Pure infiniteness of C^*-algebras associated to Fell bundles

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based on joint work with Wojciech Szymański (arXiv:1505.05202)
modulo 'work in progress'

Plan

1) Introduction
2) Fell bundles and reduced cross-sectional C^*-algebras
3) Pure infiniteness criterion
4) Dynamical conditions implying pure infiniteness

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Cuntz comparison and properly infinite elements

Let A be a C^*-algebra. Notation: $a \approx_\varepsilon b \overset{def}{\Longleftrightarrow} \|a-b\| < \varepsilon$.

Def. For $a, b \in A^+$ we write

$$a \precsim b \; (Cuntz 1978) ow \; \forall_{\varepsilon>0} \, \exists_{x\in A} \quad x^*bx \approx_\varepsilon a.$$

We say that $a, b \in A^+$ are **Cuntz equivalent** if both $a \precsim b$ and $b \precsim a$ holds.

Def. (Rørdam, Kirchberg 2000) For $a \in A^+ \setminus \{0\}$ we say

a is **infinite** if there is $b \in A^+ \setminus \{0\}$ such that $a \oplus b \precsim a \oplus 0$ in $M_2(A)$,
a is **properly infinite** if $a \oplus a \precsim a \oplus 0$ in $M_2(A)$.

Lem. Let $a \in A^+ \setminus \{0\}$.

$$a \text{ is infinite} \iff \exists_{b\in A^+\setminus\{0\}} \forall_{\varepsilon>0} \, \exists_{x,y\in aA} \quad x^*x \approx_\varepsilon a, \quad y^*y \approx_\varepsilon b, \quad x^*y \approx_\varepsilon 0,$$

$$a \text{ is properly infinite} \iff \forall_{\varepsilon>0} \, \exists_{x,y\in aA} \quad x^*x \approx_\varepsilon a, \quad y^*y \approx_\varepsilon a, \quad x^*y \approx_\varepsilon 0.$$

Prop. $a \in A^+ \setminus \{0\}$ is properly infinite if and only if

for every ideal I in A the image of a in A/I is either zero or infinite.

Purely infinite C^*-algebras

Def. (Cuntz 1981)

simple C^*-algebra A is **purely infinite** $\iff$ every non-zero hereditary C^*-subalgebra of A contains an infinite projection.

Def. (Rørdam, Kirchberg 2000)

C^*-algebra A is **purely infinite** $\iff$ every $a \in A^+ \setminus \{0\}$ is properly infinite.

Def. (Rørdam, Kirchberg 2002)

C^*-algebra A is **strongly purely infinite** $\iff$ every pair $a, b \in A^+ \setminus \{0\}$ satisfies

$$\forall_{\varepsilon>0}\ \exists_{x\in aA,\, y\in bA} \qquad x^*x \approx_\varepsilon a, \qquad y^*y \approx_\varepsilon b, \qquad x^*y \approx_\varepsilon 0$$

A has the **ideal property (IP)** if projections in A separate ideals in A

Thm. (Pasnicu, Rørdam 2007) The following conditions are equivalent:

i) A is purely infinite and has (IP)

ii) A is strongly purely infinite and has (IP)

iii) for every ideal I in A every non-zero hereditary C^*-subalgebra in A/I contains an infinite projection

Purely infinite crossed products (overview)

Crossed products $A \rtimes_\alpha \mathbb{N}$ by an endomorphism

Authors	Date	Algebra A	Dynamics
Rørdam	1995	simple, real rank zero, comparability property	corner endomorphism
Ortega,Pardo	2014	separable, real rank zero	residual Rokhlin* property residually contracts projections

Reduced crossed products $A \rtimes_{\alpha,r} G$ by group actions

Authors	Date	Algebra A	Dynamics
Laca Spielberg	1996	$A = C(X)$	topologically free, strong boundary action
Jolissaint Robertson	2000	unital with infinite corners separable	properly outer n-filling
Rørdam Sierakowski	2012	$A = C_0(X)$ real rank zero	residually topologically free, exact, paradoxical
Giordano Sierakowski	2014	as above but for partial actions	
Kirchberg Sierakowski	2015 preprint	separable or commutative	residually properly outer, exact, G-separating

