Topological aperiodicity for product systems of C^*-correspondences

Bartosz Kwaśniewski, IMADA, Odense

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based on joint work with Wojciech Szymański, IMADA, Odense

Universal C^*-algebras and uniqueness problem

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Def. The universal C^*-algebra generated by $\mathcal{G}$ subject to $\mathcal{R}$ is a C^*-algebra $C^*(\mathcal{G}, \mathcal{R}) := C^*(\iota(\mathcal{G}))$ where ι is a representation of $(\mathcal{G}, \mathcal{R})$ such that if π is a representation of $(\mathcal{G}, \mathcal{R})$ then

$$\iota(g) \longmapsto \pi(g), \qquad g \in \mathcal{G},$$

extends to an epimorphism $C^*(\mathcal{G}, \mathcal{R}) \to C^*(\pi(\mathcal{G}))$.

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Def. $(\mathcal{G}, \mathcal{R})$ has uniqueness property if for any two faithful representations π_1, π_2 of $(\mathcal{G}, \mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

extends to $*$-isomorphism $*\text{-}Alg(\pi_1(\mathcal{G})) \cong *\text{-}Alg(\pi_2(\mathcal{G}))$, which in the presence of 'amenability' is equivalent to $C^*(\pi_1(\mathcal{G})) \cong C^*(\pi_2(\mathcal{G}))$.

Crossed products by group actions

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Fact.

If $A \rtimes_\alpha^r G$ is the reduced crossed product the natural surjection

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in general is not injective.

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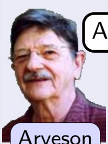
Aperiodicity



Arveson

Classical examples: quantum statistics

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canonical anticommutation relations (CAR) algebra

$\mathcal{G} = \{a(f) : f \in H$ - Hilbert space$\}$, $\mathcal{R}$ - conj. linear structure of H plus

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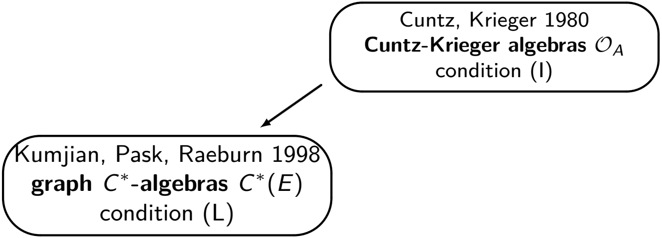

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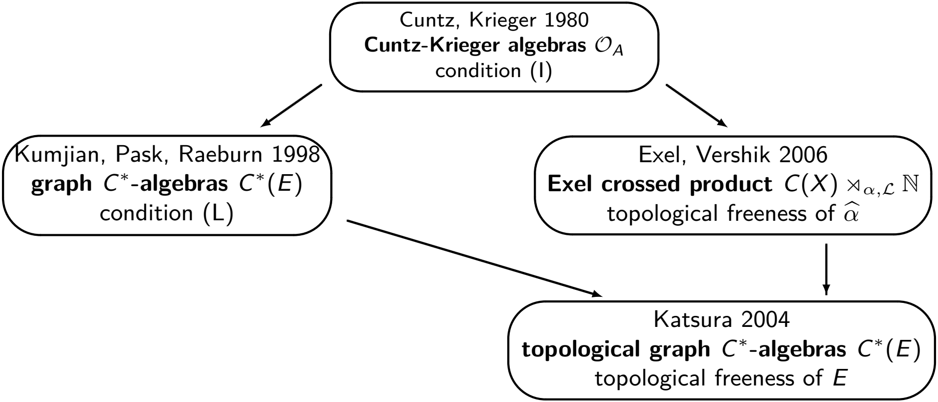

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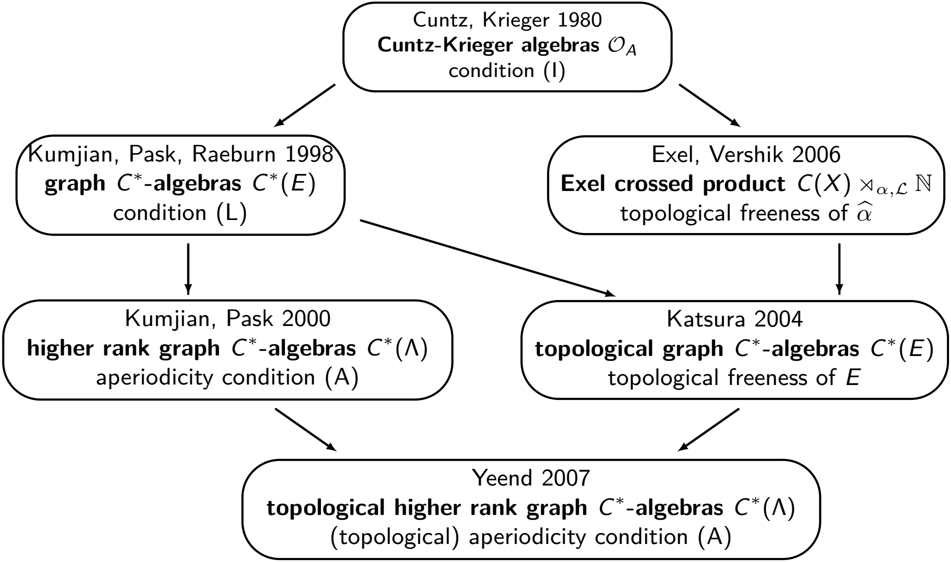

