Topological aperiodicity for product systems of C^*-correspondences

Bartosz Kwaśniewski, IMADA, Odense /IM UwB

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based on joint work with Wojciech Szymański, IMADA, Odense

Universal C^*-algebras and uniqueness problem

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Def. The universal C^*-algebra generated by $\mathcal{G}$ subject to $\mathcal{R}$ is a C^*-algebra $C^*(\mathcal{G}, \mathcal{R}) := C^*(\iota(\mathcal{G}))$ where ι is a representation of $(\mathcal{G}, \mathcal{R})$ such that if π is a representation of $(\mathcal{G}, \mathcal{R})$ then

$$\iota(g) \longmapsto \pi(g), \qquad g \in \mathcal{G},$$

extends to an epimorphism $C^*(\mathcal{G}, \mathcal{R}) \to C^*(\pi(\mathcal{G}))$.

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Def. $(\mathcal{G}, \mathcal{R})$ has uniqueness property if for any two faithful representations π_1, π_2 of $(\mathcal{G}, \mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

extends to $*$-isomorphism $*\text{-}Alg(\pi_1(\mathcal{G})) \cong *\text{-}Alg(\pi_2(\mathcal{G}))$, which in the presence of 'amenability' is equivalent to $C^*(\pi_1(\mathcal{G})) \cong C^*(\pi_2(\mathcal{G}))$.

Crossed products by group actions

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Fact.

If $A \rtimes_\alpha^r G$ is the reduced crossed product the natural surjection

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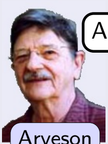

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Arveson

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$\mathcal{G} = \{a(f) : f \in H$ - Hilbert space$\}$, $\mathcal{R}$ - conj. linear structure of H plus

$$a(f)^* a(h) + a(h)a(f)^* = \langle f, h\rangle 1$$
$$a(f)a(h) + a(h)a(f) = 0$$

uniqueness property ✓ (P. Jordan & E. Wigner 1928, I. Segal 1963)

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$\mathcal{G} = \{W(f) : f \in H$ - Hilbert space$\}$, $\mathcal{R}$ - consists of

$$W(-f) = W(f)^*$$
$$W(f)W(h) = e^{-i\, Im\langle f,h\rangle} W(f+h)$$

uniqueness property ✓ (J. Sławny 1971)

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$\{A(i,j)\}_{i,j=1}^{n} \in \{0,1\}^{n\times n}$, S_i partial isometries with orthogonal ranges

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uniqueness property $\iff$ condition (I) (J. Cuntz, W. Krieger 1980)

Cuntz-Krieger uniqueness theorem industry

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Cuntz-Krieger algebras $\mathcal{O}_A$
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Exel, Vershik 2006
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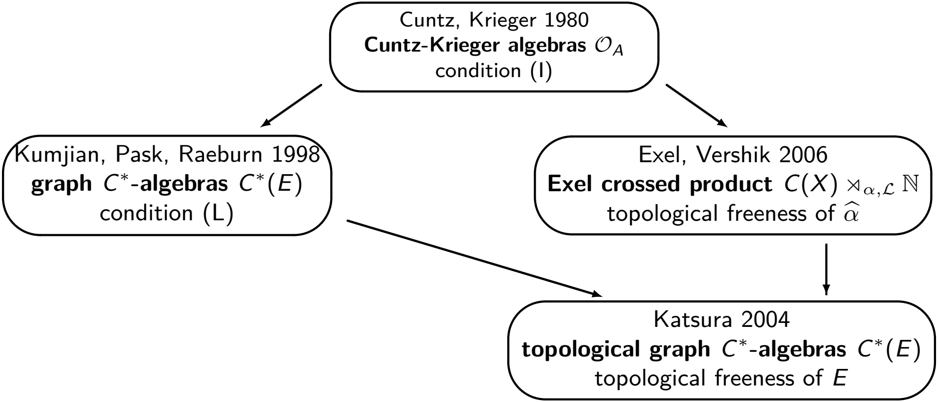

