From dynamical systems and operator theory to operator algebras and back

Bartosz Kwaśniewski, IMADA, Odense
IMADA seminar, October 30, 2014

OperaDynaDual (ODD) project

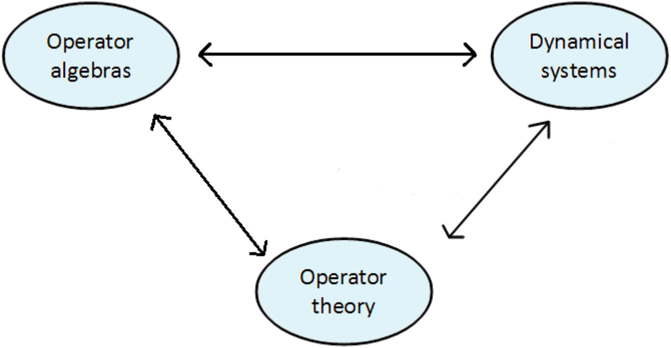
MARIE CURIE ACTIONS
SEVENTH FRAMEWORK PROGRAMME

From dynamical systems and operator theory to operator algebras and back

Bartosz Kwaśniewski, IMADA, Odense
IMADA seminar, October 30, 2014

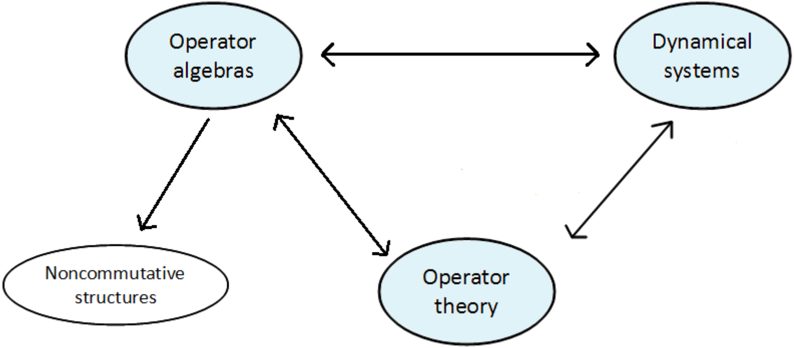
From dynamical systems and operator theory to operator algebras and back

Bartosz Kwaśniewski, IMADA, Odense
IMADA seminar, October 30, 2014



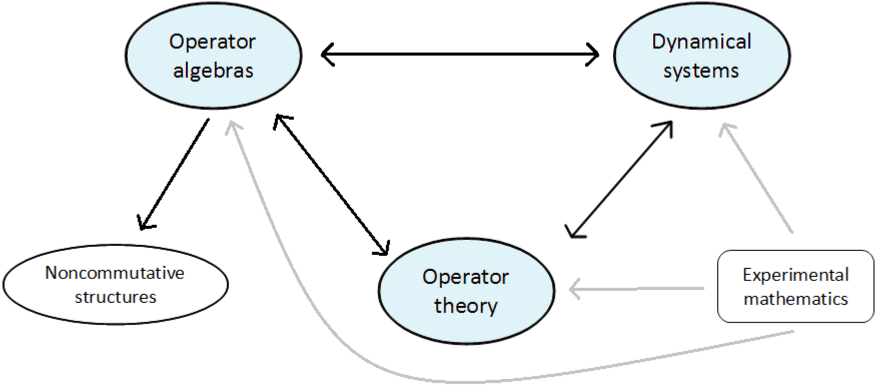
From dynamical systems and operator theory to operator algebras and back

Bartosz Kwaśniewski, IMADA, Odense
IMADA seminar, October 30, 2014



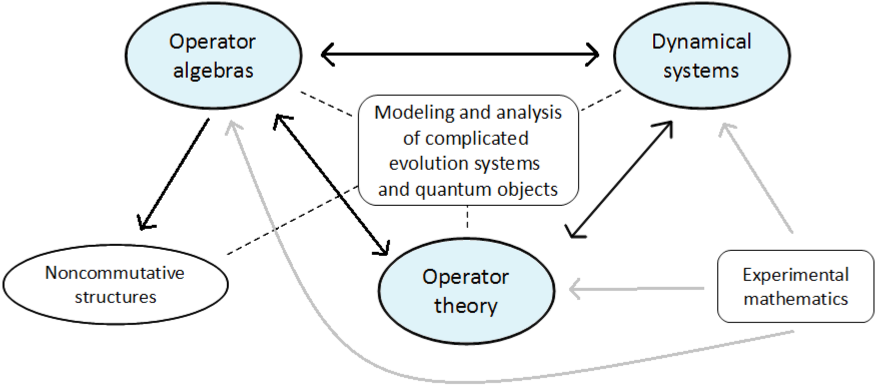
From dynamical systems and operator theory to operator algebras and back

Bartosz Kwaśniewski, IMADA, Odense
IMADA seminar, October 30, 2014



From dynamical systems and operator theory to operator algebras and back

Bartosz Kwaśniewski, IMADA, Odense
IMADA seminar, October 30, 2014



Preliminaries on bounded operators and C^*-algebras

H - Hilbert space

Preliminaries on bounded operators and C^*-algebras

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)

