Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 3(4) 2005 718–765

Covariance algebra of a partial dynamical system

Bartosz Kosma Kwaśniewski*

Institute of Mathematics,
University in Bialystok,
ul. Akademicka 2, PL-15-424 Bialystok, Poland

Received 30 October 2004; accepted 11 August 2005

Abstract: A pair (X, α) is a partial dynamical system if X is a compact topological space and $\alpha : \Delta \to X$ is a continuous mapping such that Δ is open. Additionally we assume here that Δ is closed and $\alpha(\Delta)$ is open. Such systems arise naturally while dealing with commutative C^*-dynamical systems.
In this paper we construct and investigate a universal C^*-algebra $C^*(X, \alpha)$ which agrees with the partial crossed product [10] in the case α is injective, and with the crossed product by a monomorphism [22] in the case α is onto.
The main method here is to use the description of maximal ideal space of a coefficient algebra, cf. [16, 18], in order to construct a larger system $(\widetilde{X}, \widetilde{\alpha})$ where $\widetilde{\alpha}$ is a partial homeomorphism. Hence one may apply in extenso the partial crossed product theory [10, 13]. In particular, one generalizes the notions of topological freeness and invariance of a set, which are being used afterwards to obtain the Isomorphism Theorem and the complete description of ideals of $C^*(X, \alpha)$.

Keywords: Crossed product, C^*-dynamical system, covariant representation, topological freeness
MSC (2000): 47L65, 46L45, 37B99

Introduction

In quantum theory the term covariance algebra (crossed product) means an algebra generated by an algebra of observables and by operators which determine the time evolution of a quantum system (a C^*-dynamical system), thereby the covariance algebra is an object which carries all the information about the quantum system, see [6, 18] (and the sources cited there) for this and other connections with mathematical physics. In pure mathematics C^*-algebras associated to C^*-dynamical systems proved to be useful in different

* E-mail: bartoszk@math.uwb.edu.pl

fields: classification of operator algebras [25, 6, 23]; K-theory for C^*-algebras [4, 10, 20]; functional and functional differential equations [2, 3]; or even in number theory [17]. This multiplicity of applications and the complexity of the matter attracted many authors and caused an abundence of various approaches [25, 10, 20, 26, 1, 22, 12, 14]. In the present article we propose another approach which on one hand may seem to embrace a very special case but on the other hand:

1) unlike the other authors investigating crossed products (see discussions below) we do not require here any kind (substitute) of reversibility of the given system,
2) we obtain a rather thorough description of the associated covariance algebra, and also strong tools to study it,
3) we find points of contact of different approaches and thereby clarify the relations between them.

In order to give the motivation of the construction of the crossed product developed in the paper we would like to present a simple example.

Example. Consider the Hilbert space $H = L^2_\mu(\mathbb{R})$ where μ is the Lebesgue measure. Let $\mathcal{A} \subset L(H)$ be the C^*-algebra of operators of multiplication by continuous bounded functions on $\mathbb{R}$ that are constant on $\mathbb{R}_- = \{x : x \leq 0\}$. Set the unitary operator $U \in L(H)$ by the formula

$$(Uf)(x) = f(x-1), \qquad f(\,\cdot\,) \in H.$$

Routine verification shows that the mapping

$$\mathcal{A} \ni a \mapsto UaU^*$$

is an endomorphism of $\mathcal{A}$ of the form

$$UaU^*(x) = a(x-1), \qquad a(\,\cdot\,) \in \mathcal{A}, \tag{1}$$

and

$$U^*aU(x) = a(x+1), \qquad a(\,\cdot\,) \in \mathcal{A}. \tag{2}$$

Clearly the mapping $\mathcal{A} \ni a \mapsto U^*aU$ *does not* preserve $\mathcal{A}$.

Let $C^*(\mathcal{A}, U)$ be the C^*-algebra generated by $\mathcal{A}$ and U. It is easy to show that

$$C^*(\mathcal{A}, U) = C^*(\mathcal{B}, U)$$

where $\mathcal{B} \subset L(H)$ is the C^*-algebra of operators of multiplication by continuous bounded functions on $\mathbb{R}$ that have limits at $-\infty$.

In addition we have that

$$U\mathcal{B}U^* \subset \mathcal{B} \qquad \text{and} \qquad U^*\mathcal{B}U \subset \mathcal{B}$$

and the corresponding actions $\delta(\,\cdot\,) = U(\,\cdot\,)U^*$ and $\delta_*(\,\cdot\,) = U^*(\,\cdot\,)U$ on $\mathcal{B}$ are given by formulae (1) and (2).

Moreover

$$C^*(\mathcal{A}, U) = C^*(\mathcal{B}, U) \cong \mathcal{B} \times_\delta \mathbb{Z}.$$

Where in the right hand side stands the standard crossed product of $\mathcal{B}$ by the *automorphism* δ.

This example shows a natural situation when the crossed product $\mathcal{B} \times_\delta \mathbb{Z}$ is 'invisible' at the begining (on the initial algebra $\mathcal{A}$, δ acts as an endomorphism and δ_* even does not preserve $\mathcal{A}$) but after implementing a natural extension of $\mathcal{A}$ up to $\mathcal{B}$, δ becomes an *automorphism* and leads to the crossed product structure. The aim of the paper is to investigate the general constructions of this type. We will also find out that the arising constructions are rather natural and one can come across them 'almost anywhere', in particular they present the passage from the irreversible topological Markov chains to the reverible ones (see Propostition 2.8) and the maximal ideal spaces of the algebras of $\mathcal{B}$ type possess the solenoid structure (see Examples 2.12, 6.15).

We deal here with C^*-dynamical systems where dynamics is implemented by a single endomorphism, hence a C^*-*dynamical system* is identified with a pair $(\mathcal{A}, \delta)$ where $\mathcal{A}$ is a unital C^*-algebra and $\delta : \mathcal{A} \to \mathcal{A}$ is a *-endomorphism. Additionally we assume that $\mathcal{A}$ is commutative. Hence, in fact, we deal with topological dynamical systems. Indeed, using the Gelfand transform one can identify $\mathcal{A}$ with the algebra $C(X)$ of continuous functions on the maximal ideal space X of $\mathcal{A}$, and within this identification the endomorphism δ generates (see, for example, [16]) a continuous *partial mapping* $\alpha : \Delta \to X$ where $\Delta \subset X$ is closed and open (briefly clopen), and the following formula holds

$$\delta(a)(x) = \begin{cases} a(\alpha(x)) \ , & x \in \Delta \\ 0 \ , & x \notin \Delta \end{cases}, \qquad a \in C(X). \tag{3}$$

Therefore we have one-to-one correspondence between the commutative unital C^*-dynamical systems $(\mathcal{A}, \delta)$ and pairs (X, α), where X is compact and α is a partial continuous mapping in which the domain is clopen. We shall call (X, α) a *partial dynamical system.*

Usually covariance algebra is another name for the crossed product which in turn is defined in various ways [25, 23, 10, 22], though it seems more appropriate to define such objects as C^*-algebras with a universal property with respect to covariant representations [26, 1]. In the literature, cf. [25, 10, 20, 1], a *covariant representation* of $(\mathcal{A}, \delta)$ is meant to be a triple (π, U, H) where H is a Hilbert space, $\pi : \mathcal{A} \to L(H)$ is a representation of $\mathcal{A}$ by bounded operators on H, and $U \in L(H)$ is such that

$$\pi(\delta(a)) = U\pi(a)U^*, \qquad \text{for all } \ a \in \mathcal{A},$$

plus eventually some other conditions imposed on U and π. If π is faithful we shall call (π, U, H) a *covariant faithful representation.* The covariant representations of a C^*-dynamical system give rise to a category $\mathrm{Cov}(\mathcal{A}, \delta)$ where objects are the C^*-algebras $C^*(\pi(\mathcal{A}), U)$, generated by $\pi(\mathcal{A})$ and U, while morphisms are the usual *-morphisms

$\phi : C^*(\pi(\mathcal{A}), U) \to C^*(\pi'(\mathcal{A}), U')$ such that

$$\phi(\pi(a)) = \pi'(a), \text{ for } a \in \mathcal{A}, \qquad \text{and} \qquad \phi(U) = U'$$

(here (π, U, H) and (π', U', H') denote covariant representations of $(\mathcal{A}, \delta)$). In many cases the main interest is concentrated on the subcategory CovFaith$(\mathcal{A}, \delta)$ of Cov$(\mathcal{A}, \delta)$ for which objects are algebras $C^*(\pi(\mathcal{A}), U)$ where now π is faithful. The fundamental problem then is to describe a universal object in Cov$(\mathcal{A}, \delta)$, or in CovFaith$(\mathcal{A}, \delta)$, in terms of the C^*-dynamical system $(\mathcal{A}, \delta)$. If such an object exists then it is unique up to isomorphism, and it shall be called a *covariance algebra*.

It is well known [25] that, in the case that δ is an automorphism, the classic crossed product $\mathcal{A} \rtimes_\delta \mathbb{Z}$ is the covariance algebra of the C^*-dynamical system $(\mathcal{A}, \delta)$. Being motivated by the paper [8], in which J. Cuntz discussed a concept of the crossed product by an endomorphism which is not an automorphism, many authors proposed theories of generalized crossed products with some kind of universality (see [24, 10, 20, 22, 12]). For instance, G. Murphy in [22] has proved that a corner of the crossed product of a certain direct limit is a covariance algebra of a system $(\mathcal{A}, \delta)$ where δ is a monomorphism (in fact he has proved far more general result, see [22, Theorem 2.3]). R. Exel in [10] introduced a partial crossed product which can be applied also in the case δ is not injective, though generating a partial automorphism (see also [13, 20]). Nevertheless, in general the inter-relationship between the C^*-dynamical system and its covariance algebra is still not totally-established.

In the approach developed in this paper we explore the leading concept of the coefficient algebra, introduced in [18]. The elements of this algebra play the role of Fourier's coefficients in the covariance algebra, hence the name. The authors of [18] studied the C^*-algebra $C^*(\mathcal{A}, U)$ generated by a *-algebra $\mathcal{A} \subset L(H)$ and a partial isometry $U \in L(H)$. They have defined $\mathcal{A}$ (in a slightly different yet equivalent form) to be a *coefficient algebra* of $C^*(\mathcal{A}, U)$ whenever $\mathcal{A}$ possess the following three properties

$$U^*U \in \mathcal{A}', \qquad U\mathcal{A}U^* \subset A, \qquad U^*\mathcal{A}U \subset \mathcal{A}, \tag{4}$$

where $\mathcal{A}'$ denotes the commutant of $\mathcal{A}$. Let us indicate that this concept appears, in more or less explicit form, in all the aforesaid articles: If U is unitary then (4) holds iff $\delta(\cdot) = U(\cdot)U^*$ is an automorphism of $\mathcal{A}$, and thus in this case $\mathcal{A}$ can be regarded as a coefficient algebra of the crossed product $\mathcal{A} \rtimes_\delta \mathbb{Z}$. For example in the paper [8], the UHF algebra $\mathcal{F}_n$ is a coefficient algebra of the Cuntz algebra $\mathcal{O}_n$. Also the algebra A considered by Paschke in [24] is a coefficient algebra of the C^*-algebra $C^*(A, S)$ generated by A and the isometry S. The algebra $C^{\tilde{\alpha}}$ defined in [22] as the fixed point algebra for dual action can be thought of as a generalized coefficient algebra of the crossed product $C^*(A, M, \alpha)$ of A by the semigroup M of injective endomorphisms.

Thanks to [16], the main tool we are given is the description of maximal ideal spaces of certain coefficient algebras. More precisely, for any partial isometry U and unital commutative C^*-algebra $\mathcal{A}$ such that $U^*U \in \mathcal{A}'$ and $U\mathcal{A}U^* \subset \mathcal{A}$ we infer that $(\mathcal{A}, \delta)$ is a C^*-dynamical system, where $\delta(\cdot) = U(\cdot)U^*$. However $\mathcal{A}$ does not need to fulfill the third property from (4). The solution then is to pass to a bigger C^*-algebra $\mathcal{B}$ generated by $\{\mathcal{A}, U^*\mathcal{A}U, U^{2*}\mathcal{A}U^2, ...\}$. Then $(\mathcal{B}, \delta)$ is a C^*-dynamical system and $\mathcal{B}$ is a coefficient algebra of $C^*(\mathcal{B}, U) = C^*(\mathcal{A}, U)$, see [18]. In this case the authors of [16] managed to 'estimate' the maximal ideal space $M(\mathcal{B})$ of $\mathcal{B}$ in terms of $(\mathcal{A}, \delta)$, or better to say, in terms of the generated partial dynamical system (X, α). Fortunately, the full description of $M(\mathcal{B})$ is obtained [16, 3.4] by a slight strengthening of assumptions - namely by assuming that the projection U^*U belongs not only to commutant $\mathcal{A}'$ but to $\mathcal{A}$ itself. The partial dynamical system $(M(\mathcal{B}), \widetilde{\alpha})$, corresponding to $(\mathcal{B}, \delta)$, is thus completely determined by (X, α). Two important facts are to be noticed: $\widetilde{\alpha}$ is a partial homeomorphism, and $U^*U \in \mathcal{A}$ implies that the image $\alpha(\Delta)$ of the partial mapping α is open, see Section 1 for details.

In Section 2, to an arbitrary partial dynamical system (X, α) such that $\alpha(\Delta)$ is open we associate another partial dynamical system $(\widetilde{X}, \widetilde{\alpha})$ such that:

1) $\widetilde{\alpha}$ is a partial homeomorphism,

2) there exist a continuous surjection $\Phi : \widetilde{X} \to X$ such that the equality $\Phi \circ \widetilde{\alpha} = \alpha \circ \Phi$ holds wherever it makes sense (see diagram (18)),

3) if α is injective then Φ becomes a homeomorphism, that is $(\widetilde{X}, \widetilde{\alpha}) \cong (X, \alpha)$.

This authorizes us to call $(\widetilde{X}, \widetilde{\alpha})$ the reversible extension of (X, α). In the case α is onto, $\widetilde{X}$ is a projective limit (see Proposition 2.10) and thus $(\widetilde{X}, \widetilde{\alpha})$ generalizes the known construction.

In Section 3 we find out that all the objects of CovFaith$(\mathcal{A}, \delta)$ have the same (up to isomorphism) coefficient C^*-algebra whose maximal ideal space is $\widetilde{X}$. We denote this C^*-algebra by $\mathcal{B}$. Then we construct a coefficient *-algebra $\mathcal{B}_0$ (the closure of $\mathcal{B}_0$ is $\mathcal{B}$) with the help of which we express the interrelations between the covariant representations of $(\mathcal{A}, \delta)$ and $(\mathcal{B}, \widetilde{\delta})$ where $\widetilde{\delta}$ is an endomorphism associated to the partial homeomorphism $\widetilde{\alpha}$. In particular we show that, if δ is injective, or equivalently α is onto, then we have a natural one-to-one correspondence between aforementioned representations. In general this correspondence is maintained only if we constrain ourselves to covariant faithful representations.

In Section 4 we define $C^*(\mathcal{A}, \delta) = C^*(X, \alpha)$ to be the partial crossed product of $\mathcal{B} = C(\widetilde{X})$ by a partial automorphism generated by the partial homeomorphism $\widetilde{\alpha}$. We show that $C^*(\mathcal{A}, \delta)$ is the universal object in CovFaith$(\mathcal{A}, \delta)$, and in the case that δ is injective, it is also universal when considered as an object of Cov$(\mathcal{A}, \delta)$. Therefore we call it a covariance algebra.

Section 5 is devoted to two important notions in C^*-dynamical systems theory: topological freeness and invariant sets. Classically, these notions were related only to homeomorphisms, but recently they have been adopted (generalized), by authors of [13], to work with partial homeomorphisms, see also [19]. Inspired by this line of development we present here the definitions of topological freeness and invariance under a partial mapping which include also noninjective partial mappings. Let us mention that, for instance, in [15] appears also the definition of topologically free irreversible dynamical system, but the authors of [15] attach to dynamical systems *different* C^*-algebras than we do, hence they are in need of a different definition. We show that there exists a natural bijection between closed α-invariant subsets of X and closed $\widetilde{\alpha}$-invariant subsets of $\widetilde{X}$ and that the partial dynamical system (X,α) is topologically free if and only if its reversible extension $(\widetilde{X},\widetilde{\alpha})$ is topologically free.

Section 6 contains two important results. Namely, we establish a one-to-one correspondence between the ideals in $C^*(X,\alpha)$ and closed invariant subsets of X generalizing Theorem 3.5 from [13]. Then we present a version of the Isomorphism Theorem, cf. [2, Theorem 7.1], [3, Chapter 2], [13, Theorem 2.6], [18, Theorem 2.13], which says that all objects of CovFaith$(\mathcal{A},\delta)$ are isomorphic to $C^*(\mathcal{A},\delta)$ whenever the corresponding system (X,α) is topologically free.

1 Preliminaries. Maximal ideal space of a coefficient C^*-algebra

We start this section by fixing some notation. Afterwards, we present and discuss briefly the results of [16] in order to present our methods and motivations. We finish this section with Theorem 1.8 to be used extensively in the sequel.

Throughout this article $\mathcal{A}$ denotes a commutative unital C^*-algebra, X denotes its maximal ideal space (i.e. a compact topological space), δ is an endomorphism of $\mathcal{A}$, while α stands for a continuous partial mapping $\alpha:\Delta\to X$ where $\Delta\subset X$ is clopen and the formula (3) holds. We adhere to the convention that $\mathbb{N}=0,1,2,...$, and when dealing with partial mappings we follow the notation of [16], i.e.: for $n>0$, we denote the domain of α^n by $\Delta_n=\alpha^{-n}(X)$ and its image by $\Delta_{-n}=\alpha^n(\Delta_n)$; for $n=0$, we set $\Delta_0=X$ and thus, for $n,m\in\mathbb{N}$, we have

$$\alpha^n:\Delta_n\to\Delta_{-n}, \tag{5}$$

$$\alpha^n(\alpha^m(x))=\alpha^{n+m}(x), \qquad x\in\Delta_{n+m}. \tag{6}$$

We recall that in terms of the multiplicative functionals of $\mathcal{A}$, α is given by

$$x\in\Delta_1 \iff x(\delta(1))=1, \tag{7}$$

$$\alpha(x)=x\circ\delta, \qquad x\in\Delta_1. \tag{8}$$

For the purpose of the present section we fix (only in this section) a faithful representation of $\mathcal{A}$, i.e. we assume that $\mathcal{A}$ is a C^*-subalgebra of $L(H)$ where $L(H)$ is an algebra of bounded linear operators on a Hilbert space H. Additionally we assume that endomorphism δ is given by the formula

$$\delta(a)=UaU^*, \qquad a\in\mathcal{A},$$

for some $U\in L(H)$ and so U is a partial isometry (note that there exists a correspondence between properties of U and the partial mapping α, cf. [16, 2.4]). In that case, as it makes sense, we will consider δ also as a mapping on $L(H)$. There is a point in studying together with $\delta(\cdot)=U(\cdot)U^*$ one more mapping

$$\delta_*(b)=U^*bU, \qquad b\in L(H),$$

which in general maps $a \in \mathcal{A}$ onto an element outside the algebra $\mathcal{A}$ and hence, even if we assume that $U^*U \in \mathcal{A}'$, we need to pass to a bigger algebra in order to obtain an algebra satisfying (4) .

Proposition 1.1. *[18, Proposition 4.1]* If $\delta(\cdot) = U(\cdot)U^*$ is an endomorphism of $\mathcal{A}$, $U^*U \in \mathcal{A}'$ and $\mathcal{B} = C^*(\bigcup_{n=0}^{\infty} U^{*n}\mathcal{A}U^n)$ is a C^*-algebra generated by $\bigcup_{n=0}^{\infty} U^{*n}\mathcal{A}U^n$, then $\mathcal{B}$ is commutative and both the mappings $\delta : \mathcal{B} \to \mathcal{B}$ and $\delta_* : \mathcal{B} \to \mathcal{B}$ are endomorphisms.

The elements of the algebra $\mathcal{B}$ play the role of coefficients in a C^*-algebra $C^*(\mathcal{A}, U)$ generated by $\mathcal{A}$ and U, [18, 2.3]. Hence the authors of [18] call $\mathcal{B}$ a *coefficient algebra*. It is of primary importance that $\mathcal{B}$ is commutative and that we have a description of its maximal ideal space, denoted here by $M(\mathcal{B})$, in terms of the maximal ideals in $\mathcal{A}$, see [16]. Let us recall it.

With every $\widetilde{x} \in M(\mathcal{B})$ we associate a sequence of functionals $\xi_{\widetilde{x}}^n : \mathcal{A} \to \mathbb{C}$, $n \in \mathbb{N}$, defined by the condition

$$\xi_{\widetilde{x}}^n(a) = \delta_*^n(a)(\widetilde{x}), \quad a \in \mathcal{A}. \tag{9}$$

The sequence $\xi_{\widetilde{x}}^n$ determines $\widetilde{x}$ uniquely because $\mathcal{B} = C^*(\bigcup_{n=0}^{\infty} \delta_*^n(\mathcal{A}))$. Since δ_* is an endomorphism of $\mathcal{B}$ the functionals $\xi_{\widetilde{x}}^n$ are linear and multiplicative on $\mathcal{A}$. So either $\xi_{\widetilde{x}}^n \in X$ (X is the spectrum of $\mathcal{A}$) or $\xi_{\widetilde{x}}^n \equiv 0$. It follows then that the mapping

$$M(\mathcal{B}) \ni \widetilde{x} \to (\xi_{\widetilde{x}}^0, \xi_{\widetilde{x}}^1, \ldots) \in \prod_{n=0}^{\infty} (X \cup \{0\}) \tag{10}$$

is an injection and the following statement is true, see Theorems 3.1 and 3.3 in [16].

Theorem 1.2. Let $\delta(\cdot) = U(\cdot)U^*$ be an endomorphism of $\mathcal{A}$, $U^*U \in \mathcal{A}'$, and $\alpha : \Delta_1 \to X$ be the partial mapping determined by δ. Then the mapping (10) defines a topological embedding of $M(\mathcal{B})$ into topological space $\bigcup_{N=0}^{\infty} \widehat{X}_N \cup X_\infty$. Under this embedding we have

$$\bigcup_{N=0}^{\infty} X_N \cup X_\infty \subset M(\mathcal{B}) \subset \bigcup_{N=0}^{\infty} \widehat{X}_N \cup X_\infty$$

where

$$\widehat{X}_N = \{\widetilde{x} = (x_0, x_1, \ldots, x_N, 0, \ldots) : x_n \in \Delta_n,\ \alpha(x_n) = x_{n-1},\ 1 \le n \le N\},$$

$$X_\infty = \{\widetilde{x} = (x_0, x_1, \ldots) : x_n \in \Delta_n,\ \alpha(x_n) = x_{n-1},\ 1 \le n\}.$$

$$X_N = \{\widetilde{x} = (x_0, x_1, \ldots, x_N, 0, \ldots) \in \widehat{X}_N : x_N \notin \Delta_{-1}\},$$

The topology on $\bigcup_{N=0}^{\infty} \widehat{X}_N \cup X_\infty$ is defined by a fundamental system of neighborhoods of points $\widetilde{x} \in \widehat{X}_N$ given by

$$O(a_1, \ldots, a_k, \varepsilon) = \{\widetilde{y} \in \widehat{X}_N : |a_i(x_N) - a_i(y_N)| < \varepsilon,\ i = 1, \ldots, k\}$$

and respectively of $\widetilde{x} \in X_\infty$ by

$$O(a_1, ..., a_k, n, \varepsilon) = \{\widetilde{y} \in \bigcup_{N=n}^{\infty} \widehat{X}_N \cup X_\infty : |a_i(x_n) - a_i(y_n)| < \varepsilon,\ i = 1, ..., k\}$$

where $\varepsilon > 0$, $a_i \in \mathcal{A}$ and $k, n \in \mathbb{N}$.

