Pure infiniteness and ideal structure of C^*-algebras associated with Fell bundles and product systems

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ongoing joint work with Wojciech Szymański, IMADA, Odense

Lysebu, December 9, 2014

Plan

- Amenability

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- ~~Amenability~~ $\Longrightarrow$ reduced C^*-algebras Exel

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- Exactness, intersection property — "Sierakowski"
- Ideal structure via dual system — "Archbold, Spielberg" "Kwaśniewski + Rieffel"
- Aperiodicity
  "Muhly, Solel"

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- Aperiodicity "Muhly, Solel" $+$ Paradoxicality $\Longrightarrow$ Pure infiniteness
  "Rørdam, Sierakowski"

Preliminaries (throughout G is a discrete group)

A C^*-algebra B is said to be **graded** over G if $B = \overline{\bigoplus_{g\in G} B_g}$ where $\mathcal{B} = \{B_g\}_{g\in G}$ is a family of closed subspaces such that

$$B_g^* = B_{g^{-1}} \quad \text{and} \quad B_g B_h \subseteq B_{gh}, \quad \text{for all } g, h \in G.$$

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The C^*-algebra $B = \overline{\bigoplus_{g\in G} B_g}$ is said to be **topologically graded** if

$$\|a_e\| \le \|\textstyle\sum_{g\in G} a_g\| \qquad \text{for all} \quad \textstyle\sum_{g\in G} a_g \in \bigoplus_{g\in G} B_g.$$

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For any Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ the direct sum $\bigoplus_{g\in G} B_g$ is naturally a $*$-algebra, and we put

$$C^*(\mathcal{B}) := \overline{\bigoplus_{g\in G} B_g}^{\|\cdot\|_{max}} \qquad C^*_r(\mathcal{B}) := \overline{\bigoplus_{g\in G} B_g}^{\|\cdot\|_{min}}$$

where $\|\cdot\|_{min}$ the minimal topologically graded C^*-norm on $\bigoplus_{g\in G} B_g$.

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Let J be an ideal in $C_r^*(\mathcal{B})$ we say that

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establish bijections between: induced ideals J in $C_r^*(\mathcal{B})$, ideals $\mathcal{J} = \{J_t\}_{t \in G}$ in $\mathcal{B}$, and $\mathcal{B}$-invariant ideals I in B_e.

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Problem

When all ideals in $C_r^*(\mathcal{B})$ are induced?

Exactness and intersection property

Let $\mathcal{J} = \{J_t\}_{t\in G}$ be an ideal in $\mathcal{B} = \{B_g\}_{g\in G}$. Then

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Lem.

$\mathcal{B}$ has the intersection property $\Longleftrightarrow$ any graded C^*-algebra $B = \overline{\bigoplus_{g\in G} B_g}$ is topologically graded.

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Thm. For any Fell bundle $\mathcal{B}$ we have a surjection

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It is a lattice isomorphism $\Longleftrightarrow$ $\mathcal{B}$ is exact and has the residual intersection property

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$$\widehat{h}_g := [B_g\text{-Ind}_{D_g}^{D_{g^{-1}}}]$$

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Prop. $(\{\widehat{D}_g\}_{g\in G}, \{\widehat{h}_g\}_{g\in G})$ is a partial action of G on $\widehat{B_e}$.

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Thm. If $(\{\widehat{D}_g\}_{g\in G}, \{\widehat{h}_g\}_{g\in G})$ is **topologically free**, i.e.

$$\forall_{t_1,\dots,t_n\in G} \quad \bigcap_{i=1}^{n}\{x \in \widehat{D}_{t_i^{-1}} : \widehat{h}_{t_i}(x) = x\} \text{ has empty interior in } \widehat{B}_e,$$

then $\mathcal{B} = \{B_g\}_{g\in G}$ has the intersection property.

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Cor. Suppose $(\{\widehat{D}_g\}_{g\in G}, \{\widehat{h}_g\}_{g\in G})$ is **residually topologically free**.

If $\mathcal{B}$ is exact then

$$\mathcal{I}deal(C_r^*(\mathcal{B})) \ni J \to \widehat{J \cap B_e}$$

is a bijection onto the set of all open invariant subsets in $\widehat{B_e}$.