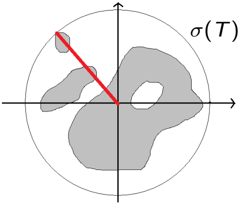
For $T \in B(H)$ one sets

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

Preliminaries on bounded operators and C^*-algebras

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)

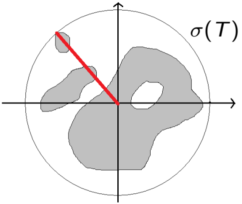


For $T \in B(H)$ one sets

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

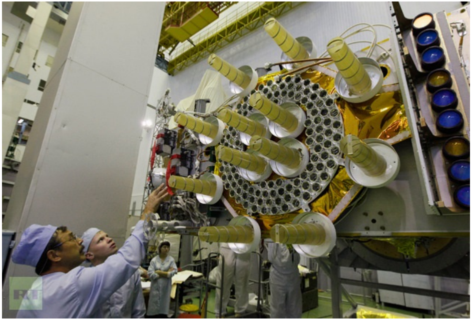
H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)



For $T \in B(H)$ one sets

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

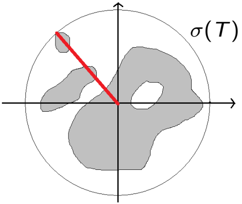


Preliminaries on bounded operators and C^*-algebras

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)

For $T \in B(H)$ one sets

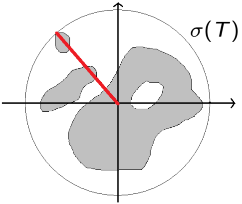


$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

Preliminaries on bounded operators and C^*-algebras

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)



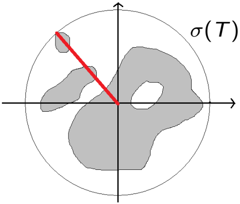
For $T \in B(H)$ one sets

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

Def. A closed $*$-subalgebra A of $B(H)$ is called a C^**-algebra*.

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)



For $T \in B(H)$ one sets

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

Def. A closed $*$-subalgebra A of $B(H)$ is called a *C^*-algebra*.

Thm. For any unital C^*-algebra $A \subseteq B(H)$ with $T \in A$ we have

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible in } A\}.$$

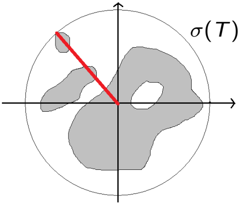
Moreover, $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

Preliminaries on bounded operators and C^*-algebras

H - Hilbert space

$B(H)$ - set of all bounded linear operators on H
(it is a normed $*$-algebra)

For $T \in B(H)$ one sets



$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible}\}, \qquad r(T) = \max_{\lambda \in \sigma(T)} |\lambda|$$

Def. A closed $*$-subalgebra A of $B(H)$ is called a C^**-algebra*.

Thm. For any unital C^*-algebra $A \subseteq B(H)$ with $T \in A$ we have

$$\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda 1 \text{ is not invertible in } A\}.$$

Moreover, $r(T) = \lim_{n\to\infty} \|T^n\|^{\frac{1}{n}}$.

Conclusion.

Any C^*-algebra containing T carries fundamental spectral information on T

Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions

T - operator of composition with ϕ

Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions

T - operator of composition with ϕ

A - algebra of operators of multiplication by functions from $C(X)$

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions

T - operator of composition with ϕ

A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

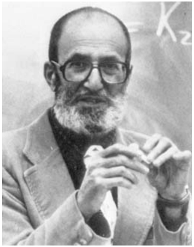
Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.
$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions
T - operator of composition with ϕ
A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$



$$\sigma(aT) = ?$$

P. R. Halmos „Ten Problems in Hilbert space" 1970

Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions

T - operator of composition with ϕ

A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions

T - operator of composition with ϕ

A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

$\alpha(a) := TaT^* = a \circ \phi$ is an automorphism of the C^*-algebra A

X - compact Hausdorff space equipped with probability borel measure μ.
$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions
T - operator of composition with ϕ
A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

$\alpha(a) := TaT^* = a \circ \phi$ is an automorphism of the C^*-algebra A

$$r(aT) = \lim_{n\to\infty} \|(aT)^n\|^{1/n} = \lim_{n\to\infty} \|a\alpha(a)...\alpha^n(a)\|^{1/n}$$

Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.

$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions

T - operator of composition with ϕ

A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

$\alpha(a) := TaT^* = a \circ \phi$ is an automorphism of the C^*-algebra A

$$r(aT) = \lim_{n\to\infty} \|(aT)^n\|^{1/n} = \lim_{n\to\infty} \|a\alpha(a)...\alpha^n(a)\|^{1/n}$$

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) = A \rtimes_\alpha \mathbb{Z}$.

Dynamics —> Operators —> C^*-algebras

X - compact Hausdorff space equipped with probability borel measure μ.
$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X)$ - Hilbert space of square integrable functions
T - operator of composition with ϕ
A - algebra of operators of multiplication by functions from $C(X)$

$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

$\alpha(a) := TaT^* = a \circ \phi$ is an automorphism of the C^*-algebra A

$$r(aT) = \lim_{n\to\infty} \|(aT)^n\|^{1/n} = \lim_{n\to\infty} \|a\alpha(a)...\alpha^n(a)\|^{1/n}$$

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) = A \rtimes_\alpha \mathbb{Z}$.