Jeong, Kodaka, Osaka 1995,1996, and Pasnicu, Phillips 2015 considered conditions implying that pure infiniteness passes to crossed products

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Fell bundles (throughout G is a discrete group)

Def. (Fell 1969)

A **Fell bundle** $\mathcal{B}$ over G is consists of Banach spaces $\{B_g\}_{g\in G}$ equipped with

$$\cdot : B_g \times B_h \longmapsto B_{gh}, \qquad * : B_g \longmapsto B_{g^{-1}}, \qquad g, h \in G,$$

such that $\bigoplus_{g\in G} B_g$ becomes a *-algebra admitting a C^*-norm.

Any completion $B = \overline{\bigoplus_{g\in G} B_g}$ in a C^*-norm is called a **$\mathcal{B}$-graded algebra.**

The **full cross sectional C^*-algebra** of $\mathcal{B}$ is $C^*(\mathcal{B}) = \overline{\bigoplus_{g\in G} B_g}^{\|\cdot\|_{max}}$

Def. (Exel, Quigg 1996)

The **reduced cross sectional C^*-algebra** of $\mathcal{B}$ is $C^*_r(\mathcal{B}) = \overline{\bigoplus_{g\in G} B_g}^{\|\cdot\|_{min}}$ where $\|\cdot\|_{min}$ is the minimal C^*-norm on $\bigoplus_{g\in G} B_g$ such that

$$\|a_e\| \leq \|\sum_{g\in G} a_g\| \qquad \text{for all} \quad \sum_{g\in G} a_g \in \bigoplus_{g\in G} B_g.$$

Rem. $C^*_r(\mathcal{B})$ is a unique $\mathcal{B}$-graded C^*-algebra equipped with a faithful conditional expectation $E : C^*_r(\mathcal{B}) \to B_e$ onto the unit fiber C^*-algebra B_e.

Various ideals

Def. Fix a Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$.

An **ideal** in $\mathcal{B}$ is $\mathcal{J} = \{J_g\}_{g\in G}$ where J_g is a closed subspace of B_g, and

$$B_g J_h \subseteq J_{gh} \qquad J_g B_h \subseteq J_{gh}, \qquad \text{for all } g, h \in G.$$

An ideal I in B_e is $\mathcal{B}$-**invariant** if $B_g I B_g^* \subseteq I$ for every $g \in G$.

An ideal J in $C_r^*(\mathcal{B})$ is **graded** if it is generated by $J \cap B_e$.

Prop. The following relations:

$$J = \overline{\bigoplus_{g\in G} J_g}, \qquad J_g = J \cap B_g = B_g I = I B_g, \qquad I = J \cap B_e$$

establish bijective correspondences between

- ideals $\mathcal{J} = \{J_g\}_{g\in G}$ in $\mathcal{B}$,
- $\mathcal{B}$-invariant ideals I in B_e,
- graded ideals J in $C_r^*(\mathcal{B})$.

$\mathcal{I}^{\mathcal{B}}(B_e) := \{I \lhd B_e : B_g I B_{g^{-1}} \subseteq I,\ g \in G\}$ - $\mathcal{B}$-invariant ideals in B_e

Exactness and intersection property (Sierakowski 2010, Abadie-Abadie)

If $\mathcal{J} = \{J_t\}_{t\in G}$ is an ideal in $\mathcal{B} = \{B_g\}_{g\in G}$, then $\mathcal{B}/\mathcal{J} := \{B_g/J_g\}_{g\in G}$ is a Fell bundle and

$$0 \longrightarrow C_r^*(\mathcal{J}) \longrightarrow C_r^*(\mathcal{B}) \longrightarrow C_r^*(\mathcal{B}/\mathcal{J}) \longrightarrow 0.$$

Def. $\mathcal{B}$ is **exact** if the above sequence is exact for every ideal $\mathcal{J}$ in $\mathcal{B}$.