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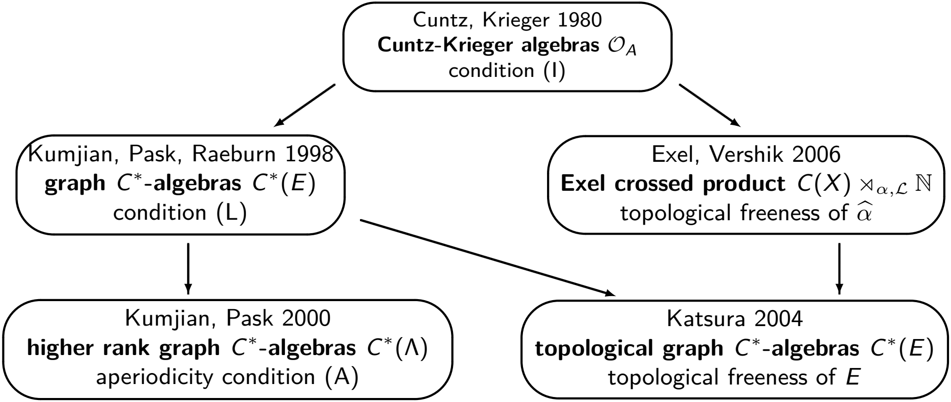

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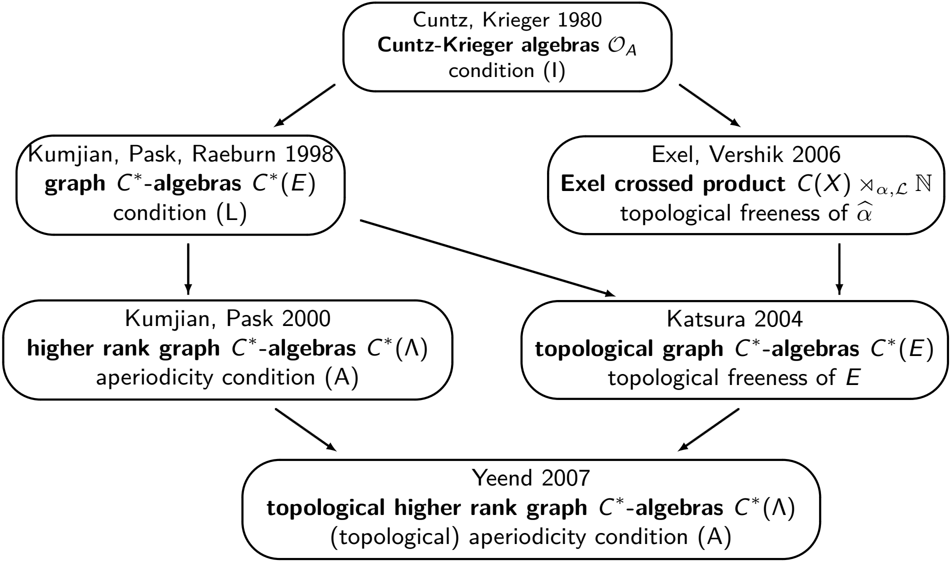

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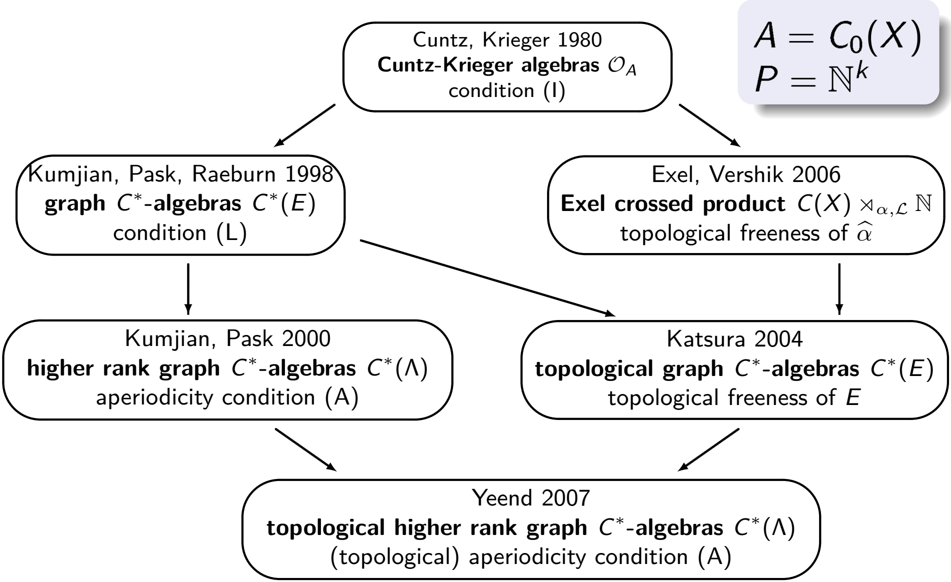

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$A = C_0(X)$
$P = \mathbb{N}^k$

Cuntz, Krieger 1980
Cuntz-Krieger algebras $\mathcal{O}_A$
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Kumjian, Pask 2000
higher rank graph C^***-algebras** $C^*(\Lambda)$
aperiodicity condition (A)

