ideal generated by

$$\{\iota(a) - k : a \in N_\varrho^\perp \text{ and } (\iota(a), k) \text{ is a redundancy}\}.$$

More generally, for $J \trianglelefteq A$ we define $C^*(A, \varrho; J)$ similarly but with

$$\{\iota(a) - k : a \in J \text{ and } (\iota(a), k) \text{ is a redundancy}\}.$$

Corollary.

- If $\varrho = \alpha$ is multiplicative, then (assuming $u = s^*$)

$$C^*(A, \alpha; J) = C^*(A, \rho; J), \qquad C^*(A, \alpha) = C^*(A, \rho).$$

- If $\varrho = \mathcal{L}$ is a transfer operator for an endomorphism α, then

$$A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \rho; J), \qquad \mathcal{O}(A, \alpha, \mathcal{L}) = C^*(A, \rho)$$

  where $J = A\alpha(A)A$ and $\mathcal{O}(A, \alpha, \mathcal{L})$ is an adjusted crossed product (Exel and Royer 2007)

Thm. For any $J \trianglelefteq A$ we have

$$C^*(A, \varrho; J) \cong \mathcal{O}(X_\varrho, J \cap J(X_\varrho)), \qquad C^*(A, \varrho) \cong \mathcal{O}_{X_\varrho}$$

where X_ϱ is the **GNS** C^***-correspondence**, i.e. Hausdorff completion of the algebraic tensor product $A \otimes A$ with $\langle a \otimes b, c \otimes d \rangle_A := b^* \varrho(a^* c) d$, where $a \cdot (b \otimes c) \cdot d := (ab) \otimes (cd)$ for $a, b, c, d \in A$.

Ex If $A = C(V)$ commutative, then $X_\varrho \sim (V, E, \mu = \{\mu_x\}_{x \in V})$ where

$$\varrho(a)(x) = \int_V a(y) d\mu_x(y), \qquad x \in V, a \in A,$$

and

$$E = \bigcup_{x \in V} \mathrm{supp}\mu_x \times \{x\} \subseteq \begin{array}{ccc} V & \times & V \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle r} \\ V & & V \end{array}$$

In special cases (V, E, μ) is:
a topological relation (Brenken 2004),
a topological quiver (Muhly, Tomoforde 2005),
a ‘Markov operator’ (Ionescu, Muhly, Vega 2012).

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Example 1 (z^2-mapping on $\mathbb{T}$)

Let $H = L_2(\mathbb{T})$, $A \subset L(H)$ consists of operators of multiplication by continuous functions:

$$A \cong C(\mathbb{T}).$$

Consider the isometry $S \in L(H)$:

$$(Sf)(z) = f(z^2), \qquad (S^*f)(z) = \frac{1}{2}\sum_{w^2=z} f(w).$$

$$a \in A \Longrightarrow \begin{cases} SaS^* \text{ - is not an operator of multiplication} & \notin A \\ S^*aS \text{ - operator of multiplication by } \frac{1}{2}\sum_{w^2=z} a(w) & \in A \end{cases}$$

Hence $\mathcal{L}(a) := S^*aS$ is a positive map on A where

$$\mathcal{L}(a)(z) = \frac{1}{2}\sum_{w^2=z} a(w), \qquad a \in A \cong C(\mathbb{T}),\ z \in \mathbb{T}.$$

Prop. $C^*(A \cup \{S\}) \cong C^*(A, \mathcal{L})$ - crossed product by a c.p. map

Also $C^*(A \cup \{S\}) \cong A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ - Exel's crossed product

$$\alpha(a)(z) = a(z^2), \qquad a \in \mathcal{A} \cong C(\mathbb{T}),\ z \in \mathbb{T}.$$

Example 2 (z^2-mapping on $\mathbb{T}$)

Let $\mathcal{H} = L_2(\mathbb{R})$, $A \subset L(\mathcal{H})$ consists of operators of multiplication by continuous periodic functions with period 1:

$$A \cong C(\mathbb{T}).$$

Consider unitary operator $U \in L(\mathcal{H})$

$$(Uf)(x) = \sqrt{2}\, f(2x), \qquad (U^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right).$$

$$a \in A \Longrightarrow \begin{cases} UaU^* \text{ - operator of multiplication by } a(2x) & \in A \\ U^* aU \text{ - operator of multiplication by } a\left(\frac{x}{2}\right) & \notin A \end{cases}$$

Hence $\alpha(a) := UaU^*$ is an endomorphism of $\mathcal{A}$ where

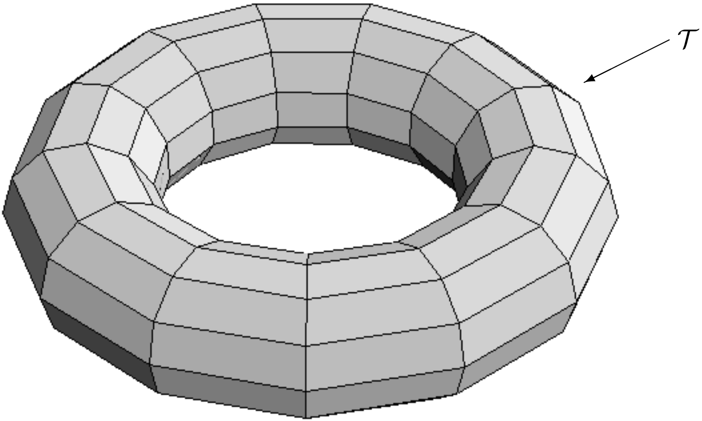
$$\alpha(a)(z) = a(z^2), \qquad a \in A \cong C(\mathbb{T}),\ z \in \mathbb{T}.$$

Prop. $C^*(A \cup \{U\}) \cong C^*(A, \alpha)$ - crossed product by an endomorphism

Also $C^*(A \cup \{U\}) \cong B \rtimes_\beta \mathbb{Z}$ - crossed product by an automorphism

$$B := C^*(\bigcup_{n=0}^{\infty} U^{*n} A U^n), \qquad \beta(b) := UbU^*, \qquad \beta^{-1}(b) = U^* bU.$$

Algebra $B := C^*(\bigcup_{n=0}^{\infty} U^{*n}AU^n)$ is commutative. Its spectrum is: Smale's Solenoid $\bigcap_{n\in\mathbb{N}} F^n(\mathcal{T})$ where $F : \mathcal{T} \to \mathcal{T}$ acts as follows



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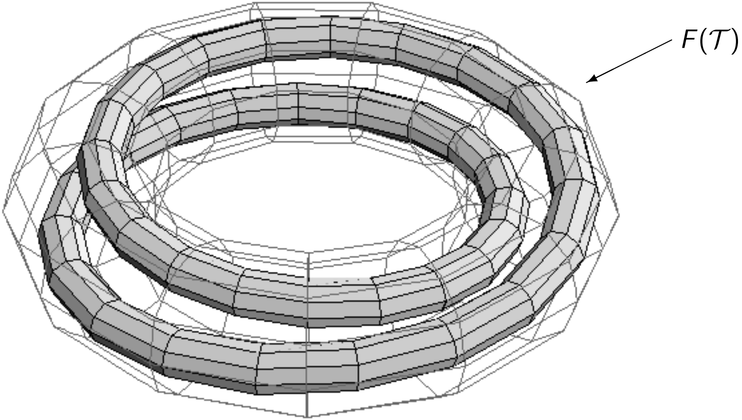

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