Purely infinite C^*-algebras associated to Fell bundles over discrete groups

Bartosz Kwaśniewski, University of Southern Denmark, Odense

based on joint work with Wojciech Szymański (arXiv:1505.05202)

'Pure infiniteness and ideal structure of C^*-algebras associated to Fell bundles'

modulo 'work in progress'

Purely infinite crossed products (overview)

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Reduced crossed products $A \rtimes_{\alpha,r} G$ by group actions

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Thm. (Pasnicu, Rørdam 2007) If A has (IP) then

A is purely infinite $\iff$ A is strongly purely infinite

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Rem. There is a faithful conditional expectation $E : C^*_r(\mathcal{B}) \to B_e$ onto the unit fiber C^*-algebra B_e.

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Prop. Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is aperiodic.

For every $b \in C^*_r(\mathcal{B})^+ \setminus \{0\}$ there is $a \in B_e^+ \setminus \{0\}$ such that $a \precsim b$

Aperiodicity and ideal structure of $C^*_r(\mathcal{B})$

Concept abstracted from the work of: Connes 1976, Elliot, 1980, Kishimoto 1981, Olesen-Pedersen, 1982, Muhly-Solel 2000, Giordano-Sierakowski 2014

Def.

A Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ is **aperiodic** if for each $g \in G \setminus \{e\}$, each $b_g \in B_g$ and every hereditary subalgebra D of B_e,

$$\inf\{\|ab_g a\| : a \in D^+,\ \|a\| = 1\} = 0.$$

$\mathcal{B}$ is **residually aperiodic** if $\mathcal{B}/\mathcal{J}$ is aperiodic for any ideal $\mathcal{J}$ in $\mathcal{B}$.

Thm. Suppose that $\mathcal{B}$ is exact and residually aperiodic.

We have a bijection between ideals in $C^*_r(\mathcal{B})$ and $\mathcal{B}$-invariant ideals in B_e:

$$C^*_r(\mathcal{B}) \rhd J \longrightarrow J \cap B_e \in \mathcal{I}^{\mathcal{B}}(B_e).$$

Prop. Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is aperiodic.

For every $b \in C^*_r(\mathcal{B})^+ \setminus \{0\}$ there is $a \in B_e^+ \setminus \{0\}$ such that $a \precsim b$

Def. (Cuntz 1978). Let $a, b \in A^+$. $a \precsim b \overset{def}{\Longleftrightarrow} \forall_{\varepsilon>0}\ \exists_{x\in A}\ \ x^* bx \approx_\varepsilon a$.

Pure infiniteness criterion

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If $RR(B_e) = 0$, each of the above conditions is equivalent to

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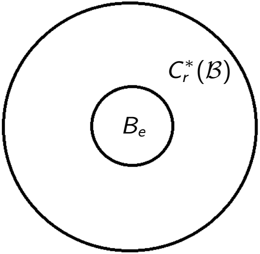

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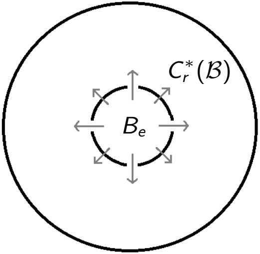

Paradoxicality

Def. (Banach-Tarski 1924, Sierakowski-Rørdam 2012)

Let $\Theta = \{\theta_g\}_{g \in G}$ be a group action on a locally compact Hausdorff Ω. A non-empty open set $V \subseteq \Omega$ is called Θ-**paradoxical** if there are open sets $V_1, \dots, V_{n+m}$ and elements $t_1, \dots, t_{n+m} \in G$, such that

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Hence a is properly infinite in $C_r^*(\mathcal{B})$.

Residual Infiniteness

Def. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is **$\mathcal{B}$-infinite** if there is $b \in B_e^+ \setminus \{0\}$ such that for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, ..., n+m$, and

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We say that $a \in B_e^+ \setminus \{0\}$ is **residually $\mathcal{B}$-infinite** if for every ideal $\mathcal{J} = \{J_g\}_{g\in G}$ the element $a + J_e$ is either zero in B_e/J_e or it is $\mathcal{B}/\mathcal{J}$-infinite.

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$C_r^*(\mathcal{B})$ is purely infinite and has (IP) whenever one of the following conditions holds:

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Let α be an exact group action on $C_0(\Omega)$ induced by residually topologically free action $\Theta = \{\theta_g\}_{g\in G}$ on a totally disconnected space Ω.

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Let α be an exact group action on $C_0(\Omega)$ induced by residually topologically free action $\Theta = \{\theta_g\}_{g\in G}$ on a totally disconnected space Ω. If every non-empty compact and open set is paradoxical, then $A \rtimes_{\alpha,r} G$ is purely infinite.

Strong boundary and n-filling actions

Def. (Laca-Spielberg 1996)

A group action $\Theta = \{\theta_t\}_{t\in G}$ on a compact Hausdorff space Ω (which is not finite as a set) is **strong boundary action** if for every two nonempty open subsets U_1, U_2 of Ω there are $g_1, g_2 \in G$ such that $\theta_{g_1}(U_1) \cup \theta_{g_2}(U_2) = \Omega$.

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A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a unital C^*-algebra A with infinite dimensional corners is called ***n*-filling**, for $n \geq 2$, if, for all elements $b_1, \ldots, b_n \in A^+$ of norm one, and for all $\varepsilon > 0$, there exist $g_1, \ldots, g_n \in G$ such that $\sum_{i=1}^{n} \alpha_{g_i}(b_i) \geq 1 - \varepsilon$.

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Lem.

Let α be an n-filling action and $\mathcal{B}$ the corresponding Fell bundle. Then $\mathcal{B}$ is minimal and any element $a \in A^+ \setminus \{0\}$ is residually $\mathcal{B}$-infinite.

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Let α be an n-filling action on A and suppose that either $A = C(\Omega)$ and the dual action is topologically free, or that A is separable and α is a properly outer action. Then $A \rtimes_{\alpha,r} G$ is simple and purely infinite.

General relationship between various actions

Paradoxical

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$$\Downarrow$$

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$$\Downarrow$$

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General relationship between various actions

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	$\Downarrow$
Paradoxical	Paradoxical + minimial
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Question:

Our theorem works for $A \rtimes_{\alpha}^{r} G$ with A being G-simple or for A with (IP). **To what extent can we extend it?**

General relationship between various actions

Separating (Kirchberg, Sierakowski)	n-filling
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Separating actions (Kirchberg-Sierakowski 2015 preprint)

Def.

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a C^*-algebra A is called G-**separating** if for every $a, b \in A_+$, $c \in A$, $\varepsilon > 0$, there exist $s, t \in A$ and $g, h \in G$ such that

$$\|s^* as - \sigma_g(a)\| < \varepsilon, \quad \|t^* at - \sigma_h(a)\| < \varepsilon, \quad \|s^* ct\| < \varepsilon.$$

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A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a commutative C^*-algebra $A = C_0(\Omega)$ is G-separating if and only if for every $U_1, U_2 \subseteq \Omega$ and compact $K_1, K_2 \subseteq \Omega$ with $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, there exist g, $h \in G$ such that

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Thm.

Let α be a G-separating action on A and suppose that either $A = C_0(\Omega)$ and the dual action is residually topologically free, or that A is separable and α is residually properly outer action. Then $A \rtimes_{\alpha,r} G$ is strongly purely infinite.