Topological aperiodicity for product systems of C^*-correspondences

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based on joint work with Wojciech Szymański, IMADA, Odense

Universal C^*-algebras and uniqueness problem

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Def. The universal C^*-algebra generated by $\mathcal{G}$ subject to $\mathcal{R}$ is a C^*-algebra $C^*(\mathcal{G}, \mathcal{R}) := C^*(\iota(\mathcal{G}))$ where ι is a representation of $(\mathcal{G}, \mathcal{R})$ such that if π is a representation of $(\mathcal{G}, \mathcal{R})$ then

$$\iota(g) \longmapsto \pi(g), \qquad g \in \mathcal{G},$$

extends to an epimorphism $C^*(\mathcal{G}, \mathcal{R}) \to C^*(\pi(\mathcal{G}))$.

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Def. $(\mathcal{G},\mathcal{R})$ has uniqueness property if for any two faithful representations π_1, π_2 of $(\mathcal{G},\mathcal{R})$ the mapping

$$\pi_1(g) \longmapsto \pi_2(g), \qquad g \in \mathcal{G},$$

extends to $*$-isomorphism $*\text{-}Alg(\pi_1(\mathcal{G})) \cong *\text{-}Alg(\pi_2(\mathcal{G}))$, which in the presence of 'amenability' is equivalent to $C^*(\pi_1(\mathcal{G})) \cong C^*(\pi_2(\mathcal{G}))$.

Crossed products by group actions

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Fact.

If $A \rtimes_\alpha^r G$ is the reduced crossed product the natural surjection

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in general is not injective.

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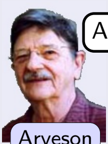

Arveson

Classical examples: quantum statistics

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Cuntz-Krieger uniqueness theorem industry

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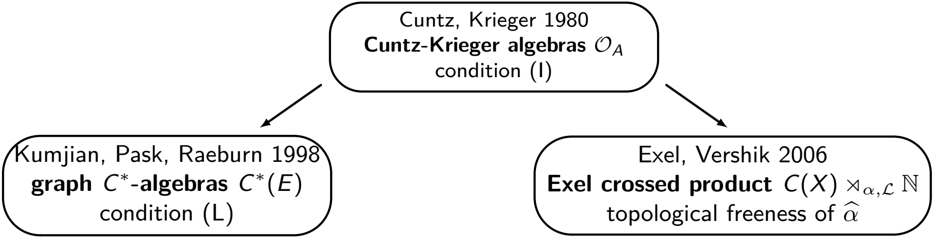

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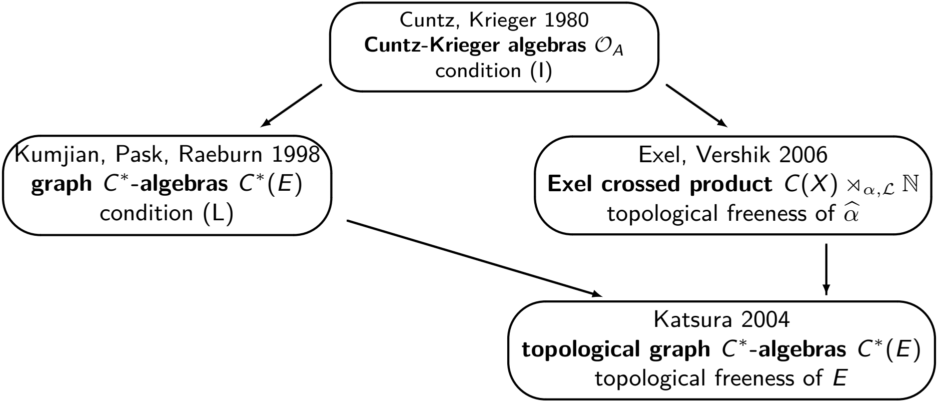

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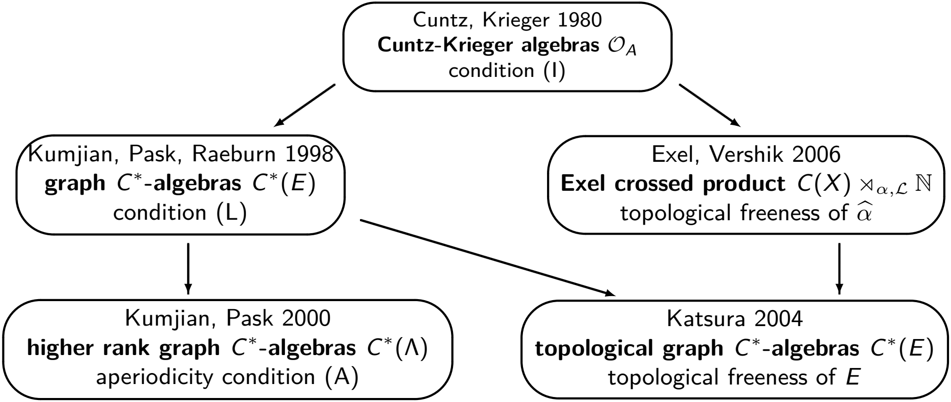