Rem. G is exact $\Longrightarrow$ $\mathcal{B}$ is exact
$\mathcal{B}$ is amenable, i.e. $C_r^*(\mathcal{B}) = C^*(\mathcal{B}) \Longrightarrow \mathcal{B}$ is exact

Def. $\mathcal{B}$ has the **intersection property** if every non-zero ideal in $C_r^*(\mathcal{B})$ has a non-zero intersection with B_e. $\mathcal{B}$ has the **residual intersection property** if $\mathcal{B}/\mathcal{J}$ has the intersection property for every ideal $\mathcal{J}$ in $\mathcal{B}$.

Thm.

Every ideal in $C_r^*(\mathcal{B})$ is graded, that is

$$C_r^*(\mathcal{B}) \rhd J \longrightarrow J \cap B_e \in \mathcal{I}^{\mathcal{B}}(B_e)$$

is a bijection $\Longleftrightarrow$ $\mathcal{B}$ is exact and has the residual intersection property.

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Aperiodicity

Concept abstracted from the work of: Connes 1976, Elliot, 1980, Kishimoto 1981, Olesen-Pedersen, 1982, Muhly-Solel 2000, Giordano-Sierakowski 2014

Def.

A Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ is **aperiodic** if for each $g \in G \setminus \{e\}$, each $b_g \in B_g$ and every hereditary subalgebra D of B_e,

$$\inf\{\|ab_g a\| : a \in D^+,\ \|a\| = 1\} = 0.$$

$\mathcal{B}$ is **residually aperiodic** if $\mathcal{B}/\mathcal{J}$ is aperiodic for any ideal $\mathcal{J}$ in $\mathcal{B}$.

Prop. Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is aperiodic.

For every $b \in C_r^*(\mathcal{B})^+ \setminus \{0\}$ there is $a \in B_e^+ \setminus \{0\}$ such that $a \precsim b$.

Cor.

If $\mathcal{B}$ is (residually) aperiodic, then $\mathcal{B}$ has the (residual) intersection property.

Pure infiniteness criterion

Thm. Suppose that $\mathcal{B}$ is exact and residually aperiodic.

$$C_r^*(\mathcal{B}) \rhd J \longrightarrow J \cap B_e \in \mathcal{I}^{\mathcal{B}}(B_e) \quad \text{is a bijection.}$$

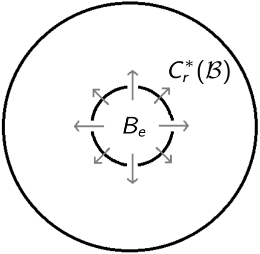
If either B_e has (IP) or $\mathcal{B}$ is minimal, i.e. are no non-trivial $\mathcal{B}$-invariant ideals in B_e, then the following statements are equivalent:

(i) $C_r^*(\mathcal{B})$ is purely infinite.

(ii) Every element in $B_e^+ \setminus \{0\}$ is properly infinite in $C_r^*(\mathcal{B})$.

If $RR(B_e) = 0$, each of the above conditions is equivalent to

(ii') Every non-zero projection in B_e is properly infinite in $C_r^*(\mathcal{B})$.



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Paradoxicality

Def. (Banach-Tarski 1924, Sierakowski-Rørdam 2012)

Let $\Theta = \{\theta_g\}_{g\in G}$ be a group action on a locally compact Hausdorff Ω. A non-empty open set $V \subseteq \Omega$ is called Θ-**paradoxical** if there are open sets $V_1, \dots, V_{n+m}$ and elements $t_1, \dots, t_{n+m} \in G$, such that

$$V = \bigcup_{i=1}^{n} V_i = \bigcup_{i=n}^{n+m} V_i, \quad \theta_{t_i}(V_i) \subseteq V \text{ and } \theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \emptyset \text{ for all } i \neq j.$$

Def. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is $\mathcal{B}$-**paradoxical** if for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, \dots, n+m$, such that

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \quad a \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \varepsilon / \max\{n^2, m^2\} \text{ for } i \neq j.$$

If the above holds for $\varepsilon = 0$ we call a **strictly** $\mathcal{B}$-**paradoxical**.

