Purely infinite C^*-algebras associated to Fell bundles over discrete groups

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'Pure infiniteness and ideal structure of C^*-algebras associated to Fell bundles'

modulo 'work in progress'

Purely infinite C^*-algebras

Let A be a C^*-algebra. Notation: $a \approx_\varepsilon b \overset{def}{\Longleftrightarrow} \|a-b\| < \varepsilon$.

Def. (Rørdam, Kirchberg 2000) For $a \in A^+ \setminus \{0\}$ we say

a is **infinite** $\Longleftrightarrow$ $\exists_{b\in A^+\setminus\{0\}} \forall_{\varepsilon>0} \exists_{x,y\in aA} \quad x^*x \approx_\varepsilon a, \quad y^*y \approx_\varepsilon b, \quad x^*y \approx_\varepsilon 0$

a is **properly infinite** $\Longleftrightarrow$ $\forall_{\varepsilon>0} \exists_{x,y\in aA} \quad x^*x \approx_\varepsilon a, \quad y^*y \approx_\varepsilon a, \quad x^*y \approx_\varepsilon 0$

Prop. $a \in A^+ \setminus \{0\}$ is properly infinite if and only if

for every ideal I in A the image of a in A/I is either zero or infinite.

Def. (Rørdam, Kirchberg 2000)

C^*-algebra A is **purely infinite** $\Longleftrightarrow$ every $a \in A^+ \setminus \{0\}$ is properly infinite.

A has the **ideal property (IP)** if projections in A separate ideals in A

Thm. (Pasnicu, Rørdam 2007) If A has (IP) then

A is purely infinite $\Longleftrightarrow$ A is strongly purely infinite

Fell bundles (throughout G is a discrete group)

Def. (Fell 1969)

A **Fell bundle** $\mathcal{B}$ over G is consists of Banach spaces $\{B_g\}_{g\in G}$ equipped with

$$\cdot : B_g \times B_h \longmapsto B_{gh}, \qquad * : B_g \longmapsto B_{g^{-1}}, \qquad g, h \in G,$$

such that $\bigoplus_{g\in G} B_g$ becomes a *-algebra admitting a C^*-norm.

The **full cross sectional C^*-algebra** of $\mathcal{B}$ is $C^*(\mathcal{B}) := \overline{\bigoplus_{g\in G} B_g}^{\|\cdot\|_{max}}$ where $\|\cdot\|_{max}$ is the maximal C^*-norm on $\bigoplus_{g\in G} B_g$.

Def. (Exel, Quigg 1996)

The **reduced cross sectional C^*-algebra** of $\mathcal{B}$ is $C^*_r(\mathcal{B}) := \overline{\bigoplus_{g\in G} B_g}^{\|\cdot\|_{min}}$ where $\|\cdot\|_{min}$ is the minimal C^*-norm on $\bigoplus_{g\in G} B_g$ such that

$$\|a_e\| \leq \|\sum_{g\in G} a_g\| \qquad \text{for all} \quad \sum_{g\in G} a_g \in \bigoplus_{g\in G} B_g.$$

Rem. There is a faithful conditional expectation $E : C^*_r(\mathcal{B}) \to B_e$ onto the unit fiber C^*-algebra B_e.

Ideals and exactness

Def. Fix a Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$.

An **ideal** in $\mathcal{B}$ is $\mathcal{J} = \{J_g\}_{g\in G}$ where J_g is a closed subspace of B_g, and

$$B_g J_h \subseteq J_{gh} \qquad J_g B_h \subseteq J_{gh}, \qquad \text{for all } g,h \in G.$$

An ideal I in B_e is $\mathcal{B}$-**invariant** if $B_g I B_g^* \subseteq I$ for every $g \in G$.

Rem. Relation $J_e = I$ establishes a bijection between ideals $\{J_g\}_{g\in G}$ in $\mathcal{B}$ and $\mathcal{B}$-invariant ideals I in B_e.

If $\mathcal{J} = \{J_g\}_{g\in G}$ is an ideal in $\mathcal{B} = \{B_g\}_{g\in G}$ then

$$0 \longrightarrow C_r^*(\mathcal{J}) \longrightarrow C_r^*(\mathcal{B}) \longrightarrow C_r^*(\mathcal{B}/\mathcal{J}) \longrightarrow 0.$$

Def. $\mathcal{B}$ is **exact** if the above sequence is exact for every ideal $\mathcal{J}$ in $\mathcal{B}$.