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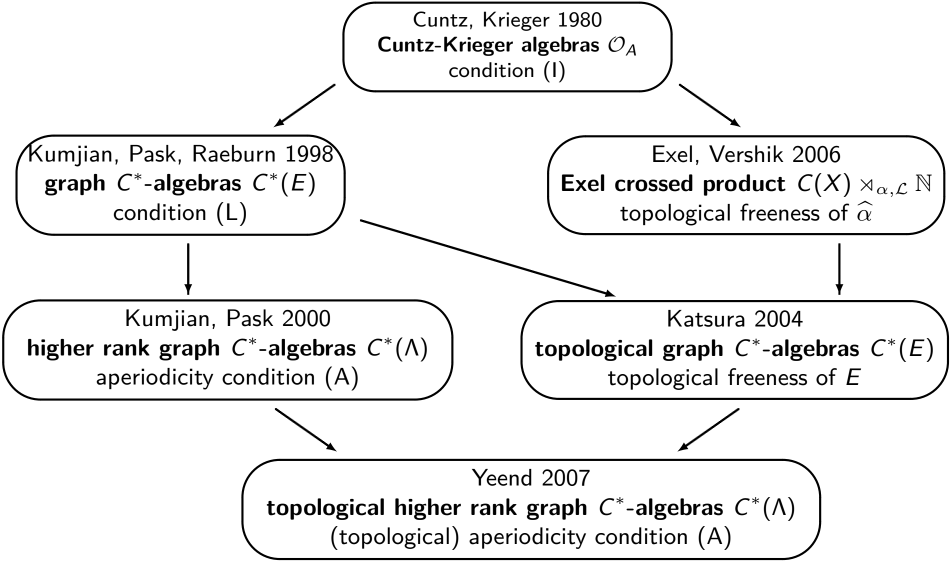

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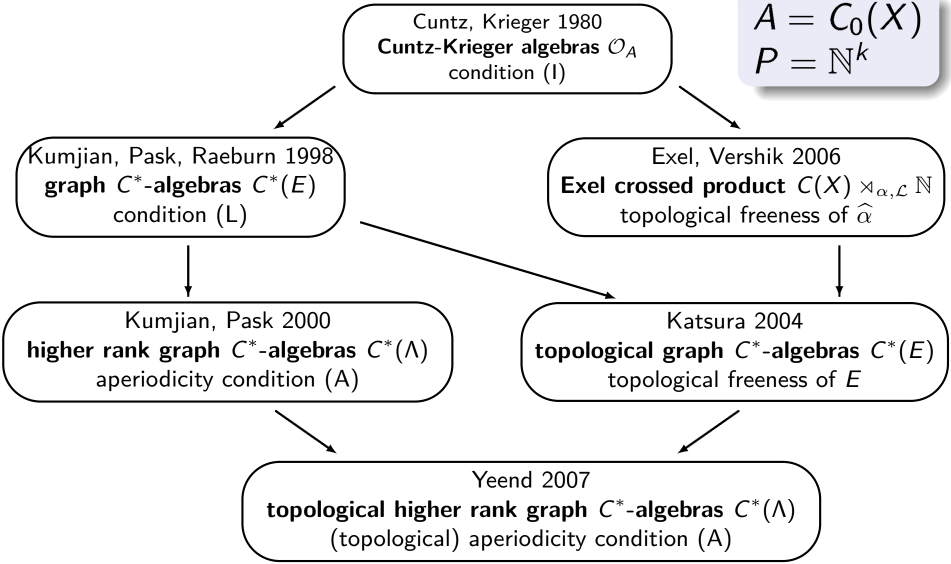

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Exel, Vershik 2006
Exel crossed product $C(X) \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$
topological freeness of $\widehat{\alpha}$

Kumjian, Pask 2000
higher rank graph C^***-algebras** $C^*(\Lambda)$
aperiodicity condition (A)

Katsura 2004
topological graph C^***-algebras** $C^*(E)$
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Gauge-(co)actions on $C^*(\mathcal{G}, \mathcal{R})$ and Fell bundles

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J. Quigg 1996

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Gauge-(co)actions on $C^*(\mathcal{G},\mathcal{R})$ and Fell bundles

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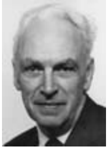

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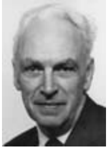

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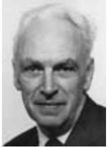

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Cuntz-Pimsner algebra is $\mathcal{O}_X := C^*(i_A(A) \cup i_X(X))$ where (i_A, i_X) is a covariant universal representation of X (M. Pimsner 1997)

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Problem

In general the structure of $\mathcal{O}_X$ is not well understood!