Remark 1.3. The topology on X is weak*. One immediately sees then (see (10)), that the topology on $\bigcup_{N\in\mathbb{N}} \widehat{X}_N \cup X_\infty$ is in fact the product topology inherited from $\prod_{n=0}^{\infty}(X \cup \{0\})$ where $\{0\}$ is clopen.
The foregoing theorem gives us an estimate of $M(\mathcal{B})$ and aiming at sharpening that result we need to strengthen the assumptions. If we replace the condition $U^*U \in \mathcal{A}'$ with the stronger one

$$U^*U \in \mathcal{A}, \tag{11}$$

then the full information on $\mathcal{B}$ is carried by the pair $(\mathcal{A}, \delta)$, cf. [16, Theorem 3.4].

Theorem 1.4. Under the assumptions of Theorem 1.2 with $U^*U \in \mathcal{A}'$ replaced by $U^*U \in \mathcal{A}$ we get

$$M(\mathcal{B}) = \bigcup_{N=0}^{\infty} X_N \cup X_\infty.$$

This motivates us to take a closer look at condition (11).
Firstly, let us observe [16, 3.5] that if $U^*U \in \mathcal{A}'$ then δ is an endomorphism of the C^*-algebra $\mathcal{A}_1 = C^*(\mathcal{A}, U^*U)$ and since $C^*(\bigcup_{n=0}^{\infty} U^{*n}\mathcal{A}_1U^n) = \mathcal{B}$ one can apply the preceding theorem to the algebra $\mathcal{A}_1$ for the full description of $M(\mathcal{B})$. This procedure turns out to be very fruitful in many situations.

Example 1.5. Let the elements of $\mathcal{A}$ be the operators of multiplication by periodic sequences of period n, on the Hilbert space $l^2(\mathbb{N})$, and let U be the co-isometry given by $[Ux](k) = x(k+1)$, for $x \in l^2(\mathbb{N})$, $k \in \mathbb{N}$. Then $X = \{x_0, ..., x_{n-1}\}$ and $\alpha(x_k) = x_{k+1\ (mod\ n)}$. If for $k = 0, ..., n-1$ we write

$$(\infty, k) = (x_k, x_{k-1}, ..., x_1, x_0, x_{n-1}, x_{n-2}, \ldots)$$

and

$$(N, k) = (\underbrace{x_k, x_{k-1}, ..., x_1, x_0, x_{n-1}, ...,}_{N} x_{n-r}, 0, 0, \ldots)$$

where $N - r \equiv k\ (mod\ n)$ (for each N there are n pairs (N, k)), then from Theorem 1.2 we get

$$\{\infty\} \times \{0, 1, ..., n-1\} \subset M(\mathcal{B}) \subset \overline{\mathbb{N}} \times \{0, 1, ..., n-1\}$$

where $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ is a compactification of the discrete space $\mathbb{N}$. In order to describe $M(\mathcal{B})$ precisely let us pass to the algebra $\mathcal{A}_1 = C^*(\mathcal{A}, U^*U)$. As U^*U is the operator of multiplication by $(0, 1, 1, ...)$ the elements of $\mathcal{A}_1$ are the operators of multiplication by sequences of the form $(a, h(0), h(1), ...)$ where a is arbitrary and $h(k+n) = h(k)$, for all $k \in \mathbb{N}$. In an obvious manner (with a slight abuse of notation) we infer the spectrum of $\mathcal{A}_1$ to be $\{y, x_0, ..., x_{n-1}\}$, and the mapping generated by δ considered as an endomorphism of $\mathcal{A}_1$ acts as follows: $\alpha(x_k) = x_{k+1\ (mod\ n)}$ and $\alpha(y) = x_0$. Abusing notation once again and putting

$$(N, k) = (\underbrace{x_k, x_{k-1}, ..., x_1, x_0, x_{n-1}, ..., x_0}_{N}, y, 0, 0, ...)$$

where $N \equiv k\ (mod\ n)$ (for each N there is now the only one pair (N, k)), in view of Theorem 1.4 we have

$$M(\mathcal{B}) = \{(N, k) \in \mathbb{N} \times \{0, 1, ..., n-1\} : N \equiv k\ (mod\ n)\} \cup \big(\{\infty\} \times \{0, 1, ..., n-1\}\big),$$

so $M(\mathcal{B})$ can be imagined as a spiral subset of the cylinder $\overline{\mathbb{N}} \times \{0, 1, ..., n-1\}$, (see Figure 1).

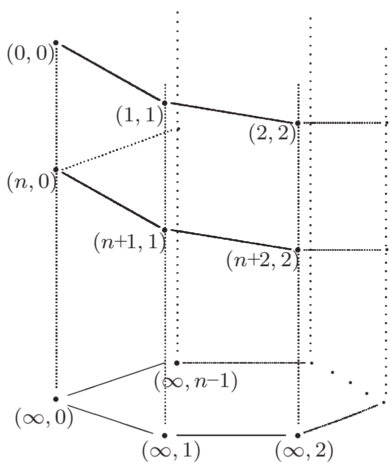


Fig. 1 Maximal ideal space of the coefficient algebra from Example 1.5.

Condition (11) is closely related to the openness of Δ_{-1} (as Δ_1 is compact and α is continuous Δ_{-1} is always closed).

Proposition 1.6. Let $P_{\Delta_{-1}}$ be the projection corresponding to the characteristic function $\chi_{\Delta_{-1}}$. If $U^*U \in \mathcal{A}$ then Δ_{-1} is open and $U^*U = P_{\Delta_{-1}}$. If $U^*U \in \mathcal{A}'$, Δ_{-1} is open and $\mathcal{A}$ acts nondegenerately on H, then $U^*U \leqslant P_{\Delta_{-1}}$.

Proof. Let $U^*U \in \mathcal{A}$. We show that the image Δ_{-1} of the set Δ_1 of functionals satisfying (7) under the mapping (8) is the set of functionals $x \in X$ satisfying $x(U^*U) = 1$.
Let $x' \in \Delta_1$, that is $x'(UU^*) = 1$. Putting $x = \alpha(x')$ we have $x(U^*U) = x'(\delta(U^*U)) = x'(UU^*UU^*) = x'(UU^*)x'(UU^*) = 1$. Now, let $x \in X$ be such that $x(U^*U) = 1$. We define on $\delta(\mathcal{A})$ a multiplicative functional $x'(b) := x(U^*bU)$, $b \in \delta(\mathcal{A})$. For $b = \delta(a)$, $a \in \mathcal{A}$, we then have

$$x'(\delta(a)) = x(U^*UaU^*U) = x(U^*U)x(a)x(U^*U) = x(a).$$

x' is therefore well defined. Since $x'(\delta(1)) = x(1) = 1$ it is non zero and there exists its extension $\overline{x}' \in X$ on $\mathcal{A}$ (see [9, 2.10.2]). It is clear that $\overline{x}' \in \Delta_1$ and $\alpha(\overline{x}') = x$. Hence $x \in \Delta_{-1}$. Thus we have proved that $x \in \Delta_{-1} \iff x(U^*U) = 1$, which means that $U^*U \in \mathcal{A}$ is the characteristic function of Δ_{-1}. It follows then that Δ_{-1} is clopen.
Now, let Δ_{-1} be open. Then $\chi_{\Delta_{-1}} \in C(X)$ and $\delta(\chi_{\Delta_{-1}}) = \chi_{\alpha^{-1}(\Delta_{-1})} = \chi_{\Delta_1}$. As $\mathcal{A}$ acts nondegenerately, rewriting this equation in terms of operators we have

$$UP_{\Delta_{-1}}U^* = UU^*. \tag{12}$$

Letting $H_i = U^*UH$ be the initial and $H_f = UU^*H$ be the final space of U we get $U^* : H_f \to H_i$ is an isomorphism and $U : H_i \to H_f$ is its inverse. Taking arbitrary $h \in H_f$ and applying the both sides of (12) to it we obtain $UP_{\Delta_{-1}}U^*h = h$, and hence $P_{\Delta_{-1}}H_i = H_i$, that is $U^*U \leqslant P_{\Delta_{-1}}$. □

Note. The inequality in the second part of the preceding proposition can not be replaced by equality. In order to see that consider for instance $\mathcal{A}$ and U from Example 1.5.
By virtue of Proposition 1.1 the mappings δ and δ_* are endomorphisms of the C^*-algebra $\mathcal{B}$. With the help of the presented theorems we can now find the form of the partial mappings they generate. We shall rely on the fact [16, 2.5] expressed by the coming proposition.

Proposition 1.7. Let $\delta(\cdot) = U(\cdot)U^*$ and $\delta_*(\cdot) = U^*(\cdot)U$ be endomorphisms of $\mathcal{A}$ and let α be the partial mapping of X generated by δ. Then Δ_1 and Δ_{-1} are clopen and $\alpha : \Delta_1 \to \Delta_{-1}$ is a homeomorphism. Moreover, the endomorphism δ_* is given on $C(X)$ by the formula

$$\delta_*(a)(x) = \begin{cases} a(\alpha^{-1}(x)) \ , & x \in \Delta_{-1} \\ 0 \ , & x \notin \Delta_{-1} \end{cases} \tag{13}$$

Finally we arrive at the closing theorem.

Theorem 1.8. Let the hypotheses of Theorem 1.2 hold. Then
i) the sets

$$\widetilde{\Delta}_1 = \{(x_0, ...) \in M(\mathcal{B}) : x_0 \in \Delta_1\},$$
$$\widetilde{\Delta}_{-1} = \{(x_0, x_1, ...) \in M(\mathcal{B}) : x_1 \neq 0\}$$

are clopen subsets of $M(\mathcal{B})$,
ii) the endomorphism δ generates on $M(\mathcal{B})$ the partial homeomorphism $\widetilde{\alpha} : \widetilde{\Delta}_1 \to \widetilde{\Delta}_{-1}$ given by the formula

$$\widetilde{\alpha}(x_0, ...) = (\alpha(x_0), x_0, ...), \qquad (x_0, ...) \in \widetilde{\Delta}_1, \tag{14}$$

iii) the partial mapping generated by δ_* is the inverse of $\widetilde{\alpha}$, that is $\widetilde{\alpha}^{-1} : \widetilde{\Delta}_{-1} \to \widetilde{\Delta}_1$ where

$$\widetilde{\alpha}^{-1}(x_0, x_1 ...) = (x_1, ...), \qquad (x_0, x_1 ...) \in \widetilde{\Delta}_{-1}. \tag{15}$$

Proof. We rewrite Proposition 1.7 in terms of Theorem 1.2. Let $\widetilde{x} = (x_0, x_1, ...) \in M(\mathcal{B})$. From (7) we get

$$\widetilde{x} \in \widetilde{\Delta}_1 \Longleftrightarrow \widetilde{x}(UU^*) = 1, \qquad \widetilde{x} \in \widetilde{\Delta}_{-1} \Longleftrightarrow \widetilde{x}(U^*U) = 1.$$

However, the definition (9) of functionals $\xi_{\widetilde{x}}^n = x_n$ implies that $\widetilde{x}(UU^*) = 1 \Longleftrightarrow x_0(UU^*) = 1$, and $\widetilde{x}(U^*U) = 1 \Longleftrightarrow x_1(1) = 1$, which proves i).
The mapping $\widetilde{\alpha}$ generated by δ on $M(\mathcal{B})$ (see (8)), is given by the composition: $\widetilde{\alpha}(\widetilde{x}) \equiv \widetilde{x} \circ \delta$. So, let $\widetilde{x} = (x_0, x_1, ...) \in \widetilde{\Delta}_1$, then the sequence of functionals $\xi_{\widetilde{x}}^n$ satisfies: $\xi_{\widetilde{x}}^n(a) = a(x_n)$, $a \in \mathcal{A}$, $n \in \mathbb{N}$. Now let us consider an analogous sequence of functionals $\xi_{\widetilde{\alpha}(\widetilde{x})}^n$ defining the point $\widetilde{\alpha}(\widetilde{x}) = (\overline{x}_0, \overline{x}_1, ...)$. For $n > 0$ we have

$$a(\overline{x}_n) = \xi_{\widetilde{\alpha}(\widetilde{x})}^n(a) = \widetilde{\alpha}(\widetilde{x})(\delta_*^n(a)) = \widetilde{x}(\delta(\delta_*^n(a))) = \widetilde{x}(UU^{*n}aU^nU^*) =$$

$$= \widetilde{x}(UU^*)\widetilde{x}(\delta_*^{n-1}(a))\widetilde{x}(U^*U) = \widetilde{x}(\delta_*^{n-1}(a)) = \xi_{\widetilde{x}}^{n-1}(a) = a(x_{n-1}),$$

while for $n = 0$ we have

$$a(\overline{x}_0) = \xi_{\widetilde{\alpha}(\widetilde{x})}^0(a) = \widetilde{\alpha}(\widetilde{x})(a) = \widetilde{x}(\delta(a)) = \xi_{\widetilde{x}}^0(\delta(a)) = \delta(a)(x_0) = a(\alpha(x_0)).$$

Thus we infer that $\widetilde{\alpha}(\widetilde{x}) = (\alpha(x_0), x_0, ...)$. By Proposition 1.7, $\widetilde{\alpha}^{-1}$ is the inverse to mapping $\widetilde{\alpha}$. Hence we get (15). □

2 Reversible extension of a partial dynamical system

One of the most important consequences of Theorems 1.4 and 1.8 is that although the algebra $\mathcal{B}$ is relatively bigger than $\mathcal{A}$ and its structure depends on U and U^* (U and U^* need not be in $\mathcal{A}$) the C^*-dynamical system $(\mathcal{B}, \delta)$ still can be reconstructed by means of the intrinsic features of $(\mathcal{A}, \delta)$ itself (provided (11) holds). Therefore in this section, we make an effort to investigate effectively this reconstruction and, as it is purely topological, we forget for the time being about its algebraic aspects.
Once having the system (X, α), we will construct a pair $(\widetilde{X}, \widetilde{\alpha})$:

- a compact space $\widetilde{X}$ - a counterpart of the maximal ideal space obtained in Theorem 1.4 (or a 'lower estimate' of it, see Theorem 1.2), and
- a partial injective mapping $\widetilde{\alpha}$ - a counterpart of the mapping from Theorem 1.8.

We then show some useful results about the structure of $(\widetilde{X}, \widetilde{\alpha})$. In particular, we calculate $(\widetilde{X}, \widetilde{\alpha})$ for topological Markov chains, and show the interrelation between $\widetilde{X}$ and projective limits.
The most interesting case occurs when the injective mapping $\widetilde{\alpha}$ has an open image. We shall call such mappings partial homeomorphisms, cf [13]. More precisely,

a *partial homeomorphism* is a partial mapping which is injective and has open image.

Let us recall that by a partial mapping of X we always mean a continuous mapping $\alpha : \Delta_1 \to X$ such that $\Delta_1 \subset X$ is clopen, and so if α is a partial homeomorphism, then

$\alpha^{-1}: \Delta_{-1} \to X$ is a partial mapping of X in our sense, that is Δ_{-1} is clopen.
As we shall see if Δ_{-1} is open then $\widetilde{\alpha}$ is a partial homeomorphism and $\widetilde{X}$ actually becomes the spectrum of a certain coefficient algebra (see Theorem 3.3). That is the reason why we shall often assume the openness of Δ_{-1}. However in this section we do not make it a standing assumption in order to get to know better the role of it and the condition (11), cf. Proposition 1.6.

2.1 The system $(\widetilde{X}, \widetilde{\alpha})$

Let us fix a partial dynamical system (X, α) and let us consider a disjoint union $X \cup \{0\}$ of the set X and the singleton $\{0\}$ (we treat here 0 as a symbol rather than the number). We define $\{0\}$ to be clopen and hence $X \cup \{0\}$ is a compact topological space. We will define $\widetilde{X}$ to be a subset

$$\widetilde{X} \subset \prod_{n=0}^{\infty}(X \cup \{0\})$$

of the product of $\aleph_0$ copies of the space $X \cup \{0\}$ where the elements of $\widetilde{X}$ represent anti-orbits of the partial mapping α. Namely we set

$$\widetilde{X} = \bigcup_{N=0}^{\infty} X_N \cup X_\infty \tag{16}$$

where

$$X_N = \{\widetilde{x} = (x_0, x_1, ..., x_N, 0, ...) : x_n \in \Delta_n,\ x_N \notin \Delta_{-1},\ \alpha(x_n) = x_{n-1},\ \ n = 1, ..., N\},$$

$$X_\infty = \{\widetilde{x} = (x_0, x_1, ...) : x_n \in \Delta_n,\ \alpha(x_n) = x_{n-1},\ \ n \geqslant 1\}.$$

The natural topology on $\widetilde{X}$ is the one induced from the space $\prod_{n=0}^{\infty}(X \cup \{0\})$ equipped with the product topology, cf. Remark 1.3. Since $\widetilde{X} \subset \prod_{n=0}^{\infty}(\Delta_n \cup \{0\})$, the topology on $\widetilde{X}$ can also be regarded as the topology inherited from $\prod_{n=0}^{\infty}(\Delta_n \cup \{0\})$.

Definition 2.1. We shall call the topological space $\widetilde{X}$ the *extension of X under α*, or briefly the *α-extension* of X.

Theorem 2.2. The subset X_∞ is compact and the subsets X_N are clopen in $\widetilde{X}$. Moreover, the following conditions are equivalent:
a) Δ_{-1} is open.
b) $\widetilde{X}$ is compact.
c) X_0 is compact.
d) X_N is compact for every $N \in \mathbb{N}$.

Proof. As the sets Δ_n, $n \in \mathbb{N}$, are clopen, by Tichonov's theorem, the space $\prod_{n=0}^{\infty}(\Delta_n \cup \{0\})$ is compact, and to prove the compactness of X_∞ it suffices to show that X_∞ is a closed subset of $\prod_{n=0}^{\infty}(\Delta_n \cup \{0\})$, or equivalently that its complement is open. To this

end, let $\widetilde{x} = (x_0, x_1, ...) \in \prod_{n=0}^{\infty}(\Delta_n \cup \{0\})$ and suppose that $\widetilde{x} \notin X_\infty$. We show that $\widetilde{x}$ has an open neighborhood contained in $\prod_{n=0}^{\infty}(\Delta_n \cup \{0\}) \setminus \widetilde{X}_\infty$.
In view of the definition of X_∞ we have two possibilities:

1) there is $n > 0$ such that $x_n = 0$,
2) there is $n > 0$ such that $x_n \in \Delta_n$, and $\alpha(x_n) \neq x_{n-1}$,

If 1) holds then the set $\widetilde{V} = \prod_{k=0}^{\infty} V_k$ where $V_k = \Delta_k \cup \{0\}$, for $k \neq n$, and $V_n = \{0\}$, is an open neighborhood of $\widetilde{x}$ and $\widetilde{V} \cap X_\infty = \emptyset$.
Let us suppose now that 2) holds. We may also suppose that $x_{n-1} \neq 0$ ($x_{n-1} \in \Delta_{n-1}$). Hence there exist two disjoint open subsets V_1, $V_2 \subset \Delta_{n-1}$ such that $\alpha(x_n) \in V_1$ and $x_{n-1} \in V_2$. Clearly, the set $\widetilde{V} = \prod_{k=0}^{\infty} V_k$ where $V_k = \Delta_k \cup \{0\}$, for $k \neq n-1, n$, and $V_{n-1} = V_2$, $V_n = \alpha^{-1}(V_1)$, is an open neighborhood of $\widetilde{x}$, and $V_1 \cap V_2 = \emptyset$ guarantees that $\widetilde{V} \cap X_\infty = \emptyset$.
Fix $N \in \mathbb{N}$. To prove that X_N is open we recall that Δ_n, $n \in \mathbb{N}$, are clopen and Δ_{-1} is closed. Hence $\Delta_n \setminus \Delta_{-1}$, $n \in \mathbb{N}$, are open, and it is easy to see that

$$X_N = \widetilde{X} \cap (\Delta_0 \times \Delta_1 \times ... \times \Delta_{N-1} \times \Delta_N \setminus \Delta_{-1} \times (\Delta_{N+1} \cup \{0\}) \times ...).$$

Hence X_N is an open subset of $\widetilde{X}$. It is also closed because its complement is the sum of two open sets: $\widetilde{V}_1 = \{(x_0, x_1...) \in \widetilde{X} : x_N = 0\}$ and $\widetilde{V}_2 = \{(x_0, x_1...) \in \widetilde{X} : x_{N+1} \neq 0\}$.
We prove now the equivalence of a), b), c) and d).
a)$\Rightarrow$b). Suppose that Δ_{-1} is open. We prove the compactness of $\widetilde{X}$ in an analogous fashion as we proved the compactness of X_∞. Let $\widetilde{x} = (x_0, x_1, ...) \in \prod_{n=0}^{\infty}(\Delta_n \cup \{0\}) \setminus \widetilde{X}$. In view of the definition of $\widetilde{X}$ we have the three possibilities:

1) there are $n, m \in \mathbb{N}$ such that $x_n = 0$ and $x_{n+m} \in \Delta_{n+m}$ ($x_{n+m} \neq 0$),
2) there is $n > 0$ such that $x_n \in \Delta_n$, and $\alpha(x_n) \neq x_{n-1}$,
3) for some $n > 0$ we have $x_n \in \Delta_n \cap \Delta_{-1}$, and $x_{n+1} = 0$.