Prop. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be the Fell bundle associated to $\Theta = \{\theta_g\}_{g\in G}$.

An element $a \in B_e^+ = C_0(\Omega)^+$ is strictly $\mathcal{B}$-paradoxical if and only if the set $V := \{x \in \Omega : a(x) > 0\}$ is Θ-paradoxical.

Paradoxicality

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Let $\Theta = \{\theta_g\}_{g\in G}$ be a group action on a locally compact Hausdorff Ω. A non-empty open set $V \subseteq \Omega$ is called Θ-**paradoxical** if there are open sets $V_1, ..., V_{n+m}$ and elements $t_1, ..., t_{n+m} \in G$, such that

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Def. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is $\mathcal{B}$-**paradoxical** if for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, ..., n+m$, such that

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \quad a \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \varepsilon / \max\{n^2, m^2\} \text{ for } i \neq j.$$

Rem. Let $B = \overline{\bigoplus_{g\in\mathcal{G}} B_g}$ be a $\mathcal{B}$-graded C^*-algebra.

If $a \in B_e^+$ is $\mathcal{B}$-paradoxical then for $x := \sum_{i=1}^{n} a_i$ and $y := \sum_{i=n+1}^{n+m} a_i$ we get

$$a \approx_{2\varepsilon} x^* x, \qquad a \approx_{2\varepsilon} y^* y, \qquad x^* y \approx_\varepsilon 0.$$

Residual Infiniteness

Def. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is $\mathcal{B}$-**infinite** if there is $b \in B_e^+ \setminus \{0\}$ such that for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, ..., n+m$, and

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \quad b \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \varepsilon / \max\{n^2, m^2\} \ \text{ for } i \neq j.$$

If the above holds for $\varepsilon = 0$ we say a is **strictly** $\mathcal{B}$**-infinite**

We say a is **residually** $\mathcal{B}$**-infinite** if for every ideal $\mathcal{J} = \{J_g\}_{g\in G}$ in $\mathcal{B}$, the element $a + J_e$ is either zero in B_e/J_e or it is $\mathcal{B}/\mathcal{J}$-infinite.

Prop. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be the Fell bundle associated to $\Theta = \{\theta_g\}_{g\in G}$.

An element $a \in B_e^+ = C_0(\Omega)^+$ is strictly $\mathcal{B}$-infinite if and only if the set $V := \{x \in \Omega : a(x) > 0\}$ is Θ-**infinite**, i.e., there are open sets $V_1, ..., V_n$ and elements $t_1, ..., t_n \in G$, such that

$$V = \bigcup_{i=1}^{n} V_i, \quad \overline{\bigcup_{i=1}^{n} \theta_{t_i}(V_i)} \subsetneq V \ \text{ and } \ \theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \emptyset \text{ for } i \neq j.$$

The main result

Thm. Suppose that $\mathcal{B} = \{B_g\}_{g \in G}$ is an exact, residually aperiodic Fell bundle.

$C_r^*(\mathcal{B})$ is purely infinite and has (IP) whenever one of the following conditions holds:

(i) B_e has (IP) and every element in $B_e^+ \setminus \{0\}$ is Cuntz equivalent to a residually $\mathcal{B}$-infinite element,

(i') $RR(B_e) = 0$ and every non-zero projection in B_e is Cuntz equivalent to a residually $\mathcal{B}$-infinite element,

(ii) there no non-trivial $\mathcal{B}$-invariant ideals in B_e and every element in $B_e^+ \setminus \{0\}$ is Cuntz equivalent to a $\mathcal{B}$-infinite element.

Cor. (Sierakowski-Rørdam)

Let α be an exact group action on $C_0(\Omega)$ induced by residually topologically free action $\Theta = \{\theta_g\}_{g \in G}$ on a totally disconnected space Ω. If every non-empty compact and open set is paradoxical, then $A \rtimes_{\alpha, r} G$ is purely infinite.