Rem. G is exact $\Longrightarrow$ $\mathcal{B}$ is exact
$\mathcal{B}$ is amenable, i.e. $C_r^*(\mathcal{B}) = C^*(\mathcal{B}) \Longrightarrow \mathcal{B}$ is exact

Aperiodicity and ideal structure of $C_r^*(\mathcal{B})$

Concept abstracted from the work of: Connes 1976, Elliot, 1980, Kishimoto 1981, Olesen-Pedersen, 1982, Muhly-Solel 2000, Giordano-Sierakowski 2014

Def.

A Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ is **aperiodic** if for each $g \in G \setminus \{e\}$, each $b_g \in B_g$ and every hereditary subalgebra D of B_e,

$$\inf\{\|ab_g a\| : a \in D^+,\ \|a\| = 1\} = 0.$$

$\mathcal{B}$ is **residually aperiodic** if $\mathcal{B}/\mathcal{J}$ is aperiodic for any ideal $\mathcal{J}$ in $\mathcal{B}$.

Thm. Suppose that $\mathcal{B}$ is exact and residually aperiodic.

We have a bijection between ideals in $C_r^*(\mathcal{B})$ and $\mathcal{B}$-invariant ideals in B_e:

$$C_r^*(\mathcal{B}) \triangleright J \longrightarrow J \cap B_e \triangleleft B_e.$$

Prop. Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is aperiodic.

For every $b \in C_r^*(\mathcal{B})^+ \setminus \{0\}$ there is $a \in B_e^+ \setminus \{0\}$ such that $a \precsim b$

Pure infiniteness criterion

Thm. Suppose that $\mathcal{B}$ is exact and residually aperiodic.

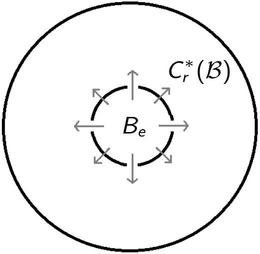
If either B_e **has (IP) or** $\mathcal{B}$ **is minimal**, i.e. are no non-trivial $\mathcal{B}$-invariant ideals in B_e, then the following statements are equivalent:

(i) $C_r^*(\mathcal{B})$ is purely infinite.

(ii) Every element in $B_e^+ \setminus \{0\}$ is properly infinite in $C_r^*(\mathcal{B})$.

If $RR(B_e) = 0$, each of the above conditions is equivalent to

(ii') Every non-zero projection in B_e is properly infinite in $C_r^*(\mathcal{B})$.



Paradoxicality

Def. (Banach-Tarski 1924, Sierakowski-Rørdam 2012)

Let $\Theta = \{\theta_g\}_{g\in G}$ be a group action on a locally compact Hausdorff Ω. A non-empty open set $V \subseteq \Omega$ is called Θ-**paradoxical** if there are open sets $V_1, \ldots, V_{n+m}$ and elements $t_1, \ldots, t_{n+m} \in G$, such that

$$V = \bigcup_{i=1}^{n} V_i = \bigcup_{i=n+1}^{n+m} V_i, \quad \theta_{t_i}(V_i) \subseteq V \;\text{ and }\; \theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \emptyset \text{ for all } i \neq j.$$

Def. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is $\mathcal{B}$-**paradoxical** if for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, \ldots, n+m$, such that

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \quad a \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \varepsilon / \max\{n^2, m^2\} \;\text{ for } i \neq j.$$

Rem. If $a \in B_e^+$ is $\mathcal{B}$-paradoxical, then for $x := \sum_{i=1}^{n} a_i$ and $y := \sum_{i=n+1}^{n+m} a_i$

$$a \approx_{2\varepsilon} x^* x, \qquad a \approx_{2\varepsilon} y^* y, \qquad x^* y \approx_\varepsilon 0.$$

Hence a is properly infinite in $C_r^*(\mathcal{B})$.

Residual Infiniteness

Def. Let $\mathcal{B} = \{B_g\}_{g\in\mathcal{G}}$ be a Fell bundle.

An element $a \in B_e^+ \setminus \{0\}$ is $\mathcal{B}$-**infinite** if there is $b \in B_e^+ \setminus \{0\}$ such that for every $\varepsilon > 0$ there are elements $a_i \in aB_{t_i}$, where $t_i \in G$ for $i = 1, \dots, n+m$, and

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \quad b \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i, \quad \text{and} \quad \|a_i^* a_j\| < \varepsilon/\max\{n^2, m^2\} \ \text{ for } i \neq j.$$

We say that $a \in B_e^+ \setminus \{0\}$ is **residually** $\mathcal{B}$-**infinite** if for every ideal $\mathcal{J} = \{J_g\}_{g\in G}$ the element $a + J_e$ is either zero in B_e/J_e or it is $\mathcal{B}/\mathcal{J}$-infinite.

Def. Let $\Theta = \{\theta_g\}_{g\in G}$ be a group action on a locally compact Hausdorff Ω.