Let us suppose that 1) holds. Then the set $\widetilde{V} = \prod_{k=0}^{\infty} V_k$ where $V_k = \Delta_k \cup \{0\}$, for $k \neq n, n+m$, and $V_n = \{0\}$, $V_{n+m} = \Delta_{n+m}$, is an open neighborhood of $\widetilde{x}$. It is clear that none of the points from $\widetilde{X}$ belongs to $\widetilde{V}$.
The same argumentation as the one concerning X_∞ shows that in the case 2) $\widetilde{x}$ lies in the interior of $\prod_{n=0}^{\infty}(\Delta_n \cup \{0\}) \setminus \widetilde{X}$.
If we suppose that 3) holds, then the set $\widetilde{V} = \prod_{k=0}^{\infty} V_k$ where $V_k = \Delta_k \cup \{0\}$, for $k \neq n, n+1$, and $V_n = \Delta_n \cap \Delta_{-1}$, $V_{n+1} = \{0\}$, is an open neighborhood of $\widetilde{x}$ (here we use the openness of Δ_{-1}). Clearly $\widetilde{V}$ does not contain any point from $\widetilde{X}$.
b)$\Rightarrow$c). X_0 is closed and hence it is compact.
c)$\Rightarrow$a). Suppose that Δ_{-1} is not open. Then $X \setminus \Delta_{-1}$ is not closed and hence it is not compact. Thus there is an open cover $\{V_i\}_{i \in I}$ of $X \setminus \Delta_{-1}$ which does not admit a finite subcover. Defining $\widetilde{V}_i = \{(x_0, x_1...) \in \widetilde{X} : x_0 \in V_i\}$, for $i \in I$, we get an open cover of X_0 which does not admit a finite subcover. Hence X_0 is not compact.
Thus we see that a)$\Leftrightarrow$b)$\Leftrightarrow$c). As b)$\Rightarrow$d) and d)$\Rightarrow$c) are obvious, the proof is complete.$\square$

It is interesting how $\widetilde{X}$ depends on α. For instance, if α is surjective then X_n, $n \in \mathbb{N}$, are empty and $\widetilde{X} = X_\infty$, in this case $\widetilde{X}$ can be defined as a projective limit, see Proposition

2.10. If α is injective then a natural continuous projection Φ of $\widetilde{X}$ onto X given by the formula

$$\Phi(x_0, x_1, ...) = x_0 \tag{17}$$

becomes a bijection and, as we will see, in the case that Δ_{-1} is open even a homeomorphism. But the farther from injectivity α is, the farther $\widetilde{X}$ is from X.

Proposition 2.3. Let (X, α) be such a system for which α is injective. Then $\Phi : \widetilde{X} \to X$ is a homeomorphism if and only if Δ_{-1} is open.

Proof. If Δ_{-1} is not open then by Theorem 2.2, $\widetilde{X}$ is not compact and hence not homeomorphic to X. Suppose then that Δ_{-1} is open. Then $\alpha : \Delta_1 \to X$ is an open mapping because α is a homeomorphism of compact set Δ_1 onto the compact set Δ_{-1}. We only need to show that the mapping Φ^{-1} is continuous or, which is the same, that Φ is open. To see this it is enough to look at a subbase for the topology in $\widetilde{X}$ in a appropriate way. Indeed, let $\widetilde{U} \subset \widetilde{X}$ be of the form

$$\widetilde{U} = \widetilde{X} \cap (U_0 \times U_1 \times ... \times U_N \times (\Delta_{N+1} \cup \{0\}) \times (\Delta_{N+2} \cup \{0\}) \times ...)$$

where $U_i \subset X \cup \{0\}$, $i = 1, ..., N$, are open. Without loss of generality we can assume that $U_i \subset \Delta_i \cup \{0\}$, $i = 1, ..., N$. There are the two possibilities:

1) If $0 \notin U_N$ then the set $U_{N-1} \cap \alpha(U_N)$ is open and

$$\widetilde{U} = \widetilde{X} \cap (U_0 \times U_1 \times ... \times \Big(U_{N-1} \cap \alpha(U_N)\Big) \times (\Delta_N \cup \{0\}) \times (\Delta_{N+1} \cup \{0\}) \times ...)$$

2) If $0 \in U_N$ then the set $U_{N-1} \cap \alpha(U_N \setminus \{0\}) \cup U_{N-1} \setminus \Delta_{-1} \cup \{0\}$ is open and

$$\widetilde{U} = \widetilde{X} \cap (U_0 \times U_1 \times ... \times \Big(U_{N-1} \cap \alpha(U_N \setminus \{0\}) \cup U_{N-1} \setminus \Delta_{-1} \cup \{0\}\Big) \times (\Delta_N \cup \{0\}) \times ...).$$

Applying the above procedure N times we conclude that $\widetilde{U} = \widetilde{X} \cap \prod_{k=0}^{\infty} V_k$ where $V_k = \Delta_k \cup \{0\}$, for $k > 0$, and $V_0 \subset X \cup \{0\}$ is a certain open set. It is obvious that $\Phi(\widetilde{U}) = V_0$. Thus Φ is an open mapping and the proof is complete. □

Example 2.4. Consider the dynamical system (X, α) where $X = [0, 1]$ and $\alpha(x) = q \cdot x$, for fixed $0 < q < 1$ and any $x \in [0, 1]$. In this situation α is injective and $\Delta_{-1} = [0, q]$ is not open. As Φ is bijective, we can identify $\widetilde{X}$ with an interval $[0, 1]$ but the topology on $\widetilde{X}$ differs from the natural topology on $[0, 1]$. It is not hard to check that the topology on $\widetilde{X} = [0, 1]$ is generated by intervals $[0, a)$, (a, b), $(b, 1]$, where $0 < a < b < 1$, and singletons $\{q^k\}$, $k > 0$. Thus $\widetilde{X}$ is not compact and Φ is not a homeomorphism.

Now, we would like to investigate a partial mapping $\widetilde{\alpha}$ on $\widetilde{X}$ associated with α. It seems very natural to look for a partial mapping $\widetilde{\alpha}$ such that $\Phi \circ \widetilde{\alpha} = \alpha \circ \Phi$ wherever the superposition $\alpha \circ \Phi$ makes sense. If such $\widetilde{\alpha}$ exists then its domain is $\widetilde{\Delta}_1 := \Phi^{-1}(\Delta_1)$ and its image is contained in $\widetilde{\Delta}_{-1} := \Phi^{-1}(\Delta_{-1})$. Moreover, as Φ is continuous, we get

$$\widetilde{\Delta}_1 = \Phi^{-1}(\Delta_1) = \{\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X} : x_0 \in \Delta_1\}$$

is clopen while

$$\widetilde{\Delta}_{-1} = \Phi^{-1}(\Delta_{-1}) = \{\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X} : x_0 \in \Delta_{-1}\}$$

is closed and if Δ_{-1} is open then $\widetilde{\Delta}_{-1}$ is open too. It is not a surprise that there always exists a homeomorphism $\widetilde{\alpha}$ such that the following diagram

$$\begin{array}{ccc} \widetilde{\Delta}_1 & \xrightarrow{\widetilde{\alpha}} & \widetilde{\Delta}_{-1} \\ \Phi \downarrow & & \downarrow \Phi \\ \Delta_1 & \xrightarrow{\alpha} & \Delta_{-1} \end{array} \tag{18}$$

commutes. However, the commutativity of the diagram (18) does not determine the homeomorphism $\widetilde{\alpha}$ uniquely.

Proposition 2.5. The mapping $\widetilde{\alpha} : \widetilde{\Delta}_1 \to \widetilde{\Delta}_{-1}$ given by the formula

$$\widetilde{\alpha}(\widetilde{x}) = (\alpha(x_0), x_0, ...), \qquad \widetilde{x} = (x_0, ...) \in \widetilde{\Delta}_1 \tag{19}$$

is a homeomorphism (and hence if Δ_{-1} is open $\widetilde{\alpha}$ is a partial homeomorphism of $\widetilde{X}$) such that the diagram (18) is commutative.

Proof. The inverse of α is given by the formula $\widetilde{\alpha}^{-1}((x_0, x_1, ...)) = (x_1, x_2, ...)$. The straightforward equations $\widetilde{\alpha}(\widetilde{\Delta}_1 \cap (U_0 \times U_1 \times ...)) = \widetilde{X} \cap (X \times U_0 \times ...)$ and $\widetilde{\alpha}^{-1}(\widetilde{\Delta}_{-1} \cap (U_0 \times U_1 \times U_2 \times ...)) = \widetilde{X} \cap ((\alpha^{-1}(U_0) \cap U_1) \times U_2 \times ...)$, and the definition of the topology in $\widetilde{X}$ imply that $\widetilde{\alpha}$ and $\widetilde{\alpha}^{-1}$ are continuous and hence they are homeomorphisms. The commutativity of the diagram (18) is obvious. □

In this manner we can attach to every pair (X, α) such that Δ_{-1} is open another system $(\widetilde{X}, \widetilde{\alpha})$ where $\widetilde{\alpha}$ is a partial homeomorphism of $\widetilde{X}$ and if α is a partial homeomorphism then systems (X, α) and $(\widetilde{X}, \widetilde{\alpha})$ are topologically equivalent via Φ, see Proposition 2.3. In particular case of a classical irreversible dynamical system, that is when α is a covering mapping of X, $\widetilde{\alpha}$ is a full homeomorphism and hence $(\widetilde{X}, \widetilde{\alpha})$ is a classical reversible dynamical system. This motivates us to coin the following definition.

Definition 2.6. Let (X, α) be a partial dynamical system and let Δ_{-1} be open. We say that the pair $(\widetilde{X}, \widetilde{\alpha})$, where $\widetilde{X}$ and $\widetilde{\alpha}$ are given by (16) and (19) respectively, is a *reversible extension* of (X, α).

Example 2.7. It may happen that two different partial dynamical systems have the same reversible extension. Let (X, α) and (X', α') be given by the relations: $X = \{x_0, x_1, x_2, y_2\}$, $\Delta_1 = X \setminus \{x_0\}$, $\alpha(y_2) = \alpha(x_2) = x_1$, $\alpha(x_1) = x_0$; $X' = \{x'_0, x'_1, x'_2, y'_1, y'_2\}$, $\Delta'_1 = X' \setminus \{x'_0\}$ and $\alpha'(x'_2) = x'_1$, $\alpha'(y'_2) = y'_1$, $\alpha'(y'_1) = \alpha'(x'_1) = x'_0$; or by the diagrams:

Then $\widetilde{X}$ consist of the following points $\widetilde{x}_0 = (x_0, x_1, x_2, 0, ...)$, $\widetilde{x}_1 = (x_1, x_2, 0, ...)$, $\widetilde{x}_2 = (x_2, 0, ...)$, $\widetilde{y}_0 = (x_0, x_1, y_2, 0, ...)$, $\widetilde{y}_1 = (x_1, y_2, 0, ...)$ and $\widetilde{y}_2 = (y_2, 0, ...)$. Similarly $\widetilde{X}'$ is the set of points $\widetilde{x}'_0 = (x'_0, x'_1, x'_2, 0, ...)$, $\widetilde{x}'_1 = (x'_1, x'_2, 0, ...)$, $\widetilde{x}'_2 = (x'_2, 0, ...)$, $\widetilde{y}'_0 = (x'_0, y'_1, y'_2, 0, ...)$, $\widetilde{y}'_1 = (y'_1, y'_2, 0, ...)$ and $\widetilde{y}'_2 = (y'_2, 0, ...)$. Hence the systems $(\widetilde{X}, \widetilde{\alpha})$ and $(\widetilde{X}', \widetilde{\alpha}')$ are given by the same diagram

$$\begin{tikzcd} \widetilde{x}_0 & \widetilde{x}_1 \arrow[l] & \widetilde{x}_2 \arrow[l] \\ \widetilde{y}_0 & \widetilde{y}_1 \arrow[l] & \widetilde{y}_2 \arrow[l] \end{tikzcd} \qquad \begin{tikzcd} \widetilde{x}'_0 & \widetilde{x}'_1 \arrow[l] & \widetilde{x}'_2 \arrow[l] \\ \widetilde{y}'_0 & \widetilde{y}'_1 \arrow[l] & \widetilde{y}'_2 \arrow[l] \end{tikzcd}$$

2.2 Topological Markov chains, projective limits and hyperbolic attractors

It is not hard to give an example of a partial dynamical system which is not a reversible extension of any 'smaller' dynamical system, though its dynamics is implemented by a partial homeomorphism. Yet many reversible dynamical systems arise from irreversible ones as their reversible extensions. In order to see that we recall first the topological Markov chains and then the hyperbolic attractors.

Let $A = (A(i,j))_{i,j\in\{1,...,N\}}$ be a square matrix with entries in $\{0,1\}$, and such that no row of A is identically zero. We associate with A two dynamical systems (X_A, σ_A) and $(\overline{X}_A, \overline{\sigma}_A)$. The *one-sided Markov subshift* σ_A acts on the compact space $X_A = \{(x_k)_{k\in\mathbb{N}} \in \{1,...,N\}^{\mathbb{N}} : A(x_k, x_{k+1}) = 1,\ k \in \mathbb{N}\}$ (the topology on X_A is the one inherited from the Cantor space $\{1,...,N\}^{\mathbb{N}}$) by the rule

$$(\sigma_A x)_k = x_{k+1}, \qquad \text{for } k \in \mathbb{N}, \text{ and } x \in X_A.$$

Unless A is a permutation matrix σ_A is not injective, and σ_A is onto if and only if every column of A is non-zero. The *two-sided Markov subshift* $\overline{\sigma}_A$ acts on the compact space $\overline{X}_A = \{(x_k)_{k\in\mathbb{Z}} \in \{1,...,N\}^{\mathbb{Z}} : A(x_k, x_{k+1}) = 1,\ k \in \mathbb{Z}\}$ and is defined by

$$(\overline{\sigma}_A x)_k = x_{k+1}, \qquad \text{for } k \in \mathbb{Z}, \text{ and } x \in \overline{X}_A.$$

Mapping $\overline{\sigma}_A$ is what is called a *topological Markov chain* and abstractly can be defined as an expansive homeomorphism of a completely disconnected compactum.

If we assume that not only the rows of A but also the columns are not identically zero then σ_A is onto and we have

Proposition 2.8. Let A have no zero columns and let $(\widetilde{X}_A, \widetilde{\sigma}_A)$ be the reversible extension of (X_A, σ_A). Then

$$(\widetilde{X}_A, \widetilde{\sigma}_A) \cong (\overline{X}_A, \overline{\sigma}_A).$$

Proof. Let $\widetilde{x} \in \widetilde{X}_A$. Then $\widetilde{x} = (x_0, x_1, ...)$ where $x_n = (x_{n,k})_{k\in\mathbb{N}} \in X_A$ is such that $\sigma_A^m(x_{n+m}) = x_n$, $n, m \in \mathbb{N}$. Thus, $x_{n+m,k+m} = x_{n,k}$, for k, n, $m \in \mathbb{N}$, or in other words

$$m_1 - n_1 = m_2 - n_2 \Longrightarrow x_{n_1,m_1} = x_{n_2,m_2}.$$

We see that the sequence $\overline{x} \in \overline{\sigma}_A$ such that $\overline{x}_k = x_{n,m}$, $m - n = k \in \mathbb{Z}$, carries the full information about $\widetilde{x}$. Hence, defining Υ by the formula

$$\Upsilon(\widetilde{x}) = (..., x_{n,0}, ..., x_{1,0}, \dot{x}_{0,0}, x_{0,1}, ..., x_{0,n}, ...) \tag{20}$$

where dot over $x_{0,0}$ denotes the zero entry, we get an injective mapping $\Upsilon : \widetilde{X}_A \to \overline{X}_A$. It is evident that Υ is surjective and can be readily checked that it is also continuous, whence Υ is a homeomorphism.
Finally let us recall that $\widetilde{\sigma}_A(\widetilde{x}) = (\sigma_A(x_0), x_0, ...)$ and $\sigma_A(x_0) = (x_{0,1}, x_{0,2}, ...)$ and thus

$$\Upsilon(\widetilde{\sigma}_A(\widetilde{x})) = (..., x_{n,0}, ..., x_{1,0}, x_{0,0}, \dot{x}_{0,1}, ..., x_{0,n}, ...) = \overline{\sigma}_A(\Upsilon(\widetilde{x}))$$

which says that $(\widetilde{X}_A, \widetilde{\sigma}_A)$ and $(\overline{\sigma}_A, \overline{X}_A)$ are topologically conjugate by Υ. $\square$

The shift σ_A has a clopen image of the form

$$\sigma_A(X_A) = \{(x_k)_{k\in\mathbb{N}} \in X_A : \sum_{x=1}^{N} A(x, x_0) > 0\}.$$

Thus, if A has at least one zero column then σ_A is not onto, and since $\overline{\sigma}_A$ is always onto, systems $(\widetilde{X}_A, \widetilde{\sigma}_A)$ and $(\overline{X}_A, \overline{\sigma}_A)$ cannot be conjugate. In fact, $\widetilde{X}_A = \bigcup_{n\in\mathbb{N}} X_{A,n} \cup X_{A,\infty}$, see Definition 2.1, and in the same manner as in the proof of Proposition 2.8 we may define a homeomorphism Υ from the subset $X_{A,\infty} = \{(x_0, x_1, ...) : x_n \in X_A,\ \sigma_A(x_n) = x_{n-1},\ n \geqslant 1\}$ onto $\overline{X}_A$ such that $\Upsilon(\widetilde{\sigma}_A(\widetilde{x})) = \overline{\sigma}_A(\Upsilon(\widetilde{x}))$ for $\widetilde{x} \in X_{A,\infty}$, that is

$$(X_{A,\infty}, \widetilde{\sigma}_A) \cong (\overline{X}_A, \overline{\sigma}_A).$$

In order to build a homeomorphic image of the whole space $\widetilde{X}_A$ we need to add a countable number of components to $\overline{X}_A$. We may do it by embedding $\overline{X}_A$ into another space $\overline{X}_{A'}$ associated with the larger alphabet $\{0, 1, ..., N\}$ and a larger matrix $A' = (A'(i,j))_{i,j\in\{0,1,...,N\}}$.

Proposition 2.9. Let $A' = (A'(i,j))_{i,j\in\{0,1,...,N\}}$ be given by

$$A'(i,j) = \begin{cases} A(i,j), & \text{if } \ i,j \in \{1, ..., N\}, \\ 1, & \text{if } \ i = 0 \text{ and either } j = 0 \text{ or } j\text{-th column of } A \text{ is zero }, \\ 0, & \ \text{otherwise}, \end{cases}$$

and let $X^+_{A'} = (x_k)_{k\in\mathbb{Z}} \in X^+_{A'} : x_0 \neq 0$. Then $\overline{\sigma}_{A'}(\overline{X}^+_{A'}) \subset \overline{X}^+_{A'}$ and

$$(\widetilde{X}_A, \widetilde{\sigma}_A) \cong (\overline{X}^+_{A'}, \overline{\sigma}_{A'}).$$

Proof. Let us treat $\overline{X}_A$ as a subset of $\{0, 1, ..., N\}^{\mathbb{Z}}$ and recall that $\widetilde{X}_A = \bigcup_{n\in\mathbb{N}} X_{A,n} \cup X_{A,\infty}$, and we have the homeomorphism $\Upsilon : X_{A,\infty} \to \overline{X}_A$, cf. (20). We put

$$\Upsilon(\widetilde{x}) = (..., 0, 0, x_{n-1,0}, ..., x_{1,0}, \dot{x}_{0,0}, x_{0,1}, ..., x_{0,n}, ...)$$

for $\widetilde{x} = (x_0, x_1, ..., x_{n-1}, 0, ...) \in X_{A,n}$ where $x_m = (x_{m,k})_{k\in\mathbb{N}} \in X_A$, $m = 0, ..., n-1$. In the same fashion as in the proof of Proposition 2.8 one checks that the mapping $\Upsilon : \widetilde{X}_A \to \{0, 1, ..., N\}^{\mathbb{Z}}$ is injective and the equality $\Upsilon(\widetilde{\sigma}_A(\widetilde{x})) = \overline{\sigma}_{A'}(\Upsilon(\widetilde{x}))$ holds for every $\widetilde{x} \in \widetilde{X}_A$. Moreover, $X_{A,n}$ is mapped by Υ onto the set

$$\{(x_k)_{k\in\mathbb{Z}} \in \{0, 1, ..., N\}^{\mathbb{Z}} : x_k = 0,\ k < -n; \sum_{i=1}^{N} A(i, x_n) = 0;\ A(x_k, x_{k+1}) = 1,\ k > -n\}$$

denoted by $\overline{X}_{A,n}$. Thus $(\widetilde{X}_A, \widetilde{\sigma}_A) \cong (\bigcup_{n\in\mathbb{N}} \overline{X}_{A,n} \cup \overline{X}_A, \overline{\sigma}_{A'})$. It is clear that $\overline{X}^+_{A'} = \bigcup_{n\in\mathbb{N}} \overline{X}_{A,n} \cup \overline{X}_A$ and hence the proof is complete. □

The proof of Proposition 2.8 is actually the proof of the probably known fact that if σ_A is onto, then $\overline{X}_A$ is the projective (inverse) limit of the projective sequence $X_A \xleftarrow{\sigma_A} X_A \xleftarrow{\sigma_A} ...$. Let us pick out the relationship between α-extensions and projective limits. For that purpose (and also for future needs) we introduce some terminology.
As $\widetilde{\alpha}$ is a partial homeomorphism, we denote by $\widetilde{\Delta}_n$ the domain of $\widetilde{\alpha}^n$, $n \in \mathbb{Z}$. For $n \in \mathbb{N}$ we have

$$\widetilde{\Delta}_n = \{\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X} : x_0 \in \Delta_n\} = \Phi^{-1}(\Delta_n),$$
$$\widetilde{\Delta}_{-n} = \{\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X} : x_n \neq 0\} \subset \Phi^{-1}(\Delta_{-n}).$$

With the help of Φ and $\widetilde{\alpha}^{-1}$, we define the family of projections $\Phi_n = \Phi \circ \widetilde{\alpha}^{-n} : \widetilde{\Delta}_{-n} \longrightarrow \Delta_n$, for $n \in \mathbb{N}$. We have

$$\Phi_n(\widetilde{x}) = (\Phi \circ \widetilde{\alpha}^{-n})(x_0, x_1, ..., x_n, ...) = x_n, \qquad \widetilde{x} \in \widetilde{\Delta}_{-n}.$$

Since $X_\infty = \bigcap_{n=1}^{\infty} \widetilde{\Delta}_{-n}$, the mappings Φ_n are well defined on X_∞, and the following statements are straightforward.

Proposition 2.10. The system $(X_\infty, \Phi_n)_{n\in\mathbb{N}}$ is the projective limit of the projective sequence $(\Delta_n, \alpha_n)_{n\in\mathbb{N}}$ where $\alpha_n = \alpha|_{\Delta_n}$: $X_\infty = \varprojlim (\Delta_n, \alpha_n)$.

Corollary 2.11. If α is onto, then $\widetilde{X} = \varprojlim (\Delta_n, \alpha_n)$.

The above statement provides us with many interesting examples of reversible extensions because projective (inverse) limit spaces commonly appear as attractors in dynamical systems (this was observed for the first time by R. F. Willams, see [27]). We recall here the classic example.

Example 2.12 (Solenoid). Let $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in the complex plane and let α be the expanding endomorphism of S^1 given by

$$\alpha(z) = z^2, \qquad z \in S^1.$$

Then the projective limit $\varprojlim (S^1, \alpha)$ is homeomorphic to Smale's solenoid, that is an attractor of the mapping F acting on the solid torus $\mathcal{T} = S^1 \times D^2$, where $D^2 = \{z \in \mathbb{C} : |z| \leq 1\}$, by $F(z_1, z_2) = (z_1^2, \lambda z_2 + \frac{1}{2} z_1)$ where $0 < \lambda < \frac{1}{2}$ is fixed.