$\sigma(aT)$ may be described in terms of the C^*-dynamical system (A, α).

Dynamics —> Operators —> C^*-algebras (A noncommutative)

X - compact Hausdorff space equipped with probability borel measure μ.
$\phi : X \to X$ - measure preserving homeomorphism

$H := L^2_\mu(X, \mathbb{C}^n)$ -Hilbert space of vector-valued functions
T - operator of composition with ϕ
A - algebra of operators of multiplication by functions from $C(X, M_n(\mathbb{C}))$
$aT : H \to H$, $a \in A$ - weighted composition operator:

$$(aTf)(x) = a(x)f(\phi(x)), \qquad f \in H.$$

$\alpha(a) := TaT^* = a \circ \phi$ is an automorphism of the C^*-algebra A

$$r(aT) = \lim_{n\to\infty} \|(aT)^n\|^{1/n} = \lim_{n\to\infty} \|a\alpha(a)...\alpha^n(a)\|^{1/n}$$

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) = A \rtimes_\alpha \mathbb{Z}$.

$\sigma(aT)$ may be described in terms of the C^*-dynamical system (A, α).

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

We call aT an *abstract weighted shift* with weight $a \in A$.

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

We call aT an *abstract weighted shift* with weight $a \in A$.

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

We call aT an *abstract weighted shift* with weight $a \in A$.

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space (Gelfand-Naimark thm.)

Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

We call aT an *abstract weighted shift* with weight $a \in A$.

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space (Gelfand-Naimark thm.)
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

We call aT an *abstract weighted shift* with weight $a \in A$.

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space (Gelfand-Naimark thm.)
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

Thm. (Variational principle; Kitover, Lebedev 1979**)** We have

$$r(aT) = \max_{\nu \in Erg(X,\phi)} \exp \int_X \ln |a(x)| d\nu$$

where $Erg(X, \phi)$ is the set of all probability invariant ergodic measures.

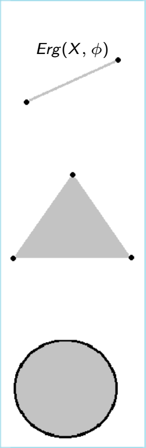
Axiomatic approach (A commutative) Antonevich, Lebedev, 1994

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

We call aT an *abstract weighted shift* with weight $a \in A$.



Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space (Gelfand-Naimark thm.)
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

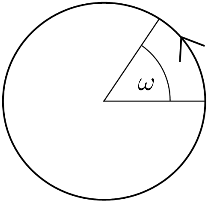
Thm. (Variational principle; Kitover, Lebedev 1979**)** We have

$$r(aT) = \max_{\nu \in Erg(X,\phi)} \exp \int_X \ln |a(x)| d\nu$$

where $Erg(X, \phi)$ is the set of all probability invariant ergodic measures.

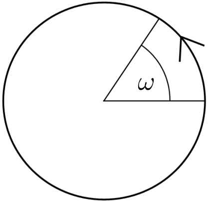
Example: rotation of the circle

Let $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\phi(z) = z \cdot e^{2\pi i \omega}$ where $\omega \in [0, 1)$



Example: rotation of the circle

Let $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\phi(z) = z \cdot e^{2\pi i \omega}$ where $\omega \in [0, 1)$

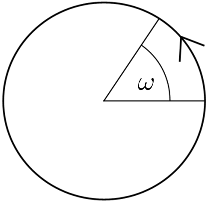


1) If $\omega = \frac{m}{n} \in \mathbb{Q}$ then

$$r(aT) = \max_{z \in S^1} \sqrt[n]{|a(z) \cdot a(\phi(z)) \cdot \ldots \cdot a(\phi^{n-1}(z))|}$$

Example: rotation of the circle

Let $X = S^1 = \{z \in \mathbb{C} : |z| = 1\}$ and $\phi(z) = z \cdot e^{2\pi i \omega}$ where $\omega \in [0, 1)$



1) If $\omega = \frac{m}{n} \in \mathbb{Q}$ then

$$r(aT) = \max_{z \in S^1} \sqrt[n]{|a(z) \cdot a(\phi(z)) \cdot \ldots \cdot a(\phi^{n-1}(z))|}$$

2) If $\omega \notin \mathbb{Q}$ then $Erg(X, \phi) = \{m\}$ - normalized Lebesgue measure on S^1

$$r(aT) = \exp \int_{S^1} \ln |a(z)| \, dm(z)$$

Problem

In general the simplex of invariant measures is very complicated!

Problem

In general the simplex of invariant measures is very complicated!

If μ is probability invariant measure then

$$r(aT) \geq \exp \int_X \ln |a(x)| \, d\mu$$

The role of uniqueness theorem

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

The role of uniqueness theorem

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

Uniqueness theorem (Arveson, O'Donovan 1975)

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) \cong A \rtimes_\alpha \mathbb{Z}$.