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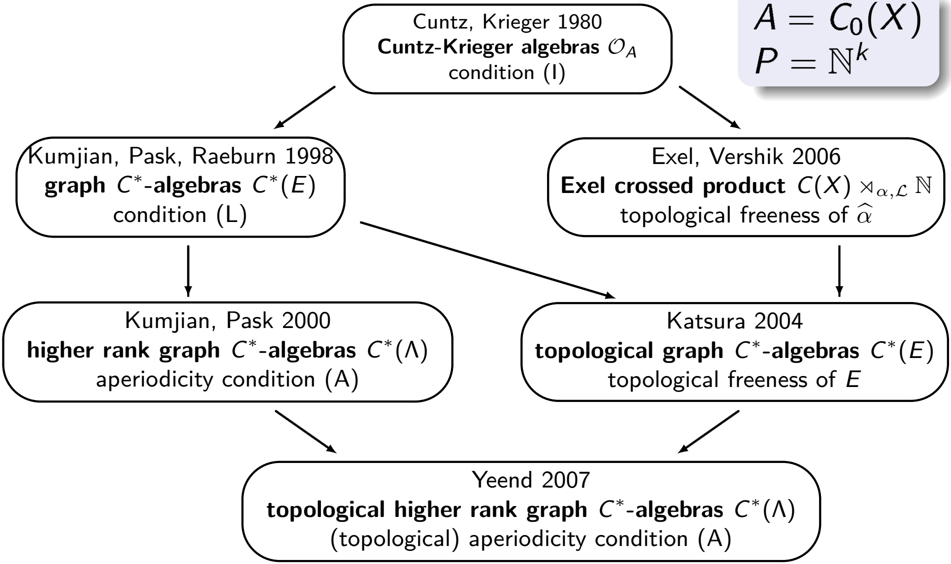

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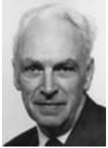

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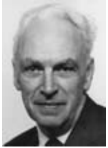
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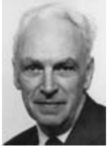

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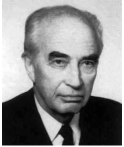

P. Dubreil

Regular C^*-correspondences and their product systems

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Representation of X in a C^*-algebra B is a pair (π, ψ) where $\pi : A \to B$ is a $*$-homomorphism and $\psi : X \to B$ linear s.t.

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Cuntz-Pimsner algebra is $\mathcal{O}_X := C^*(i_A(A) \cup i_X(X))$ where (i_A, i_X) is a covariant universal representation of X (M. Pimsner 1997)

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Problem

In general the structure of $\mathcal{O}_X$ is not well understood!

Doplicher-Roberts picture of Cuntz-Pimsner algebra $\mathcal{O}_X$

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Let $G = PP^{-1}$ be the group of fractions and recall that $(P, \leq)$ is directed where $p \leq q \iff pr = q$ for some $r \in P$.

Thm. (Doplicher-Roberts picture of Cuntz-Pimsner algebra $\mathcal{O}_X$)

For each $g = pq^{-1} \in G$, $p, q \in P$, define the Banach space direct limit

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Problem For which X, for any injective covariant representation ψ of X there is a (unique) epimorphism $\lambda_\psi : C^*(\psi(X)) \to \mathcal{O}_X^r$ such that:

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Thm. Suppose X is an imprimitivity $A-B$-bimodule

For $\pi : B \to \mathcal{B}(H)$ define $X\text{-Ind}(\pi) : A \to \mathcal{B}(X \otimes_\pi H)$ by

$$X\text{-Ind}(\pi)(a)(x \otimes_\pi h) = (ax) \otimes_\pi h.$$

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Rieffel

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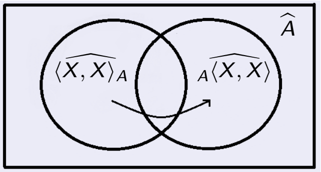

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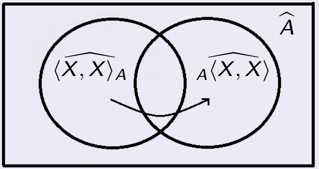

Thm. (Kwasniewski 2014)

If $[X\text{-Ind}]$ is topologically free, then $A \rtimes_X \mathbb{Z}$ possess uniqueness property

Let $\widehat{A} = \{[\pi] : \pi \in \mathrm{Irr}(A)\}$ be the spectrum of a C^*-algebra A.

Multivalued maps dual to regular C^*-correspondences

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A *dual* to α is a multivalued map $\widehat{\alpha} : \widehat{B} \to \widehat{A}$ ($\widehat{\alpha} : \widehat{B} \to 2^{\widehat{A}}$) given by

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Def.

We define *dual map* $\widehat{X} : \widehat{A} \to \widehat{A}$ *to the regular C^*-correspondence* X as the composition of multivalued maps

$$\widehat{X} = \widehat{\phi} \circ [X\text{-Ind}]$$

where $\widehat{\phi} : \widehat{\mathcal{K}(X)} \to \widehat{A}$ is dual to the left action $\phi : A \to \mathcal{K}(X)$ of A on X.

Prop. The family $\widehat{X} := \{\widehat{X}_p\}_{p \in P}$ is a semigroup of multivalued maps

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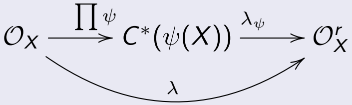

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Corollary (simplicity of $\mathcal{O}^r_X$)

Suppose that X is topologically aperiodic and *minimal*, i.e. there are no nontrivial ideals J in A such that

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Proof: $I \triangleleft \mathcal{O}_X^r$ implies $J := A \cap I$ is either A or $\{0\}$.

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