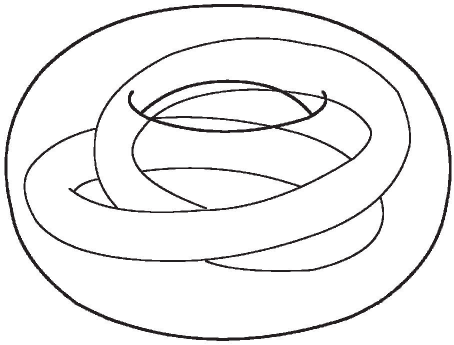


Fig. 2 The image of the solid torus under F is a solid torus which wraps twice around itself.

Namely the solenoid is the set $\mathcal{S} = \bigcap_{k\in\mathbb{N}} F^k(\mathcal{T})$, see Fig. 2, and the reversible extension of (S^1,α) is equivalent to $(\mathcal{S}, F|_{\mathcal{S}})$, see e.g. [7].

2.3 Decomposition of sets in $\widetilde{X}$

Before the end of this section we introduce a certain idea which enables us to 'decompose' a subset $\widetilde{U} \subset \widetilde{X}$ into the family $\{U_n\}_{n\in\mathbb{N}}$ of subsets of X. We shall need this device in Section 5.

Definition 2.13. Let $\widetilde{U} \subset \widetilde{X}$ be a subset of α-extension of X and let $n \in \mathbb{N}$. We shall call the set

$$U_n = \Phi_n(\widetilde{U} \cap \widetilde{\Delta}_{-n})$$

an *n-section* of $\widetilde{U}$.

It is evident that if $\{U_n\}_{n\in\mathbb{N}}$ is the family of sections of $\widetilde{U}$ then $\widetilde{U}$ is a subset of $\Big(U_0 \times (U_1 \cup \{0\}) \times ... \times (U_n \cup \{0\}) \times ...\Big) \cap \widetilde{X}$ but in general the opposite relation does not hold (see Example 2.16). Fortunately we have the following true statements.

Proposition 2.14. If α is injective on the inverse image of $\Delta_{-\infty} := \bigcap_{n\in\mathbb{N}} \Delta_{-n}$, that is for every point $x \in \Delta_{-\infty}$ we have $|\alpha^{-1}(x)| = 1$, then for every subset $\widetilde{U} \subset \widetilde{X}$ we have

$$\widetilde{U} = \Big(U_0 \times (U_1 \cup \{0\}) \times ... \times (U_n \cup \{0\}) \times ...\Big) \cap \widetilde{X} \tag{21}$$

where U_n is the n-section of $\widetilde{U}$, $n \in \mathbb{N}$.

Proof. Let $\widetilde{x} = (x_0, x_1, ...) \in (U_0 \times (U_1 \cup \{0\}) \times ... \times (U_n \cup \{0\}) \times ...) \cap \widetilde{X}$. We show that $\widetilde{x} \in \widetilde{U}$. Indeed, if $\widetilde{x} \notin X_\infty = \bigcap_{n\in\mathbb{N}} \widetilde{\Delta}_{-n}$, then $\widetilde{x} = (x_0, ..., x_N, 0, ...)$ where $x_N \in \Delta_N \setminus \Delta_{-1}$, and thus $\widetilde{x}$ is uniquely determined by x_N. As $x_N \in U_N = \Phi_N(\widetilde{U} \cap \widetilde{\Delta}_{-N})$, $\widetilde{U}$ must contain $\widetilde{x}$.

If $\widetilde{x} \in X_\infty$ then $x_n \in \Delta_{-\infty}$ for all $n \in \mathbb{N}$, and $\widetilde{x}$ is uniquely determined by $x_0 \in U_0$. Thus $\widetilde{x} \in \widetilde{U}$. □

Theorem 2.15. Let (X, α) be a partial dynamical system such that Δ_{-1} is open. Then every closed subset $\widetilde{V} \subset \widetilde{X}$ is uniquely determined by its sections $\{V_n\}_{n\in\mathbb{N}}$ via formula (21).

Proof. Let $\widetilde{V} \subset \widetilde{X}$ be closed, that is compact (see Theorem 2.2), and let $\widetilde{x} = (x_0, x_1, ...) \in (V_0 \times (V_1 \cup \{0\}) \times ... \times (V_n \cup \{0\}) \times ...) \cap \widetilde{X}$. We show that $\widetilde{x} \in \widetilde{V}$. If $\widetilde{x} \notin X_\infty$ then (see the argument in the proof of Proposition 2.14) we immediately get $\widetilde{x} \in \widetilde{V}$. Thus we only need to consider the case when $\widetilde{x} \in X_\infty$. For that purpose we define $\widetilde{D}_n = \{\widetilde{y} = (y_0, y_1, ...) \in \widetilde{X} : y_n = x_n\} \cap \widetilde{V}$, $n \in \mathbb{N}$. Clearly $\{\widetilde{D}_n\}_{n\in\mathbb{N}}$ is the decreasing family of closed nonempty subsets of the compact set $\widetilde{V}$. Hence

$$\bigcap_{n\in\mathbb{N}} \widetilde{D}_n = \{\widetilde{x}\} \in \widetilde{V}$$

and the proof is complete. □

Example 2.16. For the sake of illustration of the preceding statements and to see that they cannot be sharpen let us consider a dynamical system given by the diagram

$$\begin{tikzcd} \overset{x_1}{\bullet} \arrow[r] & \overset{x_0}{\bullet} \arrow[loop right] \end{tikzcd}$$

or equivalently by relations $X = \Delta_1 = \{x_0, x_1\}$, $\alpha(x_1) = \alpha(x_0) = x_0$ (or equivalently let $\alpha = \sigma_A$ and $X = X_A$ where $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$). Then $\Delta_{-\infty} = \{x_0\}$ and $|\alpha^{-1}(x_0)| = 2$, therefore Proposition 2.14 can not be used. The space $\widetilde{X}$ consists of elements $\widetilde{x}_n = (\underbrace{x_0, ..., x_0, x_1}_{n}, 0, ...)$, $n \in \mathbb{N}$, and $\widetilde{x}_\infty = (x_0, ..., x_0, ...)$. Hence it is convenient to identify $\widetilde{X}$ with the compactification $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ of the discrete space $\mathbb{N}$. Under this identification $\widetilde{\alpha}$ is given by

$$\widetilde{\alpha}(n) = n + 1, \ n \in \mathbb{N}, \qquad \widetilde{\alpha}(\infty) = \infty.$$

It is clear that all the sections of the subset $\mathbb{N} \subset \overline{\mathbb{N}}$ are equal to X. As $\mathbb{N}$ is not closed in $\overline{\mathbb{N}}$, Theorem 2.15 does not work here. Indeed, we have

$$\overline{\mathbb{N}} = \Big(X \times (X \cup \{0\}) \times ... \times (X \cup \{0\}) \times ...\Big) \neq \mathbb{N}.$$

3 Covariant representations and their coefficient algebra

The aim of this section is to study the interrelations between the covariant representations of C^*-dynamical systems corresponding to (X, α) and its reversible extension $(\widetilde{X}, \widetilde{\alpha})$. Our main tool will be Theorem 1.4 and hence, cf. Proposition 1.6,

from now on we shall always assume that the image Δ_{-1} of the partial mapping α is open.

First we show that the algebra $\mathcal{B} = C(\widetilde{X})$ possesses a certain universal property with respect to covariant faithful representations of (X, α). Afterwards, we construct a dense $*$-subalgebra of $\mathcal{B}$ with the help of which we investigate the structure of $\mathcal{B}$ and endomorphisms induced by $\widetilde{\alpha}$ and $\widetilde{\alpha}^{-1}$. Finally we show that there is a one-to-one correspondence between the covariant faithful representations of (X, α) and $(\widetilde{X}, \widetilde{\alpha})$, and in the case α is onto this correspondence is true for all covariant (not necessarily faithful) representations.

3.1 Definition and basic result

Let us recall that $\mathcal{A} = C(X)$ and δ is combined with α by (3). We denote by $C_K(X)$ the algebra of continuous functions on X vanishing outside a set $K \subset X$. We start with the definition of covariant representation, cf. [25, 10, 20, 1].

Definition 3.1. A *covariant representation* of a C^*-dynamical system $(\mathcal{A}, \delta)$, or of the partial dynamical system (X, α), is a triple (π, U, H) where $\pi : \mathcal{A} \to L(H)$ is a representation of $\mathcal{A}$ on Hilbert space H and $U \in L(H)$ is a partial isometry whose initial space is $\pi(C_{\Delta_{-1}}(X))H$ and whose final space is $\pi(C_{\Delta_1}(X))H$. In addition it is required that

$$U\pi(a)U^* = \pi(\delta(a)), \qquad \text{for } a \in \mathcal{A}.$$

If the representation π is faithful we call the triple (π, U, H) a *covariant faithful representation*. Let $\mathrm{CovRep}(\mathcal{A}, \delta)$ denote the set of all covariant representations and $\mathrm{CovFaithRep}(\mathcal{A}, \delta)$ the set of all covariant faithful representations of $(\mathcal{A}, \delta)$.

Remark 3.2. As Δ_1 and Δ_{-1} are clopen, the projections P_{Δ_1} and $P_{\Delta_{-1}}$ corresponding to the characteristic functions χ_{Δ_1} and $\chi_{\Delta_{-1}}$ belong to $\mathcal{A}$. Thus for every covariant representation (π, U, H) we see that $UU^* = \pi(P_{\Delta_1})$ and $U^*U = \pi(P_{\Delta_{-1}})$ belong to $\pi(\mathcal{A})$, cf. (11).

We shall see in Corollary 3.13 that every C^*-dynamical system $(\mathcal{A}, \delta)$ possesses a covariant faithful representation, and hence the sets $\mathrm{CovRep}\,(A, \delta)$ and $\mathrm{CovFaithRep}\,(\mathcal{A}, \delta)$ are non-empty.

Now we reformulate the main result of [16] in terms of covariant representations. The point is that for every covariant representation (π, U, H) of $(\mathcal{A}, \delta)$ the condition (11) holds, whence if π is faithful then by Theorem 1.4 the maximal ideal space of the C^*-algebra $C^*(\bigcup_{n=0}^{\infty} U^{*n}\pi(\mathcal{A})U^n)$ is homeomorphic to α-extension $\widetilde{X}$ of X.

Theorem 3.3. Let $(\pi, U, H) \in \mathrm{CovFaithRep}\,(\mathcal{A}, \delta)$ and let $\widetilde{X}$ be the α-extension of X. Then

$$C^*\big(\bigcup_{n=0}^{\infty} U^{*n}\pi(\mathcal{A})U^n\big) \cong C(\widetilde{X}).$$

In other words, the coefficient C^*-algebra of $C^*(\pi(\mathcal{A}), U)$ is isomorphic to the algebra of continuous functions on α-extension of X, cf. [18]. Moreover this isomorphism maps an operator of the form $\pi(a_0) + U^*\pi(a_1)U + ... + U^{*N}\pi(a_N)U^N$, where $a_0, a_1, ..., a_N \in \mathcal{A}$, onto a function $b \in C(\widetilde{X})$ such that

$$b(\widetilde{x}) = a_0(x_0) + a_1(x_1) + ... + a_N(x_N),$$

where $\widetilde{x} = (x_0, ...) \in \widetilde{X}$ and we set $a_n(x_n) = 0$ whenever $x_n = 0$.

Proof. By Theorem 1.4 the maximal ideal space of $C^*(\bigcup_{n=0}^{\infty} U^{*n}\pi(\mathcal{A})U^n)$ is homeomorphic to $\widetilde{X}$. Hence $C^*(\bigcup_{n=0}^{\infty} U^{*n}\pi(\mathcal{A})U^n) \cong C(\widetilde{X})$. Taking into account formulas (9),(10) we obtain the postulated form of this isomorphism. Indeed, if $\widetilde{x} = (x_0, x_1, ...)$ is a character on $C^*(\bigcup_{n=0}^{\infty} U^{*n}\pi(\mathcal{A})U^n)$ then

$$\widetilde{x}(\sum_{n=0}^{N} U^{*n}\pi(a_n)U^n) = \sum_{n=0}^{N} \widetilde{x}(U^{*n}\pi(a_n)U^n) = \sum_{n=0}^{N} \xi_{\widetilde{x}}^n(a_n) = \sum_{n=0}^{N} a_n(x_n)$$

where $\xi_{\widetilde{x}}^n(a) = \widetilde{x}(U^{*n}\pi(a)U^n)$, cf. (9), and $x_n = 0$ whenever $\xi_{\widetilde{x}}^n \equiv 0$. □

From the above it follows that $\mathcal{B} = C(\widetilde{X})$ can be regarded as the universal (in fact unique) coefficient C^*-algebra for covariant faithful representations. In case δ is injective (that is α is onto), the universality of $\mathcal{B}$ is much 'stronger', see Proposition 3.7.

3.2 Coefficient C^*-algebra

We shall present now a certain dense *-subalgebra of $\mathcal{B} = C(\widetilde{X})$, a coefficient C^*-algebra which frequently might be more convenient to work with. The plan is to construct an algebra $B_0 \subset l^1(\mathbb{N}, \mathcal{A})$ and then take the quotient of it by certain ideal. The result will be naturally isomorphic to a *-subalgebra of $\mathcal{B}$.
First, let us observe that if we set $\mathcal{A}_n := \delta^n(1)\mathcal{A}$, $n \in \mathbb{N}$, then we obtain a decreasing family $\{\mathcal{A}_n\}_{n\in\mathbb{N}}$, of closed two-sided ideals in $\mathcal{A}$. Since the operator $\delta^n(1)$ corresponds to the characteristic function $\chi_{\Delta_n} \in C(X)$, one can consider $\mathcal{A}_n$ as $C_{\Delta_n}(X)$. Let B_0 denote the set consisting of sequences $a = \{a_n\}_{n\in\mathbb{N}}$ where $a_n \in \mathcal{A}_n$, $n \in \mathbb{N}$, and only a finite number of functions a_n is non zero. Namely

$$B_0 = \{a \in \prod_{n=0}^{\infty} \mathcal{A}_n : \exists_{N>0} \forall_{n>N} \quad a_n \equiv 0\}.$$

Let $a = \{a_n\}_{n\geqslant 0}$, $b = \{b_n\}_{n\geqslant 0} \in B_0$ and $\lambda \in \mathbb{C}$. We define the addition, multiplication by scalar, convolution multiplication and involution on B_0 as follows

$$(a + b)_n = a_n + b_n, \tag{22}$$

$$(\lambda a)_n = \lambda a_n, \tag{23}$$

$$(a \cdot b)_n = a_n \sum_{j=0}^{n} \delta^j(b_{n-j}) + b_n \sum_{j=1}^{n} \delta^j(a_{n-j}), \tag{24}$$

$$(a^*)_n = \overline{a}_n. \tag{25}$$

These operations are well defined and seems very familiar, except maybe the multiplication of two elements from B_0. We point out here that the index in one of the sums of (24) starts running from 0.

Proposition 3.4. The set B_0 with operations (22), (23), (24), (25) becomes a commutative algebra with involution.

Proof. It is clear that operations (22), (23) define the structure of vector space on B_0 and that operation (25) is an involution. The rule (24) is less easy to show its properties. Commutativity and distributivity can be checked easily but in order to prove the associativity we must strain ourselves quite a lot.
Let a, b, $c \in B_0$. Then

$$((a\cdot b)\cdot c)_n = (a\cdot b)_n \sum_{j=0}^{n} \delta^j(c_{n-j}) + c_n \sum_{j=1}^{n} \delta^j((a\cdot b)_{n-j}),$$

$$(a\cdot (b\cdot c))_n = a_n \sum_{j=0}^{n} \delta^j((b\cdot c)_{n-j}) + (b\cdot c)_n \sum_{j=1}^{n} \delta^j(a_{n-j}),$$

where

$$(a\cdot b)_n \sum_{j=0}^{n} \delta^j(c_{n-j}) = [a_n \sum_{k=0}^{n} \delta^k(b_{n-k}) + b_n \sum_{k=1}^{n} \delta^k(a_{n-k})] \sum_{j=0}^{n} \delta^j(c_{n-j})$$
$$= a_n \sum_{k,j=0}^{n} \delta^k(b_{n-k})\delta^j(c_{n-j}) + b_n \sum_{k=0,j=1}^{n} \delta^k(a_{n-k})\delta^j(c_{n-j})$$

and

$$c_n \sum_{j=1}^{n} \delta^j((a\cdot b)_{n-j}) = c_n \sum_{j=1}^{n} \delta^j(a_{n-j} \sum_{k=0}^{n-j} \delta^k(b_{n-j-k}) + b_{n-j} \sum_{k=1}^{n-j} \delta^k(a_{n-j-k}))$$
$$= c_n \sum_{j=1}^{n} \delta^j(a_{n-j}) \sum_{k=j}^{n} \delta^k(b_{n-k}) + \delta^j(b_{n-j}) \sum_{k=j+1}^{n} \delta^k(a_{n-k})) = c_n \sum_{k=1,j=1}^{n} \delta^j(a_{n-j})\delta^k(b_{n-k}).$$

Simultaneously by analogous computation

$$a_n \sum_{j=0}^{n} \delta^j((b\cdot c)_{n-j}) = a_n \sum_{k,j=0}^{n} \delta^k(b_{n-k})\delta^j(c_{n-j})$$

$$(b\cdot c)_n \sum_{j=1}^{n} \delta^j(a_{n-j}) = b_n \sum_{k=0,j=1}^{n} \delta^k(a_{n-k})\delta^j(c_{n-j}) + c_n \sum_{k=1,j=1}^{n} \delta^j(a_{n-j})\delta^k(b_{n-k}).$$

Thus, $((a\cdot b)\cdot c)_n = (a\cdot(b\cdot c))_n$ and the n-th entry of the sequence $a\cdot b\cdot c$ is of the form

$$a_n \sum_{k,j=0}^{n} \delta^k(b_{n-k})\delta^j(c_{n-j}) + b_n \sum_{k=0,j=1}^{n} \delta^k(a_{n-k})\delta^j(c_{n-j}) + c_n \sum_{k=1,j=1}^{n} \delta^k(a_{n-k})\delta^j(b_{n-j}).$$

□

Let us define a morphism $\varphi : B_0 \to \mathcal{B}$. To this end, let $a = \{a_n\}_{n\in\mathbb{N}} \in B_0$ and $\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X}$. We set

$$\varphi(a)(\widetilde{x}) = \sum_{n=0}^{\infty} a_n(x_n), \tag{26}$$

where $a_n(x_n) = 0$ whenever $x_n = 0$. The mapping φ is well defined as only a finite number of functions a_n, $n \in \mathbb{N}$, is non zero.

Theorem 3.5. The mapping $\varphi : B_0 \to \mathcal{B}$ given by (26) is a morphism of algebras with involution and the image of φ is dense in $\mathcal{B}$, that is

$$\overline{\varphi(B_0)} = \mathcal{B}.$$

Proof. It is clear that φ is a linear mapping preserving an involution. We show that φ is multiplicative. Let a, $b \in B_0$ and $\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X}$ and let $N > 0$ be such that for every $m > N$ we have $a_m = b_m = 0$. Using the fact that $\alpha^j(x_n) = x_{n-j}$ we obtain

$$\varphi(a \cdot b)(\widetilde{x}) = \sum_{n=0}^{N} (a \cdot b)_n(x_n) = \sum_{n=0}^{N} \Big[a_n \sum_{j=0}^{n} \delta^j(b_{n-j}) + b_n \sum_{j=1}^{n} \delta^j(a_{n-j}) \Big](x_n)$$

$$= \sum_{n=0}^{N} \big[a_n(x_n) \sum_{j=0}^{n} b_{n-j}(x_{n-j}) + b_n(x_n) \sum_{j=1}^{n} a_{n-j}(x_{n-j}) \big]$$

$$= \sum_{n=0,j=0}^{N} a_n(x_n) b_j(x_j) = \sum_{n=0}^{N} a_n(x_n) \cdot \sum_{j=0}^{N} b_j(x_j) = \big[\varphi(a) \cdot \varphi(b)\big](\widetilde{x}).$$

To prove that $\varphi(B_0)$ is dense in $\mathcal{B} = C(\widetilde{X})$ we use the Stone-Weierstrass theorem. It is clear that $\varphi(B_0)$ is a self-adjoint subalgebra of $\mathcal{B}$ and as $a = (1, 0, 0, ...) \in B_0$, we get $\varphi(a) = 1 \in \mathcal{B}$, that is $\varphi(B_0)$ contains the identity. Thus, what we only need to prove is that B_0 separates points of $\widetilde{X}$.
Let $\widetilde{x} = (x_0, x_1, ...)$ and $\widetilde{y} = (y_0, y_1, ...)$ be two distinct points of $\widetilde{X}$. Then there exists $n \in \mathbb{N}$ such that $x_n \neq y_n$ and by Urysohn's lemma there exists a function $a_n \in C_{\Delta_n}(X \cup \{0\})$ such that $a_n(x_n) = 1$ and $a_n(y_n) = 0$. Taking $a \in B_0$ of the form

$$a = (\underbrace{0, ..., 0}_{n}, a_n, 0, ...)$$

we see that $\varphi(a)(\widetilde{x}) = 1$ and $\varphi(a)(\widetilde{y}) = 0$. Thus the proof is complete. □

Let us consider the quotient space $B_0/\mathrm{Ker}\,\varphi$ and the quotient mapping $\phi : B_0/\mathrm{Ker}\,\varphi \to \mathcal{B}_0$, that is $\phi(a + \mathrm{Ker}\,\varphi) = \varphi(a)$. Clearly ϕ is an injective mapping onto a dense *-subalgebra of $\mathcal{B}$. In what follows we make use of the following notation

$$\mathcal{B}_0 := \phi(B_0/\mathrm{Ker}\,\varphi)$$

$$[a] := \phi(a + \operatorname{Ker}\varphi), \qquad a \in B_0$$

Definition 3.6. We shall call $\mathcal{B}_0$ the *coefficient* **-algebra* of a dynamical system $(\mathcal{A}, \delta)$. We will write $[a] = [a_0, a_1, ...] \in \mathcal{B}_0$ for $a = (a_0, a_1, a_2, ...) \in B_0$.