The role of uniqueness theorem

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

Uniqueness theorem (Arveson, O'Donovan 1975)

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) \cong A \rtimes_\alpha \mathbb{Z}$. In particular

$$a \mapsto a, \qquad T \mapsto zT$$

defines an automorphism γ_z of $C^*(\{aT : a \in A\})$ for any $z \in S^1$.

The role of uniqueness theorem

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$

Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

Uniqueness theorem (Arveson, O'Donovan 1975)

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) \cong A \rtimes_\alpha \mathbb{Z}$. In particular

$$a \mapsto a, \qquad T \mapsto zT$$

defines an automorphism γ_z of $C^*(\{aT : a \in A\})$ for any $z \in S^1$.

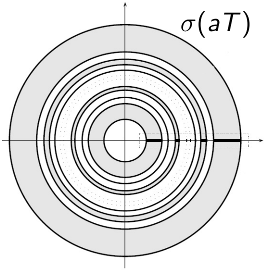
Cor. If ϕ is topologically free $\sigma(aT)$ has a circular symmetry

The role of uniqueness theorem

T - unitary operator on a Hilbert space H, $T \in B(H)$

A - commutative C^*-algebra, $A \subseteq B(H)$, $1 \in A$

$$TAT^* \subseteq A, \qquad T^*AT \subseteq A$$



Remarks

1) $\alpha(a) := TaT^*$ is an automorphism of A.
2) $A \cong C(X)$, X compact Hausdorff space
3) $\alpha(a) = a \circ \phi$ where $\phi : X \to X$ a homeomorphism.

Uniqueness theorem (Arveson, O'Donovan 1975)

If ϕ is topologically free, then $C^*(\{aT : a \in A\}) \cong A \rtimes_\alpha \mathbb{Z}$. In particular

$$a \mapsto a, \qquad T \mapsto zT$$

defines an automorphism γ_z of $C^*(\{aT : a \in A\})$ for any $z \in S^1$.

Cor. If ϕ is topologically free $\sigma(aT)$ has a circular symmetry

Proof: $aT - \lambda 1$ is invertible $\Longleftrightarrow$ $\gamma_{\overline{z}}(aT - \lambda 1) = \overline{z}(aT - \lambda z 1)$ is, $z \in S^1$.

Universal C^*-algebras and uniqueness problem

$\mathcal{G}$ - set of generators, $\mathcal{R}$ - C^*-algebraic relations on $\mathcal{G}$

Universal C^*-algebras and uniqueness problem

$\mathcal{G}$ - set of generators, $\mathcal{R}$ - C^*-algebraic relations on $\mathcal{G}$

Def. Representation of $(\mathcal{G}, \mathcal{R})$ in a C^*-algebra A is $\pi = \{\pi(g)\}_{g \in \mathcal{G}} \subseteq A$ satisfying the relations $\mathcal{R}$ in A. If $\pi(g) \neq 0$ for all $g \in \mathcal{G}$, π is faithful.

Universal C^*-algebras and uniqueness problem

$\mathcal{G}$ - set of generators, $\mathcal{R}$ - C^*-algebraic relations on $\mathcal{G}$

Def. Representation of $(\mathcal{G}, \mathcal{R})$ in a C^*-algebra A is $\pi = \{\pi(g)\}_{g \in \mathcal{G}} \subseteq A$ satisfying the relations $\mathcal{R}$ in A. If $\pi(g) \neq 0$ for all $g \in \mathcal{G}$, π is faithful.

Def. The universal C^*-algebra generated by $\mathcal{G}$ subject to $\mathcal{R}$ is a C^*-algebra $C^*(\mathcal{G}, \mathcal{R}) := C^*(\iota(\mathcal{G}))$ where ι is a representation of $(\mathcal{G}, \mathcal{R})$ such that if π is a representation of $(\mathcal{G}, \mathcal{R})$ then

$$\iota(g) \longmapsto \pi(g), \qquad g \in \mathcal{G},$$

extends to an epimorphism $\Psi_\pi : C^*(\mathcal{G}, \mathcal{R}) \to C^*(\pi(\mathcal{G}))$.

Universal C^*-algebras and uniqueness problem

$\mathcal{G}$ - set of generators, $\mathcal{R}$ - C^*-algebraic relations on $\mathcal{G}$

Def. Representation of $(\mathcal{G}, \mathcal{R})$ in a C^*-algebra A is $\pi = \{\pi(g)\}_{g \in \mathcal{G}} \subseteq A$ satisfying the relations $\mathcal{R}$ in A. If $\pi(g) \neq 0$ for all $g \in \mathcal{G}$, π is faithful.

Def. The universal C^*-algebra generated by $\mathcal{G}$ subject to $\mathcal{R}$ is a C^*-algebra $C^*(\mathcal{G}, \mathcal{R}) := C^*(\iota(\mathcal{G}))$ where ι is a representation of $(\mathcal{G}, \mathcal{R})$ such that if π is a representation of $(\mathcal{G}, \mathcal{R})$ then

$$\iota(g) \longmapsto \pi(g), \qquad g \in \mathcal{G},$$

extends to an epimorphism $\Psi_\pi : C^*(\mathcal{G}, \mathcal{R}) \to C^*(\pi(\mathcal{G}))$.