Katsura 2004
topological graph C^***-algebras** $C^*(E)$
topological freeness of E

Yeend 2007
topological higher rank graph C^***-algebras** $C^*(\Lambda)$
(topological) aperiodicity condition (A)

Regular C^*-correspondences and their product systems

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$$(\pi, \psi)^{(1)}(\phi(a)) = \pi(a), \qquad \text{for all } a \in A.$$

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Cuntz-Pimsner algebra is $\mathcal{O}_X := C^*(i_A(A) \cup i_X(X))$ where (i_A, i_X) is a covariant universal representation of X (M. Pimsner 1997)

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Cuntz-Pimsner algebra is $\mathcal{O}_X := C^*(i_X(X))$ where i_X is a universal covariant representation of the product system X (N. Fowler 2002)

Problem

In general the structure of $\mathcal{O}_X$ is not well understood!

Doplicher-Roberts picture of Cuntz-Pimsner algebra $\mathcal{O}_X$

Fix a regular product system X over a semigroup P.

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Assume P is an Ore semigroup and let $G = PP^{-1}$ be the group of fractions. Then $(P, \leq)$ is directed where

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Problem For which X, for any injective covariant representation ψ of X there is a (unique) epimorphism $\lambda_\psi : C^*(\psi(X)) \to \mathcal{O}_X^r$ such that:

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Thm. Suppose X is an imprimitivity $A-B$-bimodule

For $\pi : B \to \mathcal{B}(H)$ define $X\text{-Ind}(\pi) : A \to \mathcal{B}(X \otimes_\pi H)$ by

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Rieffel

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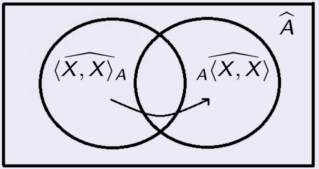

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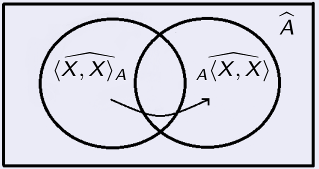

Thm. (Kwasniewski 2014)

If $[X\text{-Ind}]$ is topologically free, then $A \rtimes_X \mathbb{Z}$ possess uniqueness property

Multivalued maps dual to regular C^*-correspondences

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Def. Let $\alpha : A \to B$ be a $*$-homomorphism.

A *dual* to α is a multivalued map $\widehat{\alpha} : \widehat{B} \to \widehat{A}$ ($\widehat{\alpha} : \widehat{B} \to 2^{\widehat{A}}$) given by

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Def.

We define *dual map* $\widehat{X} : \widehat{A} \to \widehat{A}$ *to the regular* C^**-correspondence* X as the composition of multivalued maps

$$\widehat{X} = \widehat{\phi} \circ [X\text{-Ind}]$$

where $\widehat{\phi} : \widehat{\mathcal{K}(X)} \to \widehat{A}$ is dual to the left action $\phi : A \to \mathcal{K}(X)$ of A on X.

Prop. The family $\widehat{X} := \{\widehat{X}_p\}_{p\in P}$ is a semigroup of multivalued maps

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& \searrow & \underset{\lambda}{\longrightarrow} & \nearrow &
\end{array}
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Corollary (simplicity of $\mathcal{O}_X^r$)

Suppose that X is topologically aperiodic and *minimal*, i.e. there are no nontrivial ideals J in A such that

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Proof: $I \triangleleft \mathcal{O}_X^r$ implies $J := A \cap I$ is either A or $\{0\}$.

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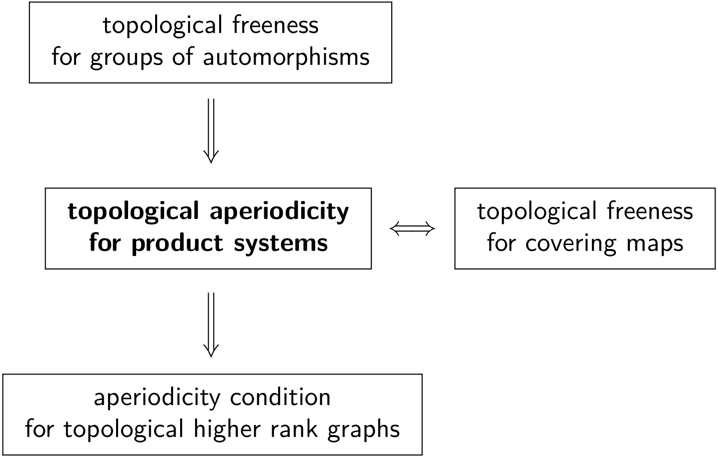

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