The natural injection $\mathcal{A} \ni a_0 \longrightarrow [a_0, 0, 0, ...] \in \mathcal{B}_0$ enables us to treat $\mathcal{A}$ as an C^*-subalgebra of $\mathcal{B}_0$ and hence also of $\mathcal{B}$:

$$\mathcal{A} \subset \mathcal{B}_0 \subset \mathcal{B}, \qquad \overline{\mathcal{B}}_0 = \mathcal{B}.$$

Using the mappings $\Phi_n : \widetilde{\Delta}_{-n} \longrightarrow \Delta_n$ (see subsection 2.2) one can embed into $\mathcal{B}_0$ not only $\mathcal{A}$ but all the subalgebras $\mathcal{A}_n = C_{\Delta_n}(X)$, $n \in \mathbb{N}$. Indeed, if we define $\Phi_{*n} : \mathcal{A}_n \to \mathcal{B}$ to act as follows

$$\Phi_{*n}(a) = [\underbrace{0, ..., 0}_{n}, a, 0, ...] = \begin{cases} a \circ \Phi_n & , \widetilde{x} \in \widetilde{\Delta}_{-n} \\ 0 & , \widetilde{x} \notin \widetilde{\Delta}_{-n} \end{cases},$$

then clearly Φ_{*n} are injective. Moreover we have $C^*(\bigcup_{n\in\mathbb{N}} \Phi_{*n}(\mathcal{A}_n)) = \mathcal{B}$, and in the case δ is a monomorphism, that is α is surjective, $\{\Phi_{*n}(\mathcal{A}_n)\}_{n\in\mathbb{N}}$ forms an increasing family of algebras and $\mathcal{B}_0 = \bigcup_{n\in\mathbb{N}} \Phi_{*n}(\mathcal{A}_n)$. We are exploiting this fact in the coming proposition.

Proposition 3.7. If δ is injective then $\mathcal{B}$ is the direct limit $\varinjlim \mathcal{A}_n$ of the sequence $(\mathcal{A}_n, \delta_n)_{n=0}^{\infty}$ where $\delta_n = \delta|_{\mathcal{A}_n}$, $n \in \mathbb{N}$.

Proof. Let $\mathcal{B} = \varinjlim \mathcal{A}_n$ be the direct limit of the sequence $(\mathcal{A}_n, \delta_n)_{n=0}^{\infty}$, and let ψ^n : $\mathcal{A}_n \to \mathcal{B}$ be the natural homomorphisms, see for instance [21]. It is straightforward to see that the diagram

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{\delta_n} & \mathcal{A}_{n+1} \\ & \searrow \Phi_{*n} & \downarrow \Phi_{*n+1} \\ & & \mathcal{B} \end{array}$$

commutes and hence there exists a unique homomorphism $\psi : \mathcal{B} \to \mathcal{B}$ such that the diagram

$$\begin{array}{ccc} \mathcal{A}_n & \xrightarrow{\psi^n} & \mathcal{B} \\ & \searrow \Phi_{*n} & \downarrow \psi \\ & & \mathcal{B} \end{array}$$

commutes. It is evident that ψ is a surjection ($\bigcup_{n\in\mathbb{N}} \Phi_{*n}(\mathcal{A}_n)$ generates $\mathcal{B}$) and as $\Phi_{*n}(\mathcal{A}_n)$ is increasing and, Φ and ψ are injective. Therefore ψ is an isomorphism and the proof is complete. $\square$

The preceding proposition points out the relationship between our approach and the approach presented (among the others) by G. J. Murphy in [22]. We shall discuss this

relationship in the sequel, see Remark 4.10.
Let us now proceed and consider endomorphisms of $\mathcal{B} = C(\widetilde{X})$ given by the formulae

$$\widetilde{\delta}(a)(x) = \begin{cases} a(\widetilde{\alpha}(\widetilde{x})) & ,\widetilde{x} \in \widetilde{\Delta}_1 \\ 0 & ,\widetilde{x} \notin \widetilde{\Delta}_1 \end{cases} \qquad \widetilde{\delta}_*(a)(x) = \begin{cases} a(\widetilde{\alpha}^{-1}(\widetilde{x})) & ,\widetilde{x} \in \widetilde{\Delta}_{-1} \\ 0 & ,\widetilde{x} \notin \widetilde{\Delta}_{-1} \end{cases} \tag{27}$$

where $\widetilde{\alpha} : \widetilde{\Delta}_1 \to \widetilde{\Delta}_{-1}$ is a canonical partial homeomorphism of $\widetilde{X}$ defined by formula (19). What is important is that characteristic functions of $\widetilde{\Delta}_1$ and $\widetilde{\Delta}_{-1}$ belong to $\mathcal{A} \subset \mathcal{B}_0$. Indeed, we have $\chi_{\widetilde{\Delta}_1} = [\chi_{\Delta_1}, 0, 0, ...]$, $\chi_{\widetilde{\Delta}_1} = [\chi_{\Delta_{-1}}, 0, 0, ...] \in \mathcal{A}$ (see remark after Definition 3.6). Furthermore, the domain $\widetilde{\Delta}_n$ of the mapping $\widetilde{\alpha}^n$, $n \in \mathbb{Z}$, is clopen and it is just an easy exercise to check that for $n \in \mathbb{N}$ we have $\chi_{\widetilde{\Delta}_n} = [\chi_{\Delta_n}, 0, 0, ...]$, $\chi_{\widetilde{\Delta}_{-n}} = [\underbrace{0, 0, ..., 0}_{n}, \Delta_n, ...] \in \mathcal{B}_0$. In particular $\chi_{\widetilde{\Delta}_{-1}} = [\chi_{\Delta_{-1}}, 0, 0, ...] = [0, \chi_{\Delta_1}, 0, ...]$. We are now ready to give an 'algebraic' description of $\widetilde{\delta}$ and $\widetilde{\delta}_*$.

Proposition 3.8. Endomorphisms $\widetilde{\delta}$ and $\widetilde{\delta}_*$ preserve *-subalgebra $\mathcal{B}_0 \subset \mathcal{B}$ and for $a = (a_0, a_1, a_2, ...) \in B_0$ we have

$$\widetilde{\delta}([a]) = [\delta(a_0) + a_1, a_2, a_3, ...], \qquad \widetilde{\delta}_*([a]) = [0, a_0\delta(1), a_1\delta^2(1), ...]. \tag{28}$$

Proof. Let $\widetilde{x} = (x_0, x_1, x_2, ...) \in \widetilde{X}$. In order to prove the first equality in (28) it is enough to notice that for $\widetilde{x} \in \widetilde{\Delta}_1$ we have

$$\delta([a])(\widetilde{x}) = [a](\widetilde{\alpha}(\widetilde{x})) = [a](\alpha(x_0), x_0, x_1, ...) = a_0(\alpha(x_0)) + a_1(x_0) + a_2(x_1) + ...$$
$$= [\delta(a_0) + a_1, a_2, a_3, ...](x_0, x_1, x_2, ...) = [\delta(a_0) + a_1, a_2, a_3, ...](\widetilde{x})$$

and for $\widetilde{x} \notin \widetilde{\Delta}_1$ both sides of the left hand side equation in (28) are equal to zero.
In the same manner we show the validity of the remaining equality. If $\widetilde{x} \in \widetilde{\Delta}_{-1}$ then

$$\delta_*([a])(\widetilde{x}) = [a](\widetilde{\alpha}^{-1}(\widetilde{x})) = [a](x_1, x_2, x_3...) = a_0(x_1) + a_1(x_2) + a_2(x_3) + ...$$
$$= [0, a_0\delta(1), a_1\delta^2(1), ...](x_0, x_1, x_2, ...) = [0, a_0\delta(1), a_1\delta^2(1), ...](\widetilde{x})$$

and if $\widetilde{x} \notin \widetilde{\Delta}_1$ then both sides of the right hand side equation in (28) are equal to zero.□

Example 3.9. The dynamical system (X, α) from Example 2.16 corresponds to the C^*-dynamical system $(\mathcal{A}, \delta)$ where $\mathcal{A} = C(\{x_0, x_1\})$ and $\delta(a) \equiv a(x_0)$. We identify $\widetilde{X}$ with $\overline{\mathbb{N}}$ as we did before. Then, since $a = [a_0, a_1, ..., a_N, 0, ...]$, $a_k \in \mathcal{A}$, $k = 0, ...N$, is a continuous function on $\widetilde{X} = \overline{\mathbb{N}}$ we can regard it as a sequence which has a limit. One readily checks that this sequence has the following form: $a(n) = \sum_{k=0}^{n-1} a_k(x_0) + a_n(x_1)$ for $n = 0, ...N$, and $a(n) = \sum_{k=0}^{N} a_k(x_0)$ for $n > N$. Hence $\mathcal{B}_0$ is the *-algebra of the eventually constant sequences, in particular $\mathcal{A}$ consist of sequences of the form $(a, b, b, b, ...)$, $a, b \in \mathbb{C}$. We have

$$\mathcal{B}_0 = \{(a(n))_{n\in\mathbb{N}} : \exists_{N\in\mathbb{N}} \forall_{n,m>N} \; a(n) = a(m)\},$$
$$\mathcal{B} = \{(a(n))_{n\in\mathbb{N}} : \exists_{a(\infty)\in\mathbb{C}} \; \lim_{n\to\infty} a(n) = a(\infty)\}$$

and within these identifications $\widetilde{\delta}$ is the forward and $\widetilde{\delta}_*$ is the backward shift.

3.3 The interrelations between covariant representations

The construction of the *-algebra $\mathcal{B}_0$ enables us to excavate the inverse of the isomorphism from Theorem 3.3, and what is more important it enables us to realize that every covariant faithful representation of $(\mathcal{A}, \delta)$ gives rise to a covariant faithful representation of $(\mathcal{B}, \widetilde{\delta})$.

Theorem 3.10. Let $(\pi, U, H) \in \mathrm{CovFaithRep}\,(\mathcal{A}, \delta)$. Then there exists an extension $\overline{\pi}$ of π onto the coefficient algebra $\mathcal{B}$ such that $\overline{\pi} : \mathcal{B} \to C^*\big(\bigcup_{n=0}^{\infty} U^{*n}\pi(\mathcal{A})U^n\big)$ is an isomorphism defined by

$$[a_0, ..., a_N, 0, ...] \longrightarrow \pi(a_0) + U^*\pi(a_1)U + ... + U^{*N}\pi(a_N)U^N.$$

Moreover, we have $(\overline{\pi}, U, H) \in \mathrm{CovFaithRep}\,(\mathcal{B}, \widetilde{\delta})$, that is

$$\overline{\pi}(\widetilde{\delta}(a)) = U\overline{\pi}(a)U^*, \qquad \overline{\pi}(\widetilde{\delta}_*(a)) = U^*\overline{\pi}(a)U, \qquad a \in \mathcal{B}. \tag{29}$$

Proof. In view of Theorem 3.3 it is immediate that $\overline{\pi}$ is an isomorphism. By Theorem 1.8 and by the form of endomorphisms $\widetilde{\delta}$ and $\widetilde{\delta}_*$, see (27), we get (29). □

We can give a statement somewhat inverse to the above.

Theorem 3.11. Let $(\overline{\pi}, U, H) \in \mathrm{CovRep}\,(\mathcal{B}, \widetilde{\delta})$ and let π be the restriction of $\overline{\pi}$ onto $\mathcal{A}$. Then $(\pi, U, H) \in \mathrm{CovRep}\,(\mathcal{A}, \delta)$. Moreover if $(\overline{\pi}, U, H)$ is in $\mathrm{CovFaithRep}\,(\mathcal{B}, \widetilde{\delta})$ then (π, U, H) is in $\mathrm{CovFaithRep}\,(\mathcal{A}, \delta)$ and the extension of π mentioned in Theorem 3.10 coincides with $\overline{\pi}$.

Proof. Recall that for $a \in \mathcal{A}$ we write $[a, 0, 0, ...] \in \mathcal{B}$ and thus (see also Proposition 3.8) we get

$$\pi(\delta(a)) = \overline{\pi}([\delta(a), 0, 0, ...]) = \overline{\pi}(\widetilde{\delta}([a, 0, 0, ...])) = U\overline{\pi}([a, 0, 0, ...])U^* = U\pi(a)U^*.$$

$$U^*U = \overline{\pi}(\chi_{\widetilde{\Delta}_1}) = \overline{\pi}([\chi_{\Delta_1}, 0, 0, ...]) = \pi(\chi_{\Delta_1}),$$

$$UU^* = \overline{\pi}(\chi_{\widetilde{\Delta}_{-1}}) = \overline{\pi}([\chi_{\Delta_{-1}}, 0, 0, ...]) = \pi(\chi_{\Delta_{-1}}).$$

Hence $(\pi, U, H) \in \mathrm{CovRep}\,(\mathcal{A}, \delta)$.
For $[a_0, ..., a_N, 0, ...] \in \mathcal{B}_0$ we have, cf. Proposition 3.8,

$$\overline{\pi}([a_0, ..., a_N, 0, ...]) = \overline{\pi}(a_0 + \widetilde{\delta}_*(a_1) + ... + \widetilde{\delta}_*^N(a_N)) = \pi(a_0) + U^*\pi(a_1)U + ... + U^{*N}\pi(a_N)U^N.$$

Hence the second part of the theorem follows. □

Corollary 3.12. There is a natural bijection between $\mathrm{CovFaithRep}\,(\mathcal{A}, \delta)$ and $\mathrm{CovFaithRep}\,(\mathcal{B}, \widetilde{\delta})$.

The endomorphism $\widetilde{\delta}$ maps isomorphically $C_{\widetilde{\Delta}_{-1}}(\widetilde{X})$ onto $C_{\widetilde{\Delta}_1}(\widetilde{X})$, whence we have a $*$-isomorphism between two closed two-sided ideals in $\mathcal{B}$. In [10] R. Exel calls such mappings partial automorphisms (of $\mathcal{B}$), see [10, Definition 3.1]. He also considers covariant representations of partial automorphisms and his definition of those objects agrees with Definition 3.1 in the case that α is a partial homeomorphism. Moreover, R. Exel proves in [10, Theorem 5.2] the existance of covariant faithful representation of a partial automorphism which automatically gives us

Corollary 3.13. The set $\mathrm{CovFaithRep}\,(\mathcal{A},\delta)$ is not empty.

Proof. The set $\mathrm{CovFaithRep}\,(\mathcal{B},\widetilde{\delta})$ is not empty by [10, Theorem 5.2]. □

The former of the preceding corollaries is not true for not faithful representations (see Example 3.16). In general the set $\mathrm{CovRep}\,(\mathcal{B},\widetilde{\delta})$ is larger than $\mathrm{CovFaithRep}\,(\mathcal{A},\delta)$, and there appears a problem with prolongation of π from $\mathcal{A}$ to $\mathcal{B}$ when π is not faithful. Fortunately in view of Proposition 3.7 we have the following true statement.

Theorem 3.14. If $\delta:\mathcal{A}\to\mathcal{A}$ is a monomorphism then for any $(\pi,U,H)\in\mathrm{CovRep}\,(\mathcal{A},\delta)$ there exist $(\overline{\pi},U,H)\in\mathrm{CovRep}\,(\mathcal{B},\widetilde{\delta})$ such that $\overline{\pi}([a_0,...,a_N,0,...])=\pi(a_0)+U^*\pi(a_1)U+ ... + U^{*N}\pi(a_N)U^N$.

Proof. Let $(\pi,U,H)\in\mathrm{CovRep}\,(\mathcal{A},\delta)$. Let us notice that as δ is injective $\Delta_{-1}=X$ and hence U is an isometry (see Remark 3.2). Now, consider the C^*-algebra $C^*\big(\bigcup_{n=0}^{\infty}U^{*n}\pi(\mathcal{A})U^n\big)$ and define the family of mappings $\pi_n:\mathcal{A}_n\to C^*\big(\bigcup_{n=0}^{\infty}U^{*n}\pi(\mathcal{A})U^n\big)$, $n\in\mathbb{N}$, by the formula

$$\pi_n(a)=U^{*n}\pi(a)U^n,\qquad a\in\mathcal{A}_n.$$

Then the following diagram

$$\begin{tikzcd} \mathcal{A}_n \arrow[r, "\delta_n"] \arrow[dr, "\pi_n"'] & \mathcal{A}_{n+1} \arrow[d, "\pi_{n+1}"] \\ & C^*\big(\bigcup_{n=0}^{\infty}U^{*n}\pi(\mathcal{A})U^n\big) \end{tikzcd}$$

commutates. Hence, according to Proposition 3.7 there exists a unique C^*-morphism $\overline{\pi}$ such that the diagram

$$\begin{tikzcd} \mathcal{A}_n \arrow[r, "\Phi_{*n}"] \arrow[dr, "\pi_n"'] & \mathcal{B} \arrow[d, "\overline{\pi}"] \\ & C^*\big(\bigcup_{n=0}^{\infty}U^{*n}\pi(\mathcal{A})U^n\big) \end{tikzcd}$$

commutes. The hypotheses now follows. □

Corollary 3.15. If δ is a monomorphism then the mapping $\overline{\pi}\to\overline{\pi}|_{\mathcal{A}}$ establishes a bijection from $\mathrm{CovRep}\,(\mathcal{B},\widetilde{\delta})$ onto $\mathrm{CovRep}\,(\mathcal{A},\delta)$.

In case δ is not a monomorphism two different covariant representations of $(\mathcal{B},\widetilde{\delta})$ may induce the same covariant representation of $(\mathcal{A},\delta)$.

Example 3.16. Not to look very far let us take the system (X,α) from Example 2.7. The corresponding C^*-dynamical system is $(\mathcal{A},\delta)$ where $\mathcal{A}=C(\{x_0,x_1,x_2,y_2\})\cong\mathbb{C}^4$ and $\delta(a)=(0,a_{x_0},a_{x_1},a_{x_1})$ for $a=(a_{x_0},a_{x_1},a_{x_2},a_{y_2})\in\mathcal{A}$. The coefficient algebra is

$$\mathcal{B}=C(\{\widetilde{x}_0,\widetilde{x}_1,\widetilde{x}_2,\widetilde{y}_0,\widetilde{y}_1,\widetilde{y}_2\})\cong\mathbb{C}^6$$

and we check that, for $a\in\mathcal{A}$,

$$[a,0,...]=(a_{x_0},a_{x_1},a_{x_2},a_{x_0},a_{x_1},a_{y_2}),\qquad [0,a\delta(1),0,...]=(0,a_{x_0},a_{x_1},0,a_{x_0},a_{x_1}),$$

$$[0,0,a\delta^2(1),0,...]=(0,0,a_{x_0},0,0,a_{x_0})\quad\text{and}\quad[0,0,...,0,a\delta^N(1),0...]\equiv 0,\ \text{for } N>2.$$

Moreover, we have $\widetilde{\delta}(a_{\widetilde{x}_0},a_{\widetilde{x}_1},a_{\widetilde{x}_2},a_{\widetilde{y}_0},a_{\widetilde{y}_1},a_{\widetilde{y}_2})=(0,a_{\widetilde{x}_0},a_{\widetilde{x}_1},0,a_{\widetilde{y}_0},a_{\widetilde{y}_1})$. It is now straightforward that $(\overline{\pi}_1,U,\mathbb{C}^2)$ and $(\overline{\pi}_2,U,\mathbb{C}^2)$ where $U=\begin{pmatrix}0&0\\1&0\end{pmatrix}$,

$$\overline{\pi}_1(a_{\widetilde{x}_0},a_{\widetilde{x}_1},a_{\widetilde{x}_2},a_{\widetilde{y}_0},a_{\widetilde{y}_1},a_{\widetilde{y}_2})=\begin{pmatrix}a_{\widetilde{x}_0}&0\\0&a_{\widetilde{x}_1}\end{pmatrix}$$

and

$$\overline{\pi}_2(a_{\widetilde{x}_0},a_{\widetilde{x}_1},a_{\widetilde{x}_2},a_{\widetilde{y}_0},a_{\widetilde{y}_1},a_{\widetilde{y}_2})=\begin{pmatrix}a_{\widetilde{y}_0}&0\\0&a_{\widetilde{y}_1}\end{pmatrix}$$

are covariant representations of $(\mathcal{B},\widetilde{\delta})$ which induce the same covariant representation $(\pi,U,\mathbb{C}^2)$ of $(\mathcal{A},\delta)$ where $\pi(a_{x_0},a_{x_1},a_{x_2},a_{y_2})=\begin{pmatrix}a_{x_0}&0\\0&a_{x_1}\end{pmatrix}$.

4 Covariance algebra

In this section we introduce the title object of the paper. We recall the definition of the partial crossed product, cf. [10, 20], and then define the covariance algebra of $(\mathcal{A},\delta)$ to be the partial crossed product associated with $(\mathcal{B},\widetilde{\delta})$. We give a number of examples of such algebras, and finally show (justify the definition) that the covariant algebra is the universal object in the category of covariant faithful representations of $(\mathcal{A},\delta)$ and in the case δ is injective also in the category of covariant (not necessarily faithful) representations of $(\mathcal{A},\delta)$.

4.1 The algebra $C^*(X,\alpha)$

Let us recall that a *partial automorphism* of a C^*-algebra $\mathcal{C}$ is a mapping $\theta : I \to J$ where I and J are closed two-sided ideals in $\mathcal{C}$ and θ is a $*$-isomorphism, cf. [10]. If a partial automorphism θ is given then for each $n \in \mathbb{Z}$ we let D_n denote the domain of θ^{-n} with the convention that $D_0 = \mathcal{C}$ and θ^0 is the identity automorphism of $\mathcal{C}$. Letting

$$L = \{a \in l^1(\mathbb{Z}, \mathcal{C}) : a(n) \in D_n\}$$

and defining the convolution multiplication, involution, and norm as follows

$$(a * b)(n) = \sum_{k=-\infty}^{\infty} \theta^k\Big(\theta^{-k}\big(a(k)\big)b(n-k)\Big)$$

$$(a^*)(n) = \theta^n(a(-n)^*)$$

$$\|a\| = \sum_{n=-\infty}^{\infty} \|a(n)\|$$

we equip L with a Banach *-algebra structure. The universal enveloping C^*-algebra of L is called the *partial crossed product* (or the *covariance algebra*) for the partial automorphism θ and is denoted by $\mathcal{C} \rtimes_\theta \mathbb{Z}$, see [10, 20].

It is clear that the partial homeomorphism $\widetilde{\alpha}$ of $\widetilde{X}$ defines the partial automorphism $\widetilde{\delta} : C_{\widetilde{\Delta}_{-1}}(\widetilde{X}) \to C_{\widetilde{\Delta}_1}(\widetilde{X})$ of the coefficient C^*-algebra $\mathcal{B} = C(\widetilde{X})$ (we shall not distinguish between the endomorphism $\widetilde{\delta}$ given by (27) and its restriction to $C_{\widetilde{\Delta}_{-1}}(\widetilde{X}) \subset \mathcal{B}$). The definition to follow anticipates Theorem 4.7.

Definition 4.1. The *covariance algebra* $C^*(X,\alpha)$ of a partial dynamical system (X,α) is the partial crossed product for the partial automorphism $\widetilde{\delta}$ of the coefficient C^*-algebra $\mathcal{B}$.
That is $C^*(X,\alpha) = \mathcal{B} \rtimes_{\widetilde{\delta}} \mathbb{Z}$ and for $C^*(X,\alpha)$ we shall also write $C^*(\mathcal{A},\delta)$.

Remark 4.2. In the case α is injective, equivalently δ is a partial automorphism, the systems $(\mathcal{A},\delta)$ and $(\mathcal{B},\widetilde{\delta})$ are equal and the covariance algebra of $(\mathcal{A},\delta)$ is simply the partial crossed product. In particular, if α is a full homeomorphism, equivalently δ is an automorphism, then $C^*(\mathcal{A},\delta)$ is the classic crossed product. As we shall see, in the case α is surjective, equivalently δ is a monomorphism, $C^*(\mathcal{A},\delta)$ is the crossed product by a monomorphism considered for instance in [22, 24, 12, 1], cf. Remark 4.10.