Strong boundary and *n*-filling actions

Def. (Laca-Spielberg 1996)

A group action $\Theta = \{\theta_t\}_{t\in G}$ on a compact Hausdorff space Ω is **strong boundary action** if for every two nonempty open subsets U_1, U_2 of Ω there are $g_1, g_2 \in G$ such that $\theta_{g_1}(U_1) \cup \theta_{g_2}(U_2) = \Omega$.

Def. (Jolissaint-Robertson 2000)

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a unital C^*-algebra A with infinite dimensional corners is called *n*-**filling**, for $n \geq 2$, if, for all elements $b_1, ..., b_n \in A^+$ of norm one, and for all $\varepsilon > 0$, there exist $g_1, ..., g_n \in G$ such that $\sum_{i=1}^n \alpha_{g_i}(b_i) \geq 1 - \varepsilon$.

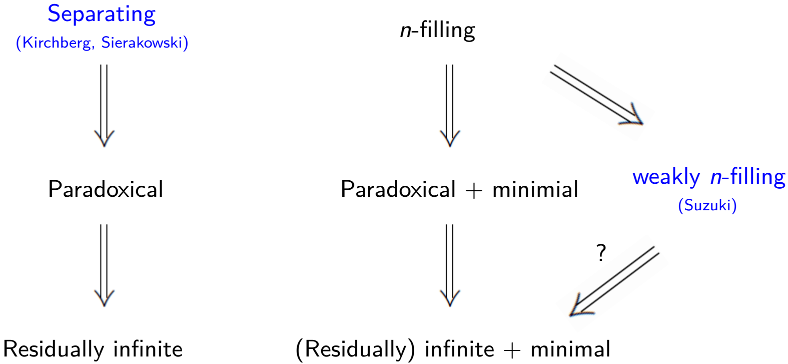
Lem.

Let α be an *n*-filling action and $\mathcal{B}$ the corresponding Fell bundle. Then $\mathcal{B}$ is minimal and any element $a \in A^+ \setminus \{0\}$ is strictly residually $\mathcal{B}$-infinite.

Cor. (Laca-Spielberg, Jollisaint-Robertson)

Let α be an *n*-filling action on A and suppose that either $A = C(\Omega)$ and the dual action is topologically free, or that A is separable and α is a properly outer action. Then $A \rtimes_{\alpha,r} G$ is simple and purely infinite.

General relationship between various actions



Separating
(Kirchberg, Sierakowski)

n-filling

Paradoxical

Paradoxical + minimial

weakly *n*-filling
(Suzuki)

?

Residually infinite

(Residually) infinite + minimal

Question:

Our theorem works for $A \rtimes_{\alpha}^{r} G$ with A being G-simple or for A with (IP). **To what extent can we extend it?**

Separating actions (Kirchberg-Sierakowski 2015 preprint)

Def.

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a C^*-algebra A is called G-**separating** if for every $a, b \in A_+$, $c \in A$, $\varepsilon > 0$, there exist $s, t \in A$ and $g, h \in G$ such that

$$\|s^* as - \sigma_g(a)\| < \varepsilon, \quad \|t^* at - \sigma_h(a)\| < \varepsilon, \quad \|s^* ct\| < \varepsilon.$$

Lem.

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a commutative C^*-algebra $A = C_0(\Omega)$ is G-separating if and only if for every $U_1, U_2 \subseteq \Omega$ and compact $K_1, K_2 \subseteq \Omega$ with $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, there exist g, $h \in G$ such that

$$\theta_g(K_1) \subseteq U_1, \qquad \theta_h(K_2) \subseteq U_2, \qquad \theta_g(K_1) \cap \theta_h(K_2) = \emptyset,$$

where $\Theta = \{\theta_t\}_{t\in G}$ is the action dual to α.

Thm.

Let α be a G-separating action on A and suppose that either $A = C_0(\Omega)$ and the dual action is residually topologically free, or that A is separable and α is residually properly outer action. Then $A \rtimes_{\alpha,r} G$ is strongly purely infinite.