Def. $(\mathcal{G}, \mathcal{R})$ has uniqueness property if for any two faithful representations π_1, π_2 of $(\mathcal{G}, \mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

extends to isomorphism $C^*(\pi_1(\mathcal{G})) \cong C^*(\pi_2(\mathcal{G}))$.

Classical examples: quantum statistics

Classical examples: quantum statistics

canonical anticommutation relations (CAR) algebra

$\mathcal{G} = \{a(f) : f \in H$ - Hilbert space$\}$, $\mathcal{R}$ - conj. linear structure of H plus

$$a(f)^* a(h) + a(h)a(f)^* = \langle f, h\rangle 1$$
$$a(f)a(h) + a(h)a(f) = 0$$

uniqueness property ✓ (P. Jordan & E. Wigner 1928, I. Segal 1963)

Classical examples: quantum statistics

canonical anticommutation relations (CAR) algebra

$\mathcal{G} = \{a(f) : f \in H$ - Hilbert space$\}$, $\mathcal{R}$ - conj. linear structure of H plus

$$a(f)^* a(h) + a(h)a(f)^* = \langle f, h\rangle 1$$
$$a(f)a(h) + a(h)a(f) = 0$$

uniqueness property ✓ (P. Jordan & E. Wigner 1928, I. Segal 1963)

canonical commutation relations (CCR) algebra

$\mathcal{G} = \{W(f) : f \in H$ - Hilbert space$\}$, $\mathcal{R}$ - consists of

$$W(-f) = W(f)^*$$
$$W(f)W(h) = e^{-i\, Im\langle f,h\rangle} W(f+h)$$

uniqueness property ✓ (J. Sławny 1971)

Unitary algebra $C^*(S)$

$$S^*S = SS^* = 1$$

no uniqueness property

Unitary algebra $C^*(S)$

$$S^*S = SS^* = 1$$

no uniqueness property

Rotation algebras $\mathcal{A}_\theta = C^*(S, T)$, $\theta \in \mathbb{R}$

S, T unitary, $ST = e^{2\pi i\theta} TS$

uniqueness property $\iff \theta \notin \mathbb{Q}$

Unitary algebra $C^*(S)$

$$S^*S = SS^* = 1$$

no uniqueness property

Rotation algebras $\mathcal{A}_\theta = C^*(S, T)$, $\theta \in \mathbb{R}$

S, T unitary, $ST = e^{2\pi i\theta} TS$

uniqueness property $\Longleftrightarrow \theta \notin \mathbb{Q}$

Toeplitz algebra $= C^*(S)$

$$S^*S = 1, \qquad SS^* \neq 1$$

uniqueness property $\checkmark$

(L. A. Coburn 1969)

Unitary algebra $C^*(S)$

$$S^*S = SS^* = 1$$

no uniqueness property

Rotation algebras $\mathcal{A}_\theta = C^*(S, T)$, $\theta \in \mathbb{R}$

S, T unitary, $ST = e^{2\pi i\theta} TS$

uniqueness property $\iff \theta \notin \mathbb{Q}$

Toeplitz algebra $= C^*(S)$

$$S^*S = 1, \qquad SS^* \neq 1$$

uniqueness property ✓

(L. A. Coburn 1969)

Cuntz algebras $\mathcal{O}_n = C^*(S_1, S_2, ..., S_n)$

$$S_i^* S_j = \delta_{i,j} 1, \quad \sum_{i=1}^{n} S_i S_i^* = 1$$

uniqueness property ✓ (J. Cuntz 1977)

Unitary algebra $C^*(S)$

$$S^*S = SS^* = 1$$

no uniqueness property

Rotation algebras $\mathcal{A}_\theta = C^*(S, T)$, $\theta \in \mathbb{R}$

S, T unitary, $ST = e^{2\pi i\theta} TS$

uniqueness property $\Longleftrightarrow \theta \notin \mathbb{Q}$

Toeplitz algebra $= C^*(S)$

$$S^*S = 1, \qquad SS^* \neq 1$$

uniqueness property ✓

(L. A. Coburn 1969)

Cuntz algebras $\mathcal{O}_n = C^*(S_1, S_2, ..., S_n)$

$$S_i^*S_j = \delta_{i,j}1, \quad \sum_{i=1}^{n} S_iS_i^* = 1$$

uniqueness property ✓ (J. Cuntz 1977)

Cuntz-Krieger algebras $\mathcal{O}_A = C^*(S_1, S_2, ..., S_n)$

$\{A(i,j)\}_{i,j=1}^{n} \in \{0,1\}^{n\times n}$, S_i partial isometries with orthogonal ranges

$$\sum_{j=1}^{n} A(i,j)S_jS_j^* = S_i^*S_i$$

Unitary algebra $C^*(S)$

$S^*S = SS^* = 1$

no uniqueness property

Rotation algebras $\mathcal{A}_\theta = C^*(S, T)$, $\theta \in \mathbb{R}$

S, T unitary, $ST = e^{2\pi i\theta}TS$

uniqueness property $\Longleftrightarrow \theta \notin \mathbb{Q}$

Toeplitz algebra $= C^*(S)$

$$S^*S = 1, \qquad SS^* \neq 1$$

uniqueness property ✓

(L. A. Coburn 1969)