Let $\sum_{k=-N}^{N} a_k u^k$ stands for the element a in $L = \{a \in l^1(\mathbb{Z},\mathcal{B}) : a(n) \in C_{\widetilde{\Delta}_n}(\widetilde{X})\}$ such that $a(k) = a_k$ for $|k| \leqslant N$, and $a(k) = 0$ otherwise. In view of the defined operations on L it is clear that u is a partial isometry, u^k is u to power k and $(u^k)^* = u^{-k}$, so this notation should not cause any confusion. Using the natural injection $\mathcal{B} \ni a \mapsto au^0 \in L$ we identify $\mathcal{B} = C(\widetilde{X})$ with the subalgebra of $C^*(\mathcal{A},\delta)$, see [10, Corollary 3.10]. Recalling the identification from Definition 3.6 we have

$$\mathcal{A} \subset \mathcal{B} \subset C^*(\mathcal{A},\delta) = \mathcal{B} \rtimes_\theta \mathbb{Z}.$$

Example 4.3. The covariance algebra of the dynamical system considered in Examples 2.16 and 3.9 is the Toeplitz algebra. Indeed, the coefficient algebra $\mathcal{B}$ consists of convergent and $\widetilde{\delta}$ is a forward shift so the partial crossed product $\mathcal{B}\rtimes_{\widetilde{\delta}}\mathbb{Z}$, cf. [10], is the Toeplitz algebra.

Example 4.4. Let us go back again to Example 2.7 (see also Example 3.16). It is immediate to see that $C^*(X,\alpha)=C^*(X',\alpha')$ and invoking [10], or [20, Example 2.5] we can identify this algebra with $M_3\oplus M_3$ where M_3 is the algebra of complex matrices 3×3. If we set $\mathcal{A}=C(X)$ and $\mathcal{A}'=C(X')$ then due to the above remark we note that $\mathcal{A}$ and $\mathcal{A}'$ consist of the matrices of the form

$$\begin{pmatrix} a_{x_0} & 0 & 0\\ 0 & a_{x_1} & 0\\ 0 & 0 & a_{x_2}\end{pmatrix}\oplus\begin{pmatrix} a_{x_0} & 0 & 0\\ 0 & a_{x_1} & 0\\ 0 & 0 & a_{y_2}\end{pmatrix},\qquad \text{and}\qquad \begin{pmatrix} a_{x_0} & 0 & 0\\ 0 & a_{x_1} & 0\\ 0 & 0 & a_{x_2}\end{pmatrix}\oplus\begin{pmatrix} a_{x_0} & 0 & 0\\ 0 & a_{y_1} & 0\\ 0 & 0 & a_{y_2}\end{pmatrix}$$

respectively. The dynamics on $\mathcal{A}$ and $\mathcal{A}'$ are implemented by the partial isometry $U := \begin{pmatrix} 0\,0\,0\\ 1\,0\,0\\ 0\,1\,0\end{pmatrix}\oplus\begin{pmatrix} 0\,0\,0\\ 1\,0\,0\\ 0\,1\,0\end{pmatrix}$. The coefficient algebras $\mathcal{B}_0=\{\bigcup_{n=0}^{\infty}U^{*n}(\mathcal{A})U^n\}$ and $\mathcal{B}_0'=\{\bigcup_{n=0}^{\infty}U^{*n}(\mathcal{A}')U^n\}$ equal with the algebra of diagonal matrices $\begin{pmatrix} a_{\widetilde{x}_0} & 0 & 0\\ 0 & a_{\widetilde{x}_1} & 0\\ 0 & 0 & a_{\widetilde{x}_2}\end{pmatrix}\oplus$ $\begin{pmatrix} a_{\widetilde{y}_0} & 0 & 0\\ 0 & a_{\widetilde{y}_1} & 0\\ 0 & 0 & a_{\widetilde{y}_2}\end{pmatrix}$, and it is straightforward that $M_3\oplus M_3=C^*(\mathcal{A},U)=C^*(\mathcal{A}',U)$ and $\mathcal{A}\subset\mathcal{A}'$. In fact $\mathcal{A}$ is the smallest C^*-subalgebra of $M_3\oplus M_3$ such that $U^*U\in\mathcal{A}$, $U(\cdot)U^*$ is an endomorphism of $\mathcal{A}$ and $C^*(\mathcal{A},U)=M_3\oplus M_3$.

Example 4.5. It is known, cf. [10, 20], that an arbitrary finite dimensional C^*-algebra can be expressed as a covariance algebra (partial crossed product) of a certain dynamical system. The foregoing example inspires us to present the smallest such system in the sense that space X has the least number of points. Let $\mathcal{A}=M_{n_1}\oplus ...\oplus M_{n_k}$ where $1\leqslant n_1\leqslant ...\leqslant n_k$ be a finite dimensional C^*-algebra and let us first assume that is there is no factor M_1 in the decomposition of $\mathcal{A}$, that is $n_1>1$. We set $X=\{x_1,x_2,...,x_{n_k-1},y_{n_1},y_{n_2},...,y_{n_k}\}$, so $|X|=n_k+k-1$, and $\alpha(x_m)=x_{m-1}$, for $m=2,...,n_k-1$; $\alpha(y_{n_m})=x_{n_m-1}$, for $m=1,...,k$,

$$\begin{tikzcd}
 & x_{n_k-1} \arrow[r] & \cdots \arrow[r] & x_{n_m-1} \arrow[r] & \cdots \arrow[r] & x_{n_1-1} \arrow[r] & \cdots \arrow[r] & x_1 \\
y_{n_k} \arrow[ur] & & y_{n_m} \arrow[ur] & & y_{n_1} \arrow[ur] & & &
\end{tikzcd}$$

It is clear that $C^*(X,\alpha) = \mathcal{A}$, see [20, Example 2.5]. In order to include algebras containing a number, say l, of one-dimensional factors one should simply add l points to the above diagram.

Example 4.6. Let a be the bilateral weighted shift on a separable Hilbert space H and let a have the closed range. We have the polar decomposition $a = U|a|$, where $|a|$ is a diagonal operator and U is the bilateral shift. If we denote by $\mathcal{A}$ the commutative C^*-algebra $C^*(1, \{U^n|a|U^{*n}\}_{n\in\mathbb{N}})$, then $\delta(\cdot) = U(\cdot)U^*$ is a unital injective endomorphism of $\mathcal{A}$ and hence the dynamical system (X,α) corresponding to $(\mathcal{A},\delta)$ is such that $\alpha : X \to X$ is onto. It is immediate that the coefficient algebra $\mathcal{B} = C(\widetilde{X})$ has the form $C^*(1, \{U^n|a|U^{*n}\}_{n\in\mathbb{Z}})$. Thus due to [23, Theorem 2.2.1], $C^*(a) = \mathcal{B} \rtimes_{\widetilde{\delta}} \mathbb{Z}$ and so

$$C^*(a) = C^*(X,\alpha).$$

Following [23] we present now the canonical form of (X,α). Let Y denote the spectrum of $|a|$ and let $T : X \to \prod_{n=0}^{\infty} Y$ be defined by $T(x) = (x(|a|), x(\delta(|a|)), ..., x(\delta^n(|a|)), ...)$. Then similarly to [23] we infer that T is a homeomorphism of X onto $T(X)$, where $T(X)$ is given the topology induced by the product topology on $\prod_{n=0}^{\infty} Y$, and under T, α becomes a shift on the product space $T(X)$.

4.2 Universality of $C^*(X,\alpha)$

Now we are in position to prove the main result of this section which justifies the anticipating Definition 4.1. We shall base the proof on the results from the previous section and some known facts concerning the partial crossed product [10]. We adopt the commonly used notation $U^{-n} = U^{*n}$ where U is a partial isometry and $n \in \mathbb{N}$.

Theorem 4.7. Let $(\pi, U, H) \in \mathrm{CovFaithRep}\,(\mathcal{A},\delta)$ or $(\pi, U, H) \in \mathrm{CovRep}\,(\mathcal{A},\delta)$ in the case δ is a monomorphism. Then the formula

$$(\overline{\pi} \times U)\Big(\sum_{n=-N}^{N} a(n)u^n\Big) = \sum_{n=-N}^{N} \Big(\sum_{k=0}^{\infty} U^{*k}\pi(a_k^{(n)})U^k\Big)U^n \tag{30}$$

where $a(n) = [a_0^{(n)}, a_1^{(n)}, ..., a_k^{(n)}, ..] \in \mathcal{B}_0$, establishes an epimorphism of the covariance algebra $C^*(\mathcal{A},\delta)$ onto the C^*-algebra $C^*(\pi(\mathcal{A}),U)$ generated by $\pi(\mathcal{A})$ and U.

Proof. In both cases, $(\pi, U, H) \in \mathrm{CovFaithRep}\,(\mathcal{A},\delta)$ or $(\pi, U, H) \in \mathrm{CovRep}\,(\mathcal{A},\delta)$ and δ is injective, (π, U, H) extends to the covariant representation $(\overline{\pi}, U, H)$ of $(\mathcal{B},\widetilde{\delta})$, see Theorems 3.10, 3.14. Since $C^*(\pi(\mathcal{A}),U) = \mathcal{B} \rtimes_{\widetilde{\delta}} \mathbb{Z}$ we obtain, by [10, Proposition 5.5], that

$$(\overline{\pi} \times U)(\sum_{n=-N}^{N} a(n)u^n) = \sum_{n=-N}^{N} \overline{\pi}(a(n))U^n.$$

establishes the representation of $C^*(\mathcal{A}, \delta)$, and by Theorems 3.10 and 3.14, $(\overline{\pi} \times U)$ is in fact given by (30). □

Due to Definition 3.1, Corollary 3.13, and the preceding Theorem 4.7 we can alternatively define the covariance algebra to be the universal unital C^*-algebra generated by a copy of $\mathcal{A}$ and a partial isometry u subject to relations

$$u^*u \in \mathcal{A}, \qquad \delta(a) = uau^*, \qquad a \in \mathcal{A},$$

see also Proposition 1.6. In particular the above relations imply that $uu^* = P_{\Delta_1}$ and $u^*u = P_{\Delta_{-1}}$. Thus δ is injective iff u is isometry, and δ is an automorphism iff u is unitary.

Theorem 4.8. Let σ be a representation of $C^*(\mathcal{A}, \delta)$ on a Hilbert space H. Let π denote the restriction of σ onto $\mathcal{A}$ and let $U = \sigma(u)$. Then $(\pi, U, H) \in \mathrm{CovRep}\,(\mathcal{A}, \delta)$.

Proof. By [10, Theorem 5.6] we have $(\overline{\pi}, U, H) \in \mathrm{CovRep}\,(\mathcal{B}, \widetilde{\delta})$ where $\overline{\pi}$ is σ restricted to $\mathcal{B}$. Hence by Theorem 3.11 we get $(\pi, U, H) \in \mathrm{CovRep}\,(\mathcal{A}, \delta)$. □

Corollary 4.9. If δ is a monomorphism then the correspondance $(\pi, U, H) \longleftrightarrow (\overline{\pi} \times U)$, cf. Theorem 4.7, is a bijection between $\mathrm{CovRep}\,(\mathcal{A}, \delta)$ and the set of all representations of $C^*(\mathcal{A}, \delta)$.

Proof. In virtue of Theorem 4.7 the mapping $(\pi, U, H) \longleftrightarrow (\overline{\pi} \times U)$ is a well defined injection and by Theorem 4.8 it is also a surjection. □

Remark 4.10. Corollary 4.9 can be considered as a special case of Theorem 2.3 from the paper [22] where (twisted) crossed products by injective endomorphisms were investigated. However, our approach is slightly different. Oversimplifying; Murphy defines the algebra $C^*(\mathcal{A}, \delta)$ as pZp where Z is the full crossed product of direct limit $B = \varinjlim \mathcal{A}$ and p is a certain projection from B, whereas we include the projection p in the direct limit $\mathcal{B} = \varinjlim \mathcal{A}_n \subset B$, see Proposition 3.7, and hence take the partial crossed product.

5 Invariant subsets and the topological freeness of partial mappings

The present section is devoted to the generalization of two important notions of the theory of crossed products. We start with a definition of α-invariant sets. With help of this notion we will describe (in the next section) the ideal structure of the covariance algebra. Next we introduce a definition of topological freeness - a property which is a powerful instrument when used to construct faithful representations of covariance algebra. Thanks to that in Section 6 we prove a version of the Isomorphism Theorem.

5.1 Definition of the invariance under α and $\widetilde{\alpha}$ and their interrelationship

The Definition 5.1 to follow might look strange at first however the author's impression is that among the others this one is the most natural generalization of that from [13, Definition 2.7] for the case considered here. One may treat α-invariance as the invariance under the partial action of $\mathbb{N}$ on X.

Definition 5.1. Let α be a partial mapping of X. A subset V of X is said to be *invariant* under the partial mapping α, or shorter α-*invariant*, if

$$\alpha^n(V \cap \Delta_n) = V \cap \Delta_{-n}, \qquad n = 0, 1, 2.... \tag{31}$$

When α is injective then we have another mapping $\alpha^{-1} : \Delta_{-1} \to X$ and life is a bit easier.

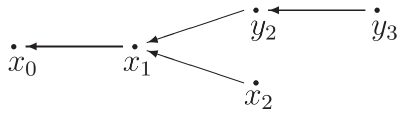
Proposition 5.2. Let α be an injective partial mapping and let $V \subset X$. Then V is α-invariant if and only if one of the following conditions holds
i) V is α^{-1}-invariant
ii) for each $n = 0, \pm 1, \pm 2, ...,$ we have $\alpha^n(V \cap \Delta_n) \subset V$
iii) $\alpha(V \cap \Delta_1) \subset V \quad$ and $\quad \alpha^{-1}(V \cap \Delta_{-1}) \subset V,$
iv) $\alpha(V \cap \Delta_1) = V \cap \Delta_{-1}.$

Proof. The equivalence of invariance of V under α and α^{-1} is straightforward, and so are implications i)$\Rightarrow$ ii)$\Rightarrow$ iii).
To prove iii) $\Rightarrow$ iv) let us observe that since $\alpha(V \cap \Delta_1) \subset \Delta_{-1}$ and $\alpha^{-1}(V \cap \Delta_{-1}) \subset \Delta_1$ we get $\alpha(V \cap \Delta_1) \subset V \cap \Delta_{-1}$ and $\alpha^{-1}(V \cap \Delta_{-1}) \subset V \cap \Delta_1$. The latter relation implies that $V \cap \Delta_{-1} = \alpha(\alpha^{-1}(V \cap \Delta_{-1})) \subset \alpha(V \cap \Delta_1)$ and so $\alpha(V \cap \Delta_1) = V \cap \Delta_{-1}$.
The only thing left to be shown is that iv) implies α-invariance of V. We prove this by induction. Let us assume that $\alpha^k(V \cap \Delta_k) = V \cap \Delta_{-k}$, for $k = 0, 1, ..., n-1$. By injectivity it is equivalent to $\alpha^{-k}(V \cap \Delta_{-k}) = V \cap \Delta_k$, for $k = 0, 1, ..., n-1$. As $\Delta_n \subset \Delta_{n-1}$ and $\Delta_{-n} \subset \Delta_{-(n-1)}$ we have $\alpha^n(V \cap \Delta_n) \subset V \cap \Delta_{-n}$ and $\alpha^{-n}(V \cap \Delta_{-n}) \subset V \cap \Delta_n$. Applying α^n to the latter relation we get $V \cap \Delta_{-n} \subset \alpha^n(V \cap \Delta_n)$ and hence $\alpha^n(V \cap \Delta_n) = V \cap \Delta_{-n}$.
□

Item ii) tells us that Definition 5.1 extends [13, Definition 2.7] in the case of a single partial mapping. Let us note that if $\Delta_1 \neq X$ and α is not injective, then none of items ii)-iv) is equivalent to (31) and therefore none of them could be used as a definition of α-invariance.

Example 5.3. Indeed, let $X = \{x_0, x_1, x_2, y_2, y_3\}$ and let α be defined by the relations

$\alpha(y_3) = y_2$, $\alpha(y_2) = \alpha(x_2) = x_1$ and $\alpha(x_1) = x_0$:



Then the set $V_1 = \{x_0, x_1, x_2\}$ fulfills item iv) but it is not invariant under α in the sense of Definition 5.1. Whereas the set $V_2 = \{x_0, x_1, y_2, y_3\}$ is α-invariant but it does not fulfill items ii) and iii).

We shall show that in general there are less α-invariant sets in X than $\widetilde{\alpha}$-invariant sets in $\widetilde{X}$, cf. Theorem 5.5 and a remark below, but fortunately there is a natural one-to-one correspondence between closed sets invariant under α and closed sets invariant under $\widetilde{\alpha}$, see Theorem 5.7. We start with a lemma.

Lemma 5.4. Let α be a partial mapping of X and let $U \subset X$ be invariant under α. Then we have

$$\alpha^k(\Delta_{n+k} \cap U) = U \cap \Delta_n \cap \Delta_{-k}, \qquad n, k = 0, 1, 2, \ldots .$$

Proof. As $\alpha^k(U \cap \Delta_k) = U \cap \Delta_{-k}$ and $U \cap \Delta_{n+k} \subset U \cap \Delta_k$ we have $\alpha^k(\Delta_{n+k} \cap U) \subset U \cap \Delta_n \cap \Delta_{-k}$, for $k, n \in \mathbb{N}$. On the other hand, for every $x \in U \cap \Delta_n \cap \Delta_{-k}$ there exists $y \in U \cap \Delta_k$ such that $\alpha^k(y) = x$. Since $x \in \Delta_n$ we we have $y \in U \cap \Delta_{n+k}$ and hence $U \cap \Delta_n \cap \Delta_{-k} \subset \alpha^k(\Delta_{n+k} \cap U)$. $\square$

Theorem 5.5. Let (α, X) be a partial dynamical system and let $(\widetilde{\alpha}, \widetilde{X})$ be its reversible extension. Let $\Phi : \widetilde{X} \to X$ be the projection defined by (17). Then the map

$$\widetilde{X} \supset \widetilde{U} \longrightarrow U = \Phi(\widetilde{U}) \subset X \tag{32}$$

is a surjection from the family of $\widetilde{\alpha}$-invariant subsets of $\widetilde{X}$ onto the family of α-invariant subsets of X. Furthermore if $\{U_n\}_{n\in\mathbb{N}}$ are the sections of $\widetilde{U}$ (see Definition 2.13)), then $\widetilde{U}$ is $\widetilde{\alpha}$-invariant if and only if U_0 is α-invariant and

$$U_n = U_0 \cap \Delta_n, \qquad n = 0, 1, 2, \ldots .$$

Proof. Let $\widetilde{U}$ be $\widetilde{\alpha}$-invariant and let $U = \Phi(\widetilde{U})$. Then by (19) and $\widetilde{\alpha}$-invariance of $\widetilde{U}$, for each $n \in \mathbb{N}$, we have

$$\alpha^n(U \cap \Delta_n) = \Phi(\widetilde{\alpha}^n(\widetilde{U} \cap \widetilde{\Delta}_n)) = \Phi(\widetilde{U} \cap \widetilde{\Delta}_{-n}) = U \cap \Delta_{-n}$$

and hence U is invariant under α and the mapping (32) is well defined. Moreover, if U_n, $n \in \mathbb{N}$, are the sections of $\widetilde{U}$, then by invariance of $\widetilde{U}$ under $\widetilde{\alpha}^{-1}$ (see Proposition 5.2) we get

$$U_n = \Phi(\widetilde{\alpha}^{-n}(\widetilde{U} \cap \widetilde{\Delta}_{-n})) = \Phi(\widetilde{U} \cap \widetilde{\Delta}_n) = U \cap \Delta_n$$

where $U = U_0$ is α-invariant.

Now, we show that the mapping (32) is surjective. Let U be any nonempty set invariant under α and let us consider $\widetilde{U}$ of the form

$$\widetilde{U} := \Big(U \times (U \cap \Delta_1 \cup \{0\}) \times ... \times (U \cap \Delta_n \cup \{0\}) \times ...\Big) \cap \widetilde{X}.$$

What we need to prove is that $\Phi(\widetilde{U}) = U$ (note that in general $\widetilde{U}$ may occur to be empty). It is clear that $\Phi(\widetilde{U}) \subset U$. In order to prove that $U \subset \Phi(\widetilde{U})$ we fix an arbitrary point $x_0 \in U$ and suppose that there does not exist $\widetilde{x}$ in $\widetilde{U}$ such that $\Phi(\widetilde{x}) = x_0$. We will construct an infinite sequence $(x_0, x_1, x_2, ...)$ in $\widetilde{U}$ and thereby obtain a contradiction.
Indeed, we must have $x_0 \in U \cap \Delta_{-1}$, for otherwise we can take $\widetilde{x} = (x_0, 0, 0, ...) \in \widetilde{U}$. Hence by Lemma 5.4 there exists $x_1 \in U \cap \Delta_1$ such that $\alpha(x_1) = x_0$. Suppose now we have chosen $n-1$ points $x_1, ..., x_n$ such that $x_k \in U \cap \Delta_k$ and $\alpha(x_k) = x_{k-1}$ for $k = 1, ..., n$, then x_n must be in $U \cap \Delta_n \cap \Delta_{-1}$, for otherwise we can take $\widetilde{x} = (x_0, x_1, ..., x_n, 0, ...) \in \widetilde{U}$. Hence by Lemma 5.4 there exists $x_{n+1} \in U \cap \Delta_{n+1}$ such that $\alpha(x_{n+1}) = x_n$. This ensure us that there is a sequence $\widetilde{x} = (x_0, x_1, x_2, ...)$ such that $x_n \in U \cap \Delta_n$ and $\alpha(x_n) = x_{n-1}$, for all $n = 1, 2, ...$. Thus $\widetilde{x} \in \widetilde{U}$ and we arrive at the contradiction.
By virtue of item v) in Proposition 5.2 in order to prove the $\widetilde{\alpha}$-invariance of $\widetilde{U}$ it suffices to show that

$$\widetilde{\alpha}(\widetilde{U} \cap \widetilde{\Delta}_1) \subset \widetilde{U} \qquad \text{and} \qquad \widetilde{\alpha}^{-1}(\widetilde{U} \cap \widetilde{\Delta}_{-1}) \subset \widetilde{U}$$

but this follows immediately from the form of $\widetilde{U}$, $\widetilde{\alpha}$, $\widetilde{\alpha}^{-1}$ and from α-invariance of U. Thus according to the first part of the proof we conclude that, for each $n \in \mathbb{N}$, the n-section U_n of $\widetilde{U}$ is equal to $U \cap \Delta_n$. The proof is complete. □

Corollary 5.6. If α is injective on the inverse image of $\Delta_{-\infty} = \bigcap_{n\in\mathbb{N}} \Delta_{-n}$ (for $x \in \Delta_{-\infty}$ we have $|\alpha^{-1}(x)| = 1$) then the mapping (32) from the previous theorem is a bijection.