Aperiodicity

Def. $\mathcal{B} = \{B_g\}_{g\in G}$ is **aperiodic** if for each $b_g \in B_g$, $g \in G \setminus \{e\}$, and every hereditary subalgebra A of B_e,

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Lem. $\mathcal{B} = \{B_g\}_{g\in G}$ is aperiodic $\iff$ for any $b = \oplus_{g\in G} b_g \in \bigoplus_{g\in G} B_g$, $b_e \geq 0$, and any $\varepsilon > 0$ there is $x \in \overline{b_e B_e b_e}$ such that $x \geq 0$, $\|x\| = 1$ and

$$\|xb_e x - xbx\| < \varepsilon, \qquad \|xb_e x\| > \|b_e\| - \varepsilon$$

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Thm. Suppose $\mathcal{B}$ is exact and residually aperiodic.

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If B_e is of real rank zero, the above conditions are equivalent to

(ii') Every non-zero projection in B_e is properly infinite in $C_r^*(\mathcal{B})$.

Paradoxicality - pure infiniteness

Def. (Rørdam, Sierakowski; Giordano, Sierakowski)

Let $(\{\Omega_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a topological space Ω. A non-empty compact open set $V \subseteq \Omega$ is called G-**paradoxical** if there are open sets $V_1, ..., V_{n+m}$ and elements $t_1, ..., t_{n+m} \in G$, such that

(1) $V = \bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i$,

(2) $V_i \subseteq \Omega_{t_i^{-1}}$ and $\theta_{t_i}(V_i) \subseteq V$ for all $i = 1, ..., n+m$,

(3) $\theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \emptyset$ for all $i \neq j$.

Paradoxicality - pure infiniteness

Def. (Rørdam, Sierakowski; Giordano, Sierakowski)

Let $(\{\Omega_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action on a topological space Ω. A non-empty compact open set $V \subseteq \Omega$ is called G-**paradoxical** if there are open sets $V_1, ..., V_{n+m}$ and elements $t_1, ..., t_{n+m} \in G$, such that

(1) $V = \bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i$,

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Def. (BKK, Szymański) Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is $\mathcal{B}$-**paradoxical** if for any $\varepsilon > 0$ there are elements $a_1, ..., a_{n+m} \in B_e^+$ such that for all $\delta > 0$ there are $b_i \in B_{t_i}$, $\|b_i\| \leq 1$, $t_i \in G$, for $i = 1, ..., n+m$, such that

(1) $a \approx_\varepsilon \sum_{i=1}^{n} a_i \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i$,

(2) $b_i^* b_i a_i \approx_\delta a_i$ and $b_i a_i \in aB_{t_i}$ for all $i = 1, ..., n+m$,

(3) $b_i^* b_j \approx_\delta 0$ for all $i \neq j$.

Paradoxicality - pure infiniteness

Lem. Let $(\{\Omega_g\}_{g\in G}, \{\theta_g\}_{g\in G})$ be a partial action and $\mathcal{B} = \{B_g\}_{g\in \mathcal{G}}$ associated Fell bundle. For any non-zero projection $p \in B_e = C_0(\Omega_e)$

$$V = \{x \in \Omega_e : p(x) > 0\} \text{ is } G\text{-paradoxical } \Longrightarrow p \text{ is } \mathcal{B}\text{-paradoxical.}$$

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Thm. Suppose $\mathcal{B}$ is exact and residually aperiodic.

If either of the conditions hold

(i) B_e has the ideal property and every element in $B_e^+ \setminus \{0\}$ is $\mathcal{B}$-paradoxical.

(ii) B_e is of real rank zero and non-zero projection in B_e is $\mathcal{B}$-paradoxical.

then $C_r^*(\mathcal{B})$ is purely infinite and has the ideal property.

Motivation: Cuntz-Pimsner algebra $\mathcal{O}_X$ of a product system

Fix a regular product system $X = \bigsqcup_{p\in P} X_p$ over a left Ore semigroup P
Let $G = PP^{-1}$ be the group of fractions.

Thm. (BKK, Szymanski) Doplicher-Roberts picture of $\mathcal{O}_X$

- The C^*-(pre)category $\{\mathcal{K}(X_p, X_q)\}_{q,p\in P}$ is naturally equipped with a right tensoring structure $\{\otimes 1_r\}_{p\in P}$.
- For each $g = pq^{-1} \in G$, $p, q \in P$, the direct limit

$$B_g := \varinjlim \left(\{\mathcal{K}(pr, qr)\}_{r\in P}, \{\otimes_{r^{-1}s} 1\}_{\substack{r,s\in P\\ r\leq s}}\right)$$

  is well defined and the family $\{B_g\}_{g\in G}$ is naturally a Fell bundle.
- We have

$$\mathcal{O}_X \cong C^*(\{B_g\}_{g\in G}).$$

  The universal covariant representation $i_X : X \to \mathcal{O}_X$ is injective.