Cuntz algebras $\mathcal{O}_n = C^*(S_1, S_2, ..., S_n)$

$$S_i^*S_j = \delta_{i,j}1, \quad \sum_{i=1}^{n} S_iS_i^* = 1$$

uniqueness property ✓ (J. Cuntz 1977)

Cuntz-Krieger algebras $\mathcal{O}_A = C^*(S_1, S_2, ..., S_n)$

$\{A(i,j)\}_{i,j=1}^{n} \in \{0,1\}^{n\times n}$, S_i partial isometries with orthogonal ranges

$$\sum_{j=1}^{n} A(i,j)S_jS_j^* = S_i^*S_i$$

uniqueness property $\Longleftrightarrow$ condition (I) (J. Cuntz, W. Krieger 1980)

Cuntz-Krieger uniqueness theorem industry

Cuntz, Krieger 1980
Cuntz-Krieger algebras $\mathcal{O}_A$
condition (I)

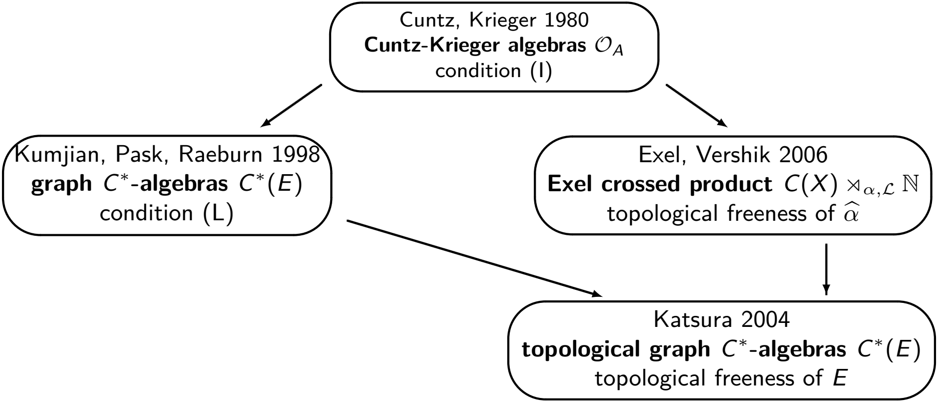
Cuntz-Krieger uniqueness theorem industry

Cuntz, Krieger 1980
Cuntz-Krieger algebras $\mathcal{O}_A$
condition (I)

↓

Kumjian, Pask, Raeburn 1998
graph C^***-algebras** $C^*(E)$
condition (L)

Cuntz-Krieger uniqueness theorem industry



Cuntz-Krieger uniqueness theorem industry

Cuntz, Krieger 1980
Cuntz-Krieger algebras $\mathcal{O}_A$
condition (I)

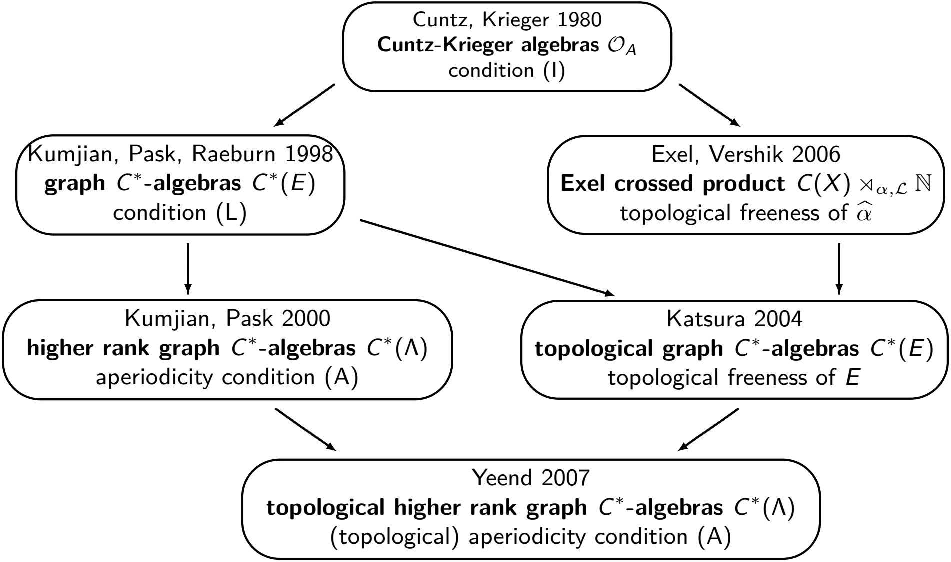
Kumjian, Pask, Raeburn 1998
graph C^***-algebras** $C^*(E)$
condition (L)

Exel, Vershik 2006
Exel crossed product $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$
topological freeness of $\widehat{\alpha}$

Kumjian, Pask 2000
higher rank graph C^***-algebras** $C^*(\Lambda)$
aperiodicity condition (A)

Katsura 2004
topological graph C^***-algebras** $C^*(E)$
topological freeness of E

Cuntz-Krieger uniqueness theorem industry



Cuntz-Krieger uniqueness theorem industry

$$A = C_0(X)$$
$$P = \mathbb{N}^k$$

Cuntz, Krieger 1980
Cuntz-Krieger algebras $\mathcal{O}_A$
condition (I)