Proof. It suffices to apply Proposition 2.14. □

Under the hypotheses of Theorem 5.5, surjection considered there might not be a bijection. For instance in Example 2.16 we have three α-invariant sets: $X, \{x_0\}, \emptyset$, and four $\widetilde{\alpha}$-invariant sets: $\overline{\mathbb{N}}, \mathbb{N}, \infty, \emptyset$. However the mapping (32) is always bijective when restricted to closed invariant sets.

Theorem 5.7. The mapping (32) is a bijection from the family of $\widetilde{\alpha}$-invariant closed sets onto the family of α-invariant closed sets.

Proof. By Theorems 5.5 and 2.15, for every $\widetilde{\alpha}$-invariant closed subset $\widetilde{U}$ such that $\Phi(\widetilde{U}) = U$ we have

$$\widetilde{U} = \Big(U \times (U \cap \Delta_1 \cup \{0\}) \times ... \times (U \cap \Delta_n \cup \{0\}) \times ...\Big) \cap \widetilde{X}, \tag{33}$$

that means $\widetilde{U}$ is uniquely determined by U. Since $\widetilde{X}$ and X are compact, and $\Phi : \widetilde{X} \to X$ is continuous the set U is closed. Hence Φ maps injectively the family of closed $\widetilde{\alpha}$-invariant sets into the family of closed α-invariant sets.

On the other hand, if U is closed and α-invariant then by definition of the product topology, the set $\widetilde{U}$ given by (33) is also closed and according to Theorem 5.5, $\Phi(\widetilde{U}) = U$. Thus the proof is complete. □

The next important notion which we shall need to obtain a simplicity criteria for covariance algebra (see Corollary 6.6) is the notion of minimality, cf. [13].

Definition 5.8. A partial continuous mapping α (or a partial dynamical system (X, α)) is said to be *minimal* if there are no α-invariant closed subsets of X other than $\emptyset$ and X.

Proposition 5.9. A partial dynamical system (X, α) is minimal if and only if its reversible extension $(\widetilde{X}, \widetilde{\alpha})$ is minimal.

Proof. An easy consequence of Theorem 5.7. □

When α is injective, the binary operations "$\cup$" and "$\cap$", or equivalently partial order relation " $\subset$ ", define the lattice structure on the family of α-invariant sets, see [13]. The situation changes when α is not injective. Of course " $\subset$ " is still a partial order relation which determines the lattice structure, but it may happen that the intersection of two α-invariant sets is no longer α-invariant.

Example 5.10. Let (X, α) and (X', α') be dynamical systems from Example 2.7. It is easy to verify that there are four sets invariant under α: X, $V_1 = \{x_0, x_1, x_2\}$, $V_2 = \{x_0, x_1, y_2\}$ and $\emptyset$; and four sets invariant under α': X', $V_1' = \{x_0', x_1', x_2'\}$, $V_2' = \{x_0', y_1', y_2'\}$, $\emptyset$. Hence neither $V_1 \cap V_2$ nor $V_1' \cap V_2'$ is invariant. However there are four invariant subsets: $\widetilde{X}$, $\widetilde{V}_1 = \{\widetilde{x}_0, \widetilde{x}_1, \widetilde{x}_2\}$, $\widetilde{V}_2 = \{\widetilde{y}_0, \widetilde{y}_1, \widetilde{y}_2\}$, $\emptyset$ on the reversible extension level $((\widetilde{X}, \widetilde{\alpha}) = (\widetilde{X}', \widetilde{\alpha}'))$ and $\widetilde{V}_1 \cap \widetilde{V}_2 = \emptyset$ is invariant of course.

Definition 5.11. We denote by $clos_\alpha(X)$ the lattice of α-invariant closed subsets of X where the lattice structure is defined by the partial order relation " $\subset$ ".

According to Theorem 5.7, Φ determines the lattice isomorphism $clos_{\widetilde{\alpha}}(\widetilde{X}) \cong clos_\alpha(X)$.

5.2 Topological freeness

Recall now that a partial action of a group G on a topological space X is said to be topologically free if the set of fixed points F_t, for each partial homeomorphism α_t with $t \neq e$, has an empty interior [13]. In view of that, the next definition constitutes a generalization of topological freeness notion to the class of systems where dynamics are implemented by one, not necessarily injective, partial mapping.

Definition 5.12. Let $\alpha : \Delta \to X$ be a continuous partial mapping of Hausdorff's topological space X. For each $n > 0$, we set $F_n = \{x \in \Delta_n : \alpha^n(x) = x\}$. It is said that the action of α (or briefly α) is *topologically free*, if every open nonempty subset $U \subset F_n$ possess 'an exit', that is there exists a point $x \in U$ such that one of the equivalent conditions hold

i) for some $k = 1, 2, ..., n$ we have $|(\alpha^{-k}(x))| > 1$,
ii) for some $k = 1, 2, ..., n$ $\alpha^{-1}(\alpha^k(x)) \neq \{\alpha^{k-1}(x)\}$.

We supply now some characteristics of this topological freeness notion.

Proposition 5.13. The following conditions are equivalent

i) α is topologically free,

ii) for each $n > 0$ and every open nonempty subset $U \subset F_n$ there exist points $x \in U$, $y \in \Delta_1 \setminus F_n$ and a number $k = 1, 2, ..., n$, such that

$$\alpha(y) = \alpha^k(x).$$

iii) for each $n > 0$, the set

$$\{x \in \Delta_{n-1} : \alpha^{k-n}(x) = \{\alpha^k(x)\} \text{ for } \ k = 0, 1, ..., n-1\}$$

has an empty interior.

Proof. *i)* $\Rightarrow$ *ii)*. Let $U \subset F_n$ be an open nonempty set. Let $x \in U$ and $k = 1, ..., n$ be such that item ii) from Definition 5.12 holds. We take $y \in \alpha^{-1}(\alpha^k(x))$ such that $y \neq \alpha^{k-1}(x)$. Then $\alpha(y) = \alpha^k(x)$ and since $\alpha^n(y) = \alpha^{k-1}(x)$ we have $y \notin F_n$.
ii) $\Rightarrow$ *iii)*. Suppose that for some $n > 0$ there exists a nonempty open subset U of $\{x \in \Delta_{n-1} : \alpha^{k-n}(x) = \{\alpha^k(x)\} \text{ for } \ k = 0, 1, ..., n-1\}$. It is clear that $U \subset F_n$ and hence for some $k_0 = 1, 2, ..., n$, there exists $y \in \Delta_1 \setminus F_n$ such that $\alpha(y) = \alpha^{k_0}(x)$. Taking $k = k_0 - 1$ we obtain that $y \in \alpha^{k-n}(x) = \{\alpha^k(x)\}$ and thus we arrive at a contradiction since $\alpha^k(x) \in F_n$ and $y \notin F_n$.
iii) $\Rightarrow$ *i)*. Suppose that α is not topologically free. Then there exists an open nonempty set $U \subset F_n$ such that for all $x \in U$ and $k = 1, ..., n$, we have $|\alpha^{-k}(x)| = 1$. It is not hard to see that $U \subset \{x \in \Delta_{n-1} : \alpha^{k-n}(x) = \{\alpha^k(x)\} \text{ for } \ k = 0, 1, ..., n-1\}$ and thereby we arrive at the contradiction. □

The role similar to the one which topological freeness plays in the theory of crossed products is the role played in the theory of C^*-algebras associated with graphs by the condition that every circuit in a graph has an exit (see, for example, [5, 11]). The connection between these two properties is not only of theoretical character, see for instance [11, Proposition 12.2]). Taking this into account the following two simple examples might be of interest. Before that let us establish the indispensable notation.
Let $A = (A(i,j))_{i,j \in \{1,...,N\}}$ be the matrix with entries in $\{0, 1\}$. It can be regarded as an *adjacency matrix* of a directed graph $Gr(A)$: the vertices of $Gr(A)$ are numbers 1,...,N and edges are pairs (x, y) of vertices such that $A(x, y) = 1$. By a *path* in $Gr(A)$ we mean a sequence $(x_0, x_1, ...x_n)$ of vertices such that $A(x_k, x_{k+1}) = 1$ for all k. A *circuit*, or a

loop, is a finite path $(x_0, ..., x_n)$ such that $A(x_n, x_0) = 1$. Finally a circuit $(x_0, x_1, ..., x_n)$ is said to have an *exit* if, for some k, there exists $y \in \{1, ..., N\}$ with $A(x_k, y) = 1$ and $y \neq x_{k+1\,(mod\ n)}$.

Example 5.14. Let (X, α) be a partial dynamical system such that $X = \{1, ..., N\}$ is finite. If we define A by the relation: $A(x, y) = 1$ iff $\alpha(y) = x$, then α is topologically free if and only if every loop in $Gr(A)$ has an exit. This is an easy consequence of the fact that $Gr(A)$ is the graph of the partial mapping α with reversed edges.

We shall say that a circuit $(x_0, x_1, ..., x_n)$ has an *entry* if, for some k, there exists $y \in \{1, ..., N\}$ with $A(y, x_k) = 1$ and $y \neq x_{k-1}\,(mod\ n)$.

Example 5.15. Let (X_A, σ_A) be a dynamical system where σ_A is a one-sided Markov subshift associated with a matrix A, see page 733. One-sided subshift σ_A acts topologically free if and only if every circuit in $Gr(A)$ has an exit or an entry. Indeed, if there exists a loop $(y_0, ..., y_n)$ in $Gr(A)$ which has no exit and no entry then $U = \{(x_k)_{k\in\mathbb{N}} \in X_A : x_0 = y_0\} = \{(y_0, y_1, ..., y_n, y_0, ...)\}$ is an open singleton, $U \subset F_{n+1}$ and U has no 'exit' in the sense of Definition 5.12, thereby σ_A is not topologically free.
On the other hand, if every loop in $Gr(A)$ has an exit or an entry, and if $x = (x_1, ..., x_n, x_1, ...)$ is an element of an open subset $U \subset F_n$ for some $n > 0$, then the loop $(x_1, ..., x_n)$ must have an entry, as it clearly has no exit. Hence $(x_1, ..., x_n)$ we have $|(\sigma_A^{-k}(x))| > 1$, for some $k = 1, 2, ..., n$, and thus σ_A is topologically free.

We end this section with the result which, in a sense, justifies Definition 5.12, and is the main tool used to prove the Isomorphism Theorem.

Theorem 5.16. Let $F_n = \{x \in \Delta_n : \alpha^n(x) = x\}$ and $\widetilde{F}_n = \{\widetilde{x} \in \widetilde{\Delta}_n : \widetilde{\alpha}^n(\widetilde{x}) = \widetilde{x}\}$, $n \in \mathbb{N} \setminus \{0\}$. We have

$$\widetilde{F}_n = \{(x_0, x_1, ...) \in \widetilde{X} : x_k \in F_n,\ k \in \mathbb{N}\}, \qquad n = 1, 2, ..., \tag{34}$$

and α is topologically free if and only if $\widetilde{\alpha}$ is topologically free.

Proof. Throughout the proof we fix an $n > 0$. It is clear that F_n and $\widetilde{F}_n$ are invariant under α and $\widetilde{\alpha}$ respectively (see Definition 5.1), and that $\Phi(\widetilde{F}_n) = F_n$. By virtue of Theorem 5.5 we have

$$\widetilde{F}_n = \Big(F_n \times (F_n \cap \Delta_1 \cup \{0\}) \times ... \times (F_n \cap \Delta_k \cup \{0\}) \times ...\Big) \cap \widetilde{X}.$$

But, since $F_n \subset \bigcap_{k\in\mathbb{Z}} \Delta_k$ we obtain $\widetilde{F}_n = (F_n \times F_n \times ... \times F_n \times ...) \cap \widetilde{X}$ and hence (34) holds.
Now suppose that α is topologically free and on the contrary that there exists an open nonempty subset $\widetilde{U} \subset \widetilde{F}_n$. Without loss of generality, we can assume that it has the form

$$\widetilde{U} = \Big(U_0 \times U_1 ... \times U_m \times \Delta_{m+1} \cup \{0\} \times \Delta_{m+2} \cup \{0\} \times ...\Big) \cap \widetilde{X}$$

where $U_0, U_1, ..., U_m$ are open subsets of X, and as $\widetilde{U} \subset \widetilde{F}_n = (F_n \times F_n \times ... \times F_n \times ...) \cap \widetilde{X}$, they are in fact subsets of F_n, and it is readily checked that

$$\widetilde{U} = \Big(F_n \times F_n ... \times \bigcap_{k=0}^{m} \alpha^{-k}(U_{m-k}) \times \Delta_{m+1} \cup \{0\} \times \Delta_{m+2} \cup \{0\} \times ...\Big) \cap \widetilde{X}.$$

The set $U := \bigcap_{k=0}^{m} \alpha^{-k}(U_{m-k})$ is an open and nonempty subset of F_n. Hence, due to the topological freeness of α there exists $y \notin F_n$ and $k = 1, ..., n$, such that $\alpha(y) = \alpha^k(x)$ for some $x \in U$. Taking any element $\widetilde{x} = (x_0, x_1, ...) \in \widetilde{X}$ such that $x_m := x$, $x_{m+i} := \alpha^{n-i}(x)$ for $i = 1, ..., n-k$, and $x_{m+n-k+1} = y$ we arrive at the contradiction, because $\widetilde{x} \in \widetilde{U}$ and $\widetilde{x} \notin \widetilde{F}_n$.

Finally suppose that α is not topologically free. Then there exists an open nonempty subset $U \subset F_n$ such that, for all $x \in U$, $|\alpha^{-k}(x)| = 1$ and so

$$\widetilde{U} = \{(x, \alpha^{n-1}(x), \alpha^{n-2}(x), ..., \alpha^1(x), x, \alpha^{n-1}(x), ...) \in \widetilde{X} : x \in U\} = (U \times (X \cup \{0\}) \times ...) \cap \widetilde{X}$$

is an open nonempty subset of $\widetilde{F}_n$. Hence $\widetilde{\alpha}$ is not topologically free and the proof is complete. □

6 Ideal structure of covariance algebra and the Isomorphism Theorem

It is well-known that every closed ideal of $\mathcal{A} = C(X)$ is of the form $C_U(X)$ where $U \subset X$ is open, and therefore we have an order preserving bijection between open sets and ideals. The Theorem 3.5 from [13] can be regarded as a generalization of this fact; it says that, under some assumptions, there exists a lattice isomorphism between open invariant sets and ideals of the partial crossed product. In this section we shall prove the new useful variant of this theorem. The novelty is that in our approach (cf. Theorem 5.7) it is more natural to investigate a correspondence between ideals of the covariance algebra and closed invariant sets.

After that we shall prove the main result of this paper, a version of the Isomorphism Theorem where the main achievement is that we do not assume any kind of reversibility of an action on a spectrum of a C^*-dynamical system.

6.1 Lattice isomorphism of closed α-invariant sets onto ideals of $C^*(X, \alpha)$

Let us start with the proposition which is an attempt of describing the concept of invariance on the algebraic level, cf. [13, Definition 2.7]. For that purpose we will abuse notation concerning endomorphism δ and denote by δ^n, $n \in \mathbb{N}$, morphisms $\delta^n : C(\Delta_{-n}) \to C(\Delta_n)$ of composition with $\alpha^n : \Delta_n \to \Delta_{-n}$. We believe that this notation does not cause confusion, although we stress that set Δ_{-n} does not have to be open and hence we can not identify $C(\Delta_{-n})$ with a subset of $C(X)$. For instance, it may happen that Δ_{-n} is not empty but have an empty interior, and then $C_{\Delta_{-n}}(X)$ is empty while $C(\Delta_{-n})$ is not. We will also abuse notation concerning subsets and write $B\cap C(\Delta_{-n})$ for $\{a \text{ restricted to } \Delta_{-n} : a \in B\}$ where $B \subset C(X)$.

Proposition 6.1. Let V be a closed subset of X and let $I = C_{X\setminus V}(X)$ be the corresponding ideal. Then for $n \in \mathbb{N}$ we have

i) $\alpha^n(V \cap \Delta_n) \subset V \cap \Delta_{-n}$ iff $a \in I \cap C(\Delta_{-n}) \Longrightarrow \delta^n(a) \in I \cap C(\Delta_n)$,

ii) $\alpha^n(V \cap \Delta_n) \supset V \cap \Delta_{-n}$ iff $\delta^n(a) \in I \Longrightarrow a \in I$, for all $a \in C(\Delta_{-n})$.

Hence V is α-invariant ($V \in clos_\alpha(X)$) if and only if

$$\forall_{n\in\mathbb{N}}\ \forall_{a\in C(\Delta_{-n})}\ a \in I \cap C(\Delta_{-n}) \Longleftrightarrow \delta^n(a) \in I \cap C(\Delta_n). \tag{35}$$

Proof. i). Let $\alpha^n(V\cap\Delta_n) \subset V\cap\Delta_{-n}$ and let $a \in I\cap C(\Delta_{-n})$ be fixed. Then for $x \in V\cap\Delta_n$ we have $\alpha^n(x) \in V \cap \Delta_{-n}$, whence $\delta^n(a) = a(\alpha^n(x)) = 0$ and $\delta^n(a) \in I \cap C(\Delta_n)$.
Now suppose $\alpha^n(V \cap \Delta_n) \nsubseteq V \cap \Delta_{-n}$. Then there exists $x_0 \in V \cap \Delta_n$ such that $\alpha^n(x_0) \notin V \cap \Delta_{-n}$. As $\alpha^n(x_0) \in \Delta_{-n}$ and V is closed, by Urysohn's lemma, there is a function $a_0 \in C(X)$ such that $a_0(\alpha^n(x_0)) = 1$ and $a_0(x) = 0$ for all $x \in V$. Thus taking a to be the restriction of a_0 to Δ_{-n} we obtain $a \in I \cap C(\Delta_{-n})$ but $\delta^n(a)(x_0) = 1$, whence $\delta^n(a) \notin I \cap C(\Delta_n)$.
ii). Let $\alpha^n(V \cap \Delta_n) \supset V \cap \Delta_{-n}$ and let $a \in C(\Delta_{-n})$ be such that $\delta^n(a) \in I \cap C(\Delta_n)$. Suppose on the contrary that $a \notin I \cap C(\Delta_{-n})$. Then $a(y_0) \neq 0$ for some $y_0 \in V \cap \Delta_{-n}$. Taking $x_0 \in V \cap \Delta_n$ such that $y_0 = \alpha^n(x_0)$ we arrive at the contradiction with $\delta^n(a) \in I \cap C(\Delta_n)$ because $\delta^n(a)(x_0) = a(y_0) \neq 0$.
If $\alpha^n(V \cap \Delta_n) \nsupseteq V \cap \Delta_{-n}$, then there exists $x_0 \in V \cap \Delta_{-n} \setminus \alpha^n(V \cap \Delta_n)$. Similarly as in the proof of item i), using Urysohn's lemma we can take $a_0 \in C(X)$ such that $a_0(x_0) = 1$ and $a_0|_{\alpha^n(V\cap\Delta_n)} \equiv 0$. Hence putting $a = a_0|_{\Delta_{-1}}$ we have $a \notin I \cap C(\Delta_{-n})$ and $\delta^n(a) \in I \cap C(\Delta_n)$.
In view of i) and ii), α-invariance of V is evidently equivalent to the condition (35). □

Definition 6.2. If I is a closed ideal of $\mathcal{A}$ satisfying (35) then we say that I is *invariant* under the endomorphism δ, or briefly δ-*invariant.*

In virtue of Proposition 6.1 it is clear that I is a δ-invariant ideal iff $I = C_{X\setminus V}(X)$ where V is a closed α-invariant set. Thus, using Theorem 5.7 one can obtain a correspondence between the invariant ideals of $\mathcal{A}$ and invariant ideals of $\mathcal{B}$. To this end we denote by $\langle I\rangle_{\mathcal{B},\widetilde{\delta}}$ the smallest $\widetilde{\delta}$-invariant ideal of $\mathcal{B}$ containing I.

Proposition 6.3. Let $I = C_{X\setminus V}(X)$ be a δ-invariant ideal of $\mathcal{A}$ and let $\widetilde{V} \in clos_{\widetilde{\alpha}}(\widetilde{X})$ be such that $\Phi(\widetilde{V}) = V$. Then

$$\langle C_{X\setminus V}(X)\rangle_{\mathcal{B},\widetilde{\delta}} = C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X}),$$

and the mapping $I \longmapsto \langle I\rangle_{\mathcal{B},\widetilde{\delta}}$ establishes an order preserving bijection between the family of δ-invariant ideals of $\mathcal{A}$ and $\widetilde{\delta}$-invariant ideals of $\mathcal{B}$. Moreover, the inverse of the mentioned bijection has the form $\widetilde{I} \longmapsto \widetilde{I} \cap \mathcal{A}$.

Proof. In order to prove the first part of the statement we show that the support

$$S = \bigcup_{f \in \langle C_{X\setminus V}(X)\rangle_{\mathcal{B},\widetilde{\delta}}} \{\widetilde{x} \in \widetilde{X} : f(\widetilde{x}) \neq 0\}$$

of the ideal $\langle C_{X\setminus V}(X)\rangle_{\mathcal{B},\widetilde{\delta}}$ is equal to $\widetilde{X} \setminus \widetilde{V}$.
Let $a \in I = C_{X\setminus V}(X)$. We identify a with $[a, 0, ...] \in \mathcal{B}$ and since $x_0 \in V$ for any $\widetilde{x} = (x_0, ...) \in \widetilde{V}$, we note that $[a, 0, ...](\widetilde{x}) = a(x_0) = 0$, that is $a = [a, 0, ...] \in C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})$. As $C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})$ is $\widetilde{\delta}$-invariant, we get $\langle C_{X\setminus V}(X)\rangle_{\mathcal{B},\widetilde{\delta}} \subset C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})$, whence $S \subset \widetilde{X} \setminus \widetilde{V}$.
Now, let $\widetilde{x} = (x_0, ..., x_k, ...) \in \widetilde{X} \setminus \widetilde{V}$. The form of $\widetilde{V}$ (compare Theorem 5.5) implies that there exists $n \in \mathbb{N}$ such that $x_n \notin V$. According to Urysohn's lemma there exists $a \in C_{X\setminus V}(X)$ such that $a(x_n) = 1$. By invariance, all the elements $\widetilde{\delta}^k(a)$ and $\widetilde{\delta}_*^k(a)$ for $k \in \mathbb{N}$, belong to $\langle I\rangle_{\mathcal{B},\widetilde{\delta}}$. In particular $\widetilde{\delta}_*^n(a) = [0, ..., a\delta^n(1), 0, ...] \in \langle I\rangle_{\mathcal{B},\widetilde{\delta}}$ where $\widetilde{\delta}_*(a)(\widetilde{x}) = a(x_n) = 1 \neq 0$. Thus $\widetilde{x} \in S$ and we get $\widetilde{X} \setminus \widetilde{V} = S$.
In virtue of Theorem 5.7 the relation $\Phi(\widetilde{V}) = V$ establishes an order preserving bijection between $clos_{\widetilde{\alpha}}(\widetilde{X})$ and $clos_{\alpha}(X)$ hence the relation $\langle C_{X\setminus V}(X)\rangle_{\mathcal{B},\widetilde{\delta}} = C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})$ establishes such a bijection too. The inverse relation $C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X}) \cap \mathcal{A} = C_{X\setminus V}(X)$ is straightforward.□

Let us recall that we identify $\mathcal{B}$ with a subalgebra of the covariance algebra $C^*(\mathcal{A}, \delta)$. Therefore for any subset K of $\mathcal{B}$ we denote by $\langle K\rangle$ an ideal of $C^*(\mathcal{A}, \delta)$ generated by K. The next statement follows from the preceding proposition and Theorem 3.5 from [13].