Doplicher-Roberts picture of Cuntz-Pimsner algebra $\mathcal{O}_X$

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Let $G = PP^{-1}$ be the group of fractions and recall that $(P, \leq)$ is directed where $p \leq q \iff pr = q$ for some $r \in P$.

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Problem For which X, for any injective covariant representation ψ of X there is a (unique) epimorphism $\lambda_\psi : C^*(\psi(X)) \to \mathcal{O}_X^r$ such that:

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Thm. Suppose X is an imprimitivity $A-B$-bimodule

For $\pi : B \to \mathcal{B}(H)$ define $X\text{-Ind}(\pi) : A \to \mathcal{B}(X \otimes_\pi H)$ by

$$X\text{-Ind}(\pi)(a)(x \otimes_\pi h) = (ax) \otimes_\pi h.$$

Then $[X\text{-Ind}] : \widehat{B} \to \widehat{A}$ is a homeomorphism.

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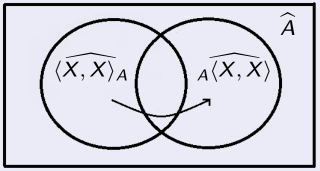

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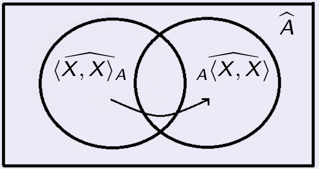
$$\begin{array}{l} \langle X, X\rangle_A := \overline{span}\{\langle x, y\rangle_A : x, y \in X\} \\ {}_A\langle X, X\rangle := \overline{span}\{{}_A\langle x, y\rangle : x, y \in X\} \end{array} \implies \textit{ideals in } A$$

X is a imprimitivity ${}_A\langle X, X\rangle$-$\langle X, X\rangle_A$ module and hence $[X\text{-Ind}] : \widehat{\langle X, X\rangle_A} \to \widehat{{}_A\langle X, X\rangle}$ is a *partial homeomorphism* of $\widehat{A}$.



Thm. (Kwasniewski 2014)

If $[X\text{-Ind}]$ is topologically free, then $A \rtimes_X \mathbb{Z}$ possess uniqueness property

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Def.

We define *dual map* $\widehat{X} : \widehat{A} \to \widehat{A}$ *to the regular* C^**-correspondence* X as the composition of multivalued maps

$$\widehat{X} = \widehat{\phi} \circ [X\text{-Ind}]$$

where $\widehat{\phi} : \widehat{\mathcal{K}(X)} \to \widehat{A}$ is dual to the left action $\phi : A \to \mathcal{K}(X)$ of A on X.

Prop. The family $\widehat{X} := \{\widehat{X}_p\}_{p\in P}$ is a semigroup of multivalued maps

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Uniqueness Theorem for $\mathcal{O}_X$. Suppose X is topologically aperiodic.

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For any injective covariant representation Ψ of X there is an epimorphism $\lambda_\psi : C^*(\psi(X)) \to \mathcal{O}_X^r$ such that the diagram

$$\begin{array}{ccccc} \mathcal{O}_X & \xrightarrow{\prod \psi} & C^*(\psi(X)) & \xrightarrow{\lambda_\psi} & \mathcal{O}_X^r \\ & & \underbrace{\qquad\qquad\qquad}_{\lambda} & & \end{array}$$

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Corollary (simplicity of $\mathcal{O}_X^r$)

Suppose that X is topologically aperiodic and *minimal*, i.e. there are no nontrivial ideals J in A such that

$$\forall_{p\in P} \quad \{a \in A : \langle X_p, aX_p\rangle_p \subseteq J\} = J.$$

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Uniqueness Theorem for $\mathcal{O}_X$. Suppose X is topologically aperiodic.

For any injective covariant representation Ψ of X there is an epimorphism $\lambda_\psi : C^*(\psi(X)) \to \mathcal{O}^r_X$ such that the diagram

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Proof: $I \triangleleft \mathcal{O}^r_X$ implies $J := A \cap I$ is either A or $\{0\}$.

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