Kumjian, Pask, Raeburn 1998
graph C^***-algebras** $C^*(E)$
condition (L)

Exel, Vershik 2006
Exel crossed product $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$
topological freeness of $\widehat{\alpha}$

Kumjian, Pask 2000
higher rank graph C^***-algebras** $C^*(\Lambda)$
aperiodicity condition (A)

Katsura 2004
topological graph C^***-algebras** $C^*(E)$
topological freeness of E

Yeend 2007
topological higher rank graph C^***-algebras** $C^*(\Lambda)$
(topological) aperiodicity condition (A)

Cuntz-Krieger uniqueness theorem industry

$$A = C_0(X)$$
$$P = \mathbb{N}^k$$

Cuntz, Krieger 1980
Cuntz-Krieger algebras $\mathcal{O}_A$
condition (I)

Kumjian, Pask, Raeburn 1998
graph C^***-algebras** $C^*(E)$
condition (L)

Exel, Vershik 2006
Exel crossed product $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$
topological freeness of $\widehat{\alpha}$

Kumjian, Pask 2000
higher rank graph C^***-algebras** $C^*(\Lambda)$
aperiodicity condition (A)

Katsura 2004
topological graph C^***-algebras** $C^*(E)$
topological freeness of E

Yeend 2007
topological higher rank graph C^***-algebras** $C^*(\Lambda)$
(topological) aperiodicity condition (A)

Example of an operator generating an endomorphism

Let $H = L_2(\mathbb{R})$, $A \subseteq B(H)$ consists of operators of multiplication by continuous functions with period 1:

$$A \cong C(S^1).$$

Example of an operator generating an endomorphism

Let $H = L_2(\mathbb{R})$, $A \subseteq B(H)$ consists of operators of multiplication by continuous functions with period 1:

$$A \cong C(S^1).$$

Let $a \in A$ and consider unitary operator $T \in B(H)$

$$(Tf)(x) = \sqrt{2}\, f(2x), \qquad (T^* f)(x) = \frac{1}{\sqrt{2}}\, f\left(\frac{x}{2}\right).$$

Example of an operator generating an endomorphism

Let $H = L_2(\mathbb{R})$, $A \subseteq B(H)$ consists of operators of multiplication by continuous functions with period 1:

$$A \cong C(S^1).$$

Let $a \in A$ and consider unitary operator $T \in B(H)$

$$(Tf)(x) = \sqrt{2}\, f(2x), \qquad (T^* f)(x) = \frac{1}{\sqrt{2}}\, f\left(\frac{x}{2}\right).$$

TaT^* - operator of multiplication by $a(2x)$

T^*aT - operator of multiplication by $a\left(\frac{x}{2}\right)$

Example of an operator generating an endomorphism

Let $H = L_2(\mathbb{R})$, $A \subseteq B(H)$ consists of operators of multiplication by continuous functions with period 1:

$$A \cong C(S^1).$$

Let $a \in A$ and consider unitary operator $T \in B(H)$

$$(Tf)(x) = \sqrt{2}\, f(2x), \qquad (T^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right).$$

$$TaT^* \text{ - operator of multiplication by } a(2x)$$

$$T^*aT \text{ - operator of multiplication by } a\left(\frac{x}{2}\right)$$

That is $\quad TAT^* \subseteq A, \qquad T^*AT \nsubseteq A.$

Example of an operator generating an endomorphism

Let $H = L_2(\mathbb{R})$, $A \subseteq B(H)$ consists of operators of multiplication by continuous functions with period 1:

$$A \cong C(S^1).$$

Let $a \in A$ and consider unitary operator $T \in B(H)$

$$(Tf)(x) = \sqrt{2}\, f(2x), \qquad (T^* f)(x) = \frac{1}{\sqrt{2}} f\left(\frac{x}{2}\right).$$

$$TaT^* \text{ - operator of multiplication by } a(2x)$$

$$T^*aT \text{ - operator of multiplication by } a\left(\frac{x}{2}\right)$$

That is $\quad TAT^* \subseteq A, \qquad T^*AT \nsubseteq A.$

Identifying A with $C(S^1)$ we have $\alpha(a) = a \circ \phi$, $a \in A = C(S^1)$ where $\phi(z) = z^2$, $z \in S^1$, so $\alpha(\cdot) = T(\cdot)T^*$ is an endomorphism of A

Example of an operator generating an endomorphism

A way out: We may pass to a bigger algebra

$$B = C^*(\bigcup_{n=0}^{\infty} T^{*n} A T^n)$$

B is commutative and $\alpha(\cdot) = T(\cdot)T^*$ is an automorphism of B.

Example of an operator generating an endomorphism

A way out: We may pass to a bigger algebra

$$B = C^*(\bigcup_{n=0}^{\infty} T^{*n} A T^n)$$

B is commutative and $\alpha(\cdot) = T(\cdot)T^*$ is an <u>automorphism</u> of B. Hence

$$B \cong C(\widetilde{X})$$

Example of an operator generating an endomorphism

A way out: We may pass to a bigger algebra

$$B = C^*(\bigcup_{n=0}^{\infty} T^{*n} A T^n)$$

B is commutative and $\alpha(\cdot) = T(\cdot)T^*$ is an automorphism of B. Hence

$$B \cong C(\widetilde{X})$$

where $\widetilde{X}$ is a compact Hausdorff space and

$$\alpha(a) = a \circ \widetilde{\phi}, \qquad a \in \mathcal{B} = C(\widetilde{X})$$

where $\widetilde{\alpha} : \widetilde{X} \to \widetilde{X}$ is a homeomorphism.