Theorem 6.4. Let $(\mathcal{A}, \delta)$ be a C^*-dynamical system such that α has no periodic points. Then the map

$$V \longmapsto \langle C_{X\setminus V}(X)\rangle$$

is a lattice anti-isomorphism from $clos_{\alpha}(X)$ onto the lattice of ideals in $C^*(\mathcal{A}, \delta)$. Moreover, for $\widetilde{V} \in clos_{\widetilde{\alpha}}(\widetilde{X})$ such that $\Phi(\widetilde{V}) = V$ the following relations hold

$$\langle C_{X\setminus V}(X)\rangle = \langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle, \quad \langle C_{X\setminus V}(X)\rangle \cap \mathcal{B} = C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X}), \quad \langle C_{X\setminus V}(X)\rangle \cap \mathcal{A} = C_{X\setminus V}(X).$$

Proof. Since α has no periodic points neither does its reversible extension $\widetilde{\alpha}$. The covariance algebra $C^*(\mathcal{A}, \delta)$ is the partial crossed product $C(\widetilde{X}) \rtimes_{\widetilde{\delta}} \mathbb{Z}$ and $\mathbb{Z}$ is an amenable group. Hence $(C(\widetilde{X}), \widetilde{\delta})$ has the approximation property, see [13]. Thus in view of Theorem 3.5 from [13], the map $\widetilde{V} \longmapsto \langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle$ is a lattice anti-isomorphism from $clos_{\widetilde{\alpha}}(\widetilde{X})$ onto the lattice of ideals of $C^*(\mathcal{A}, \delta)$, and the inverse relation is $\langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle \cap C(\widetilde{X}) = C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})$.
Now, let $\widetilde{V} \in clos_{\widetilde{\alpha}}(\widetilde{X})$ and $V \in clos_{\alpha}(X)$ be such that $\Phi(\widetilde{V}) = V$. We show that $\langle C_{X\setminus V}(X)\rangle = \langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle$.
On one hand, by Proposition 6.3 we have $\langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle \cap \mathcal{A} = \langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle \cap C(\widetilde{X}) \cap \mathcal{A} = C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X}) \cap \mathcal{A} = C_{X\setminus V}(X)$ and hence $\langle C_{X\setminus V}(X)\rangle \subset \langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle$. On the other hand, $\langle C_{X\setminus V}(X)\rangle \cap C(\widetilde{X})$ is a $\widetilde{\delta}$-invariant ideal of $C(\widetilde{X})$ containing $C_{X\setminus V}(X)$, and so it also contains $\langle C_{X\setminus V}(X)\rangle_{\mathcal{B},\widetilde{\delta}} = C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})$. Hence $\langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle \subset \langle C_{X\setminus V}(X)\rangle$.

Concluding, we have the lattice isomorphism between sets $V \in clos_\alpha(X)$ and $\widetilde{V} \in clos_{\widetilde{\alpha}}(\widetilde{X})$, and the lattice anti-isomorphism between sets $\widetilde{V} \in clos_{\widetilde{\alpha}}(\widetilde{X})$ and ideals $\langle C_{X\setminus V}(X)\rangle = \langle C_{\widetilde{X}\setminus\widetilde{V}}(\widetilde{X})\rangle$, hence $V \longmapsto \langle C_{X\setminus V}(X)\rangle$ is anti-isomorphism. □

Example 6.5. In Example 4.5 the only α-invariant sets are $\{x_1, ..., x_{m-1}, y_m\}$, $m = 1, ..., k$, and their sums. The corresponding ideals are M_{n_m}, $m = 1, ..., k$, and their direct sums.

We automatically get a simplicity criteria for the covariance algebra. We say that (X, α) forms a cycle, if $X = \{x_0, ..., x_{n-1}\}$ is finite and $\alpha(x_k) = x_{k+1(mod\ n)}$, $k = 0, ...n-1$.

Corollary 6.6. Let α be minimal. If (X, α) does not form a cycle then $C^*(\mathcal{A}, \delta)$ is simple.

Proof. It suffices to observe that if α is minimal then α has no periodic points or (X, α) forms a cycle. Hence we can apply Theorem 6.4. □

Example 6.7. If (X, α) does form a cycle then there are infinitely many ideals in $C^*(\mathcal{A}, \delta)$. Indeed if we have $\mathcal{A} = \mathbb{C}^n$ and $\delta(x_1, ..., x_n) = (x_n, x_1, ..., x_{n-1})$, it is known that the partial crossed product $\mathbb{C}^n \rtimes_\delta \mathbb{Z}_n$ is isomorphic to the algebra M_n of complex matrices $n \times n$ and hence $C^*(\mathcal{A}, \delta) \hookrightarrow C(S^1) \otimes M_n = C(S^1, M_n)$.

6.2 The Isomorphism Theorem

The Isomorphism Theorem simply states that under some conditions epimorphism from Theorem 4.7 is in fact an isomorphism. We will prove here two statements of that kind, Theorems 6.9 and 6.11, in the literature however only the latter one is named the Isomorphism Theorem. A significant role in the proofs of both of these statements plays a certain inequality which ensures the existence of conditional expectation onto the coefficient algebra, and which appears in different versions in a number of sources concerning various crossed products. For references see [18, 19, 2, 3, 23], and for the greatest similarity with the following Definition 6.8 and Theorem 6.9 see [1, Theorem 1.2].

Definition 6.8. We say that a C^*-algebra $C^*(\mathcal{C}, U)$ generated by a C^*-algebra $\mathcal{C}$ and an element U possesses the *property* $(*)$ if the following inequality holds

$$\|\sum_{k=0}^{M} U^{*k}\pi(a_k^{(0)})U^k\| \le \|\sum_{n=-N}^{N}\Big(\sum_{k=0}^{M} U^{*k}\pi(a_k^{(n)})U^k\Big)U^n\| \qquad (*)$$

for any $a_k^{(n)} \in \mathcal{C}$ and $M, N \in \mathbb{N}$.

Theorem 6.9. Let $(\pi, U, H) \in \mathrm{CovFaithRep}\,(\mathcal{A}, \delta)$. Then formula (30) establishes an isomorphism between the covariance algebra $C^*(\mathcal{A},\delta)$ and the C^*-algebra $C^*(\pi(\mathcal{A}),U)$ if and only if $C^*(\pi(\mathcal{A}),U)$ possess the property $(*)$.

Proof. *Necessity.* It suffices to observe that $C^*(\mathcal{A},\delta) = C^*(\mathcal{A},u)$ possess the property $(*)$, and this follows immediately from the fact that partial crossed products satisfy the appropriate version of this property, see [19, Remark 2.1] and [20, Proposition 3.5].
Sufficiency. By Theorem 3.10, (π,U,H) extends to the covariant faithful representation $(\overline{\pi},U,H)$ of the coefficient C^*-algebra $\mathcal{B}$. This extended representation satisfies assumptions of [19, Theorem 3.1] and as $\mathbb{Z}$ is amenable $(\overline{\pi},U,H)$ give rise to the desired isomorphism, see also [19, Remark 3.2] □

Corollary 6.10. Let $v \in \mathcal{A}$ be a partial isometry such that $uu^* \leq v^*v,\, vv^*$ where u is the universal partial isometry in $C^*(\mathcal{A},\delta)$. Then the mapping

$$\Lambda_v(u) = vu, \qquad \Lambda_v(a) = a, \quad a \in \mathcal{A},$$

extends to an automorphism of $C^*(\mathcal{A},\delta)$. In particular, taking $v = \lambda 1, \lambda \in S^1$, we have the action Λ of the unit circle S^1 on $C^*(\mathcal{A},\delta)$ for which the fixed points set is the coefficient C^*-algebra $\mathcal{B}$.

Proof. By the above theorem $C^*(\mathcal{A},\delta) = C^*(\mathcal{A},u)$ possesses the property $(*)$. Clearly, the same is true for $C^*(\mathcal{A},vu)$. Since $uu^* \leq v^*v$ we have $u = v^*vu = v^*(vu) \in C^*(\mathcal{A},uv)$, whence $C^*(\mathcal{A},\delta) = C^*(\mathcal{A},uv)$, and furthermore $(vu)^*vu = u^*v^*vu = u^*u \in \mathcal{A}$. Since $uu^* \leq vv^*$ we have $(vu)a(vu)^* = uau^*vv^* = uau^*$, that is the element vu generates the same endomorphism of $\mathcal{A}$ as u, and hence applying the preceding theorem we conclude that Λ_v extends to an automorphism of $C^*(\mathcal{A},\delta)$. The rest is straightforward. □

Now, we are in position to prove our variant of the celebrated Isomorphism Theorem.

Theorem 6.11 (Isomorphism Theorem). Let $(\mathcal{A},\delta)$ be such that α is topologically free. Then for every $(\pi,U,H) \in \mathrm{CovFaithRep}\,(\mathcal{A},\delta)$ the algebra $C^*(\pi(\mathcal{A}),U)$ possess property $(*)$. In other words, for any two covariant faithful representations (π_1,U_1,H_1) and (π_2,U_2,H_2), the mapping

$$U_1 \longmapsto U_2, \qquad \pi_1(a) \longmapsto \pi_2(a), \quad a \in \mathcal{A},$$

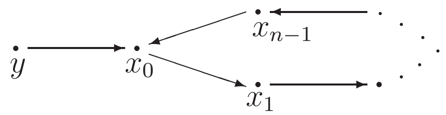
determines an isomorphism of $C^*(\pi_1(\mathcal{A}),U_1)$ onto $C^*(\pi_2(\mathcal{A}),U_2)$.

Proof. Due to Theorem 5.16, the partial homeomorphism $\widetilde{\alpha}$ is topologically free and according to Theorem 3.10 representations π_1 and π_2 give rise to covariant representations $(\overline{\pi}_1,U_1,H_1)$ and $(\overline{\pi}_2,U_2,H_2)$ of the partial dynamical system $(\mathcal{B},\widetilde{\delta})$. Thus it is enough to apply the Theorem 3.6 from [19]. □

Corollary 6.12. Let $\mathcal{A}$ act nondegenerately on a Hilbert space H, let $\delta(\cdot) = U(\cdot)U^*$ where $U \in L(H)$ is a partial isometry such that $U^*U \in \mathcal{A}$, and let the generated partial mapping α be topologically free. Then $C^*(\mathcal{A},U) \cong C^*(\mathcal{A},\delta)$.

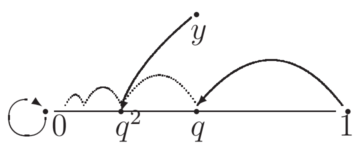
The above corollary allow us, in the presence of topological freeness, consider only abstract covariance algebras. However, in various concrete specification while using the method mentioned after Theorem 1.4, it may happen that the Isomorphism Theorem can be applied to systems $(\mathcal{A},\delta)$ such that Δ_{-1} is not open and α is not topologically free.

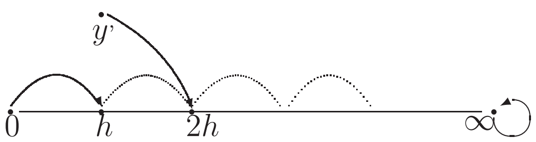
Example 6.13. Let $\mathcal{A}$ and U be as in Example 1.5, then the associated system forms a cycle and therefore it is not topologically free. However after passing to algebra $\mathcal{C} = C^*(\mathcal{A}, U^*U)$ we obtain the dynamical system $(X \cup \{y\}, \alpha)$ (see Example 1.5)



which is topologically free, cf. Example 5.14. Hence, due to the Isomorphism Theorem $C^*(\mathcal{A},U) \cong C^*(X\cup\{y\},\alpha)$. In particular, if $n=1$ then $C^*(\mathcal{A},U)$ is the Toeplitz algebra, see Examples 2.16, 3.9 and 4.3.

Example 6.14. Consider Hilbert spaces $H_1 = L^2_\mu([0,1])$ and $H_2 = L^2_\mu(\mathbb{R}_+)$ where μ is the Lebesgue measure. We fix $0<q<1$ and $0<h<\infty$. Let $\mathcal{A}_1 \subset L(H_1)$ consists of operators of multiplication by functions from $C[0,1]$ and let U_1 act according to $(U_1 f)(x) = f(q\cdot x)$, $f\in H_1$. Similarly, let elements of $\mathcal{A}_2 \subset L(H_2)$ act as operators of multiplication by functions which are continuous on $\mathbb{R}_+ = [0,\infty)$ and have limit at infinity, and let U_2 be the shift operator $(U_2 f)(x) = f(x+h)$, $f\in H_2$. Then the dynamical systems associated to C^*-dynamical systems $(\mathcal{A}_1, U_1(\cdot)U_1^*)$ and $(A_2, U_2(\cdot)U_2^*)$ are topologically conjugate but the images of the generated mappings are not open (compare with Example 2.4). Thus we can not apply the Theorem 6.11 in the form it is stated. Nevertheless, endomorphisms of bigger algebras $\mathcal{C}_1 = C^*(\mathcal{A}_1, U_1^*U_1)$ and $\mathcal{C}_2 = C^*(\mathcal{A}_2, U_2^*U_2)$ do generate dynamical systems





satisfying the assumptions of the Isomorphism Theorem. These dynamical systems are topologically conjugate by a piecewise linear mapping ϕ which maps nh into q^n, $n\in\mathbb{N}\cup\{\infty\}$, and y' into y, that is

$$\phi(x) = q^n\Big(\frac{q-1}{h}\,x+1-n(q-1)\Big), \qquad \text{for } x\in[nh, nh+1), \qquad \text{and} \quad \phi(\infty)=0,\ \ \phi(y')=y.$$

Therefore, by the Isomorphism Theorem, the mapping $\mathcal{A}_1 \ni a \mapsto a\circ\phi \in \mathcal{A}_2$, and $U_1 \mapsto U_2$, establishes the expected isomorphism; $C^*(\mathcal{A}_1,U_1) \cong C^*(\mathcal{A}_2,U_2)$.

Lastly, we would like to present an example which shows how the results achieved in this paper clarify the situation mentioned in the example from which we have started the introduction.

Example 6.15 (Solenoid). Let $H = L^2_\mu(\mathbb{R})$ where μ is the Lebesgue measure on $\mathbb{R}$, and let $\mathcal{A} \subset L(H)$ consists of the operators of multiplication by periodic continuous functions with period 1, that is $\mathcal{A} \cong C(S^1)$. Set the unitary operator $U \in L(H)$ by the formula

$$(Uf)(x) = \sqrt{2}\, f(2x).$$

Then for each $a(x) \in \mathcal{A}$, UaU^* is the operator of mulitplication by the periodic function $a(2x)$ with period $\frac{1}{2}$, and U^*aU is the operator of multiplication by the periodic function $a\left(\frac{x}{2}\right)$ with period 2. Hence

$$U\mathcal{A}U^* \subset \mathcal{A} \quad \text{and} \quad U^*\mathcal{A}U \not\subset \mathcal{A}.$$

The endomorphism $U(\cdot)U^*$ generate on the spectrum of $\mathcal{A}$ the mapping α given by $\alpha(z) = z^2$ for $z \in S^1$, and the spectrum of the algebra $\mathcal{B}$ generated by $\bigcup_{n\in\mathbb{N}} U^{*n}\mathcal{A}U^n$ is the solenoid $\mathcal{S}$: $\mathcal{B} \cong C(\mathcal{S})$, cf. Example 2.12. Further more α is topologically free and therefore we have

$$C^*(\mathcal{A}, U) \cong C^*(S^1, \alpha) = C(\mathcal{S}) \rtimes_F \mathbb{Z}$$

where in the right hand side stands the standard crossed product of $\mathcal{B} = C(\mathcal{S})$ by the *automorphism* induced by the solenoid map F, see Example 2.12.

Summary

In this paper we introduced crossed product-like realization of the universal algebra associated to 'almost' arbitrary commutative C^*-dynamical system $(\mathcal{A}, \delta)$. This new realization generalizes the known constructions for C^*-dynamical systems where dynamics is implemented by an automorphism or a monomorphism.

The primary gain of this is that we are able to describe important characteristics of the investigated object in terms of the underlying topological (partial) dynamical system (X, α), the tool which until now was used successfully only in the case of a (partial) automorphism.

Namely, we have described the ideal structure of covariance algebra by closed invariant subsets of X, in particular simplicity criteria is obtained. Moreover we have generalized the topological freeness, the condition under which all the covariant faithful representations of $(\mathcal{A}, \delta)$ are algebraically equivalent, see the Isomorphism Theorem. For applications this is probably the most important result of the paper.

The important novelty in our approach is that the construction of covariance algebra here consists of two independent steps. The advantage of this is that one may analyse covariance algebra on two levels. First, one may study the relationship between initial C^*-dynamical system and the one generated on its coefficient C^*-algebra, and then one may apply known statements and methods as the latter system is more accessible (generated mapping on the spectrum of coefficient algebra is bijective).

We indicate that recently (see [12]) a notion of crossed-product of a C^*-algebra by an endomorphism (or even partial endomorphism, see [14]) has been introduced, a construction which depends also on the choice of transfer operator. This construction is especially well adapted to deal with morphisms which generate local homeomorphisms. In particular it was used to investigate Cuntz-Krieger algebras, cf. [12, 14, 11]. However it seems that in the case that α is not injective there does not exist a transfer operator such that the aforementioned crossed-product is isomorphic to covariance algebra considered here, and in the case that α is injective the transfer operator is trivial, that is, it is α^{-1} and thus it does not add anything new to the system.

Acknowledgment

The author wishes to express his thanks to A.V. Lebedev for suggesting the problem and many stimulating conversations, to A.K. Kwaśniewski for his active interest in the preparation of this paper, and also to D. Royer for pointing an error in an earlier version of Theorem 2.2.

References

[1] S. Adji, M. Laca, M. Nilsen and I. Raeburn: "Crossed products by semigroups of endomorphisms and the Toeplitz algebras of ordered groups", *Proc. Amer. Math. Soc.*, Vol. 122(4), (1994), pp. 1133–1141.

[2] A. Antonevich: *Linear Functional Equations. Operator Approach*, Operator Theory Advances and Applications, Vol. 83, Birkhauser Verlag, Basel, 1996.

[3] A. Antonevich and A.V. Lebedev: *Functional differential equations: I. C^*-theory*, Pitman Monographs and Surveys in Pure and Applied Mathematics, Vol. 70, Longman Scientific & Technical, Harlow, Essex, England, 1994.

[4] B. Blackadar: *K-theory for Operator algebras*, Springer-Verlag, New York, 1986.

[5] T. Bates, D. Pask, I. Raeburn and W. Szymański: "C*-algebras of row-finite graphs", *New York J. Math*, Vol. 6, (2000), pp. 307–324.

[6] O. Bratteli and D. W. Robinson: *Operator algebras and Quantum Statistical Mechanics I,II*, New York 1979, 1980.

[7] M. Brin and G. Stuck: *Introduction to Dynamical Systems*, Cambridge University Press, 2003.

[8] J. Cuntz: "Simple C^*-algebras generated by isometries", *Commun. Math. Phys.*, Vol. 57, (1977), pp. 173–185.

[9] J. Dixmier: *C^*-algebres et leurs representations*, Gauter-Villars, Paris, 1969.

[10] R. Exel: "Circle actions on C^*-algebras, partial automorphisms and generalized Pimsner-Voiculescu exact sequence", *J. Funct. Analysis*, Vol. 122, (1994), pp. 361–401.

[11] R. Exel and M. Laca: "Cuntz-Krieger algebras for infinite matrices", *J. Reine Angew. Math.*, Vol. 512, (1999), pp. 119–172, http://front.math.ucdavis.edu/funct-an/9712008.

[12] R. Exel: "A new look at the crossed-product of a C^*-algebra by an endomorphism", *Ergodic Theory Dynam. Systems*, Vol 23, (2003), pp. 1733–1750, http://front.math.ucdavis.edu/math.OA/0012084.

[13] R. Exel, M. Laca and J. Quigg: "Partial dynamical systems and C^*-algebras generated by partial isometries", *J. Operator Theory*, Vol. 47, (2002), pp. 169–186, http://front.math.ucdavis.edu/funct-an/9712007.

[14] R. Exel and D. Royer: "The crossed product by a partial endomorphism" Manuscript at arXiv:math.OA/0410192 v1 6 Oct 2004, http://front.math.ucdavis.edu/math.OA/0410192.

[15] R. Exel and A. Vershik: "C*-algebras of irreversible dynamical systems", To appear in *Canadian J. Math.*, http://front.math.ucdavis.edu/math.OA/0203185.

[16] B.K. Kwaśniewski and A.V. Lebedev: "Maximal ideal space of a commutative coefficient algebra", preprint, http://front.math.ucdavis.edu/math.OA/0311416.

[17] M. Laca and I. Raeburn: "A semigroup crossed product arising in number theory", *J. London Math. Soc.*, Vol. 59, (1999), pp. 330–344.

[18] A.V. Lebedev and A. Odzijewicz: "Extensions of C^*-algebras by partial isometries", *Matem. Sbornik*, Vol. 195(7), (2004), pp. 37–70, http://front.math.ucdavis.edu/math.OA/0209049.

[19] A.V. Lebedev: "Topologically free partial actions and faithful representations of partial crossed products", preprint, http://front.math.ucdavis.edu/math.OA/0305105.

[20] K. McClanachan: "K-theory for partial crossed products by discrete groups", *J. Funct. Analysis*, Vol 130, (1995), pp. 77–117.

[21] G.J. Murphy: *C^*-algebras and operator theory*, Academic Press, 1990.

[22] G.J. Murphy: "Crossed products of C^*-algebras by endomorphisms", *Integral Equations Oper. Theory*, Vol. 24, (1996), pp. 298–319.

[23] D.P. O'Donovan: "Weighted shifts and covariance algebras", *Trans. Amer. Math. Soc.*, Vol. 208, (1975), pp. 1–25.

[24] W.L. Paschke: "The Crossed Product of a C^*-algebra by an Endomorphism", *Proc. Amer. math. Soc.*, Vol. 80, (1980), pp. 113–118.

[25] G.K. Pedersen: *C^*-algebras and their automorphism groups*, Academic Press, London, 1979.

[26] P.J. Stacey: "Crossed products of C^*-algebras by *-endomorphisms", *J. Austral. Math. Soc.*, Vol. 54, (1993), pp. 204–212.

[27] R.F. Williams: "Expanding attractors"', *IHES Publ. Math.*, Vol. 54, (1973), pp. 204–212.