A non-empty open set $V \subseteq \Omega$ is called Θ-**infinite** if there are open sets $V_1, \dots, V_n$ and elements $t_1, \dots, t_n \in G$, such that

$$V = \bigcup_{i=1}^{n} V_i, \quad \overline{\bigcup_{i=1}^{n} \theta_{t_i}(V_i)} \subsetneq V \ \text{ and } \ \theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \emptyset \ \text{ for } i \neq j.$$

The main result

Thm. Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is an exact, residually aperiodic Fell bundle.

$C_r^*(\mathcal{B})$ is purely infinite and has (IP) whenever one of the following conditions holds:

(i) B_e has (IP) and every element in $B_e^+ \setminus \{0\}$ is residually $\mathcal{B}$-infinite,

(i') $RR(B_e) = 0$ and every non-zero projection in B_e is residually $\mathcal{B}$-infinite,

(ii) $\mathcal{B}$ is minimal and every element in $B_e^+ \setminus \{0\}$ is $\mathcal{B}$-infinite.

Cor. (Sierakowski-Rørdam)

Let α be an exact group action on $C_0(\Omega)$ induced by residually topologically free action $\Theta = \{\theta_g\}_{g\in G}$ on a totally disconnected space Ω. If every non-empty compact and open set is paradoxical, then $A \rtimes_{\alpha,r} G$ is purely infinite.

Strong boundary and n-filling actions

Def. (Laca-Spielberg 1996)

A group action $\Theta = \{\theta_t\}_{t\in G}$ on a compact Hausdorff space Ω (which is not finite as a set) is **strong boundary action** if for every two nonempty open subsets U_1, U_2 of Ω there are $g_1, g_2 \in G$ such that $\theta_{g_1}(U_1) \cup \theta_{g_2}(U_2) = \Omega$.

Def. (Jolissaint-Robertson 2000)

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a unital C^*-algebra A with infinite dimensional corners is called n-**filling**, for $n \geq 2$, if, for all elements $b_1, ..., b_n \in A^+$ of norm one, and for all $\varepsilon > 0$, there exist $g_1, ..., g_n \in G$ such that $\sum_{i=1}^n \alpha_{g_i}(b_i) \geq 1 - \varepsilon$.

Lem.

Let α be an n-filling action and $\mathcal{B}$ the corresponding Fell bundle. Then $\mathcal{B}$ is minimal and any element $a \in A^+ \setminus \{0\}$ is residually $\mathcal{B}$-infinite.

Cor. (Laca-Spielberg, Jollisaint-Robertson)

Let α be an n-filling action on A and suppose that either $A = C(\Omega)$ and the dual action is topologically free, or that A is separable and α is a properly outer action. Then $A \rtimes_{\alpha,r} G$ is simple and purely infinite.

General relationship between various actions

Separating (Kirchberg, Sierakowski 2015)	n-filling
$\Downarrow$	$\Downarrow$
Paradoxical	Paradoxical + minimial
$\Downarrow$	$\Downarrow$
Residually infinite	(Residually) infinite + minimal

Question:

Our theorem works for $A \rtimes_{\alpha}^{r} G$ with A being G-simple or for A with (IP). **To what extent can we extend it?**

Separating actions (Kirchberg-Sierakowski 2015 preprint)

Def.

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a C^*-algebra A is called G-**separating** if for every $a, b \in A_+$, $c \in A$, $\varepsilon > 0$, there exist $s, t \in A$ and $g, h \in G$ such that

$$\|s^* as - \sigma_g(a)\| < \varepsilon, \quad \|t^* at - \sigma_h(a)\| < \varepsilon, \quad \|s^* ct\| < \varepsilon.$$

Lem.

A group action $\alpha = \{\alpha_t\}_{t\in G}$ on a commutative C^*-algebra $A = C_0(\Omega)$ is G-separating if and only if for every $U_1, U_2 \subseteq \Omega$ and compact $K_1, K_2 \subseteq \Omega$ with $K_1 \subseteq U_1$, $K_2 \subseteq U_2$, there exist g, $h \in G$ such that

$$\theta_g(K_1) \subseteq U_1, \qquad \theta_h(K_2) \subseteq U_2, \qquad \theta_g(K_1) \cap \theta_h(K_2) = \emptyset,$$

where $\Theta = \{\theta_t\}_{t\in G}$ is the action dual to α.

Thm.

Let α be a G-separating action on A and suppose that either $A = C_0(\Omega)$ and the dual action is residually topologically free, or that A is separable and α is residually properly outer action. Then $A \rtimes_{\alpha,r} G$ is strongly purely infinite.