Example of an operator generating an endomorphism

A way out: We may pass to a bigger algebra

$$B = C^*(\bigcup_{n=0}^{\infty} T^{*n} A T^n)$$

B is commutative and $\alpha(\cdot) = T(\cdot)T^*$ is an automorphism of B. Hence

$$B \cong C(\widetilde{X})$$

where $\widetilde{X}$ is a compact Hausdorff space and

$$\alpha(a) = a \circ \widetilde{\phi}, \qquad a \in \mathcal{B} = C(\widetilde{X})$$

where $\widetilde{\alpha} : \widetilde{X} \to \widetilde{X}$ is a homeomorphism.

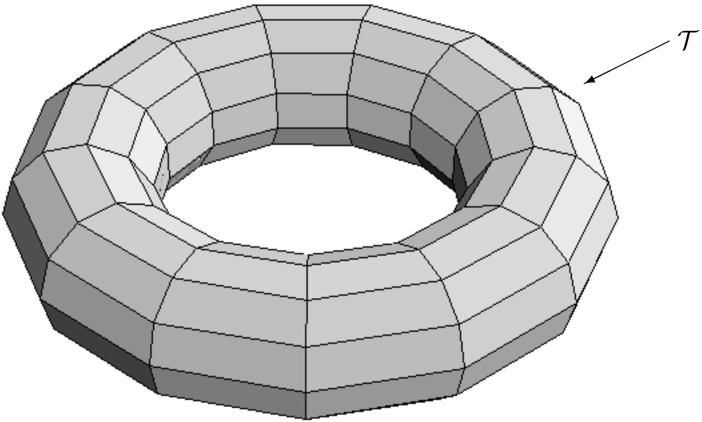
But how the system $(\widetilde{X}, \widetilde{\alpha})$ looks like?

Example of an operator generating an endomorphism

The system $(\widetilde{X}, \widetilde{\alpha})$ is Smale's Solenoid.
$\widetilde{X} \cong \bigcap_{n\in N} F^n(\mathcal{T})$

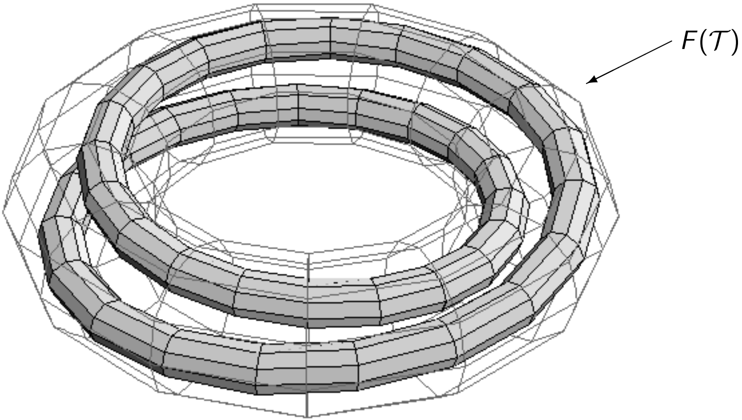
Example of an operator generating an endomorphism

The system $(\widetilde{X}, \widetilde{\alpha})$ is Smale's Solenoid.
$\widetilde{X} \cong \bigcap_{n \in N} F^n(\mathcal{T})$ where $F : \mathcal{T} \to \mathcal{T}$ acts as follows



Example of an operator generating an endomorphism

The system $(\widetilde{X}, \widetilde{\alpha})$ is Smale's Solenoid.
$\widetilde{X} \cong \bigcap_{n\in N} F^n(\mathcal{T})$ where $F : \mathcal{T} \to \mathcal{T}$ acts as follows



Example of an operator generating an endomorphism

The system $(\widetilde{X}, \widetilde{\alpha})$ is Smale's Solenoid.
$\widetilde{X} \cong \bigcap_{n\in N} F^n(\mathcal{T})$ where $F : \mathcal{T} \to \mathcal{T}$ acts as follows

Examples of similar type

There is a general procedure of extending an arbitrary endomorphism $\alpha : A \to A$ to a (partial) automorphism $\beta : B \to B$.

Examples of similar type

There is a general procedure of extending an arbitrary endomorphism $\alpha : A \to A$ to a (partial) automorphism $\beta : B \to B$.

	Irreversible system	Reversible counterpart
1	z^N on S^1	N-adic Solenoid
2	logistic maps	irreducible continua Brouwer-Janiszewski-Knaster
3	topological Markov chains	Smale horseshoes
4	maps on graphs	Plykin attractors, tillings Fibonnaci, Morse, etc.
5	maps on 2-dimensional, branched manifolds	tillings of Penrose, Amman and others

Knaster buckethandle (B-J-K continuum)

Ulam-von Neumann density on B-J-K continuum