ON q-DIFFERENCE EQUATIONS AND $\mathbf{Z}_n$ DECOMPOSITIONS OF $\exp_q$ FUNCTION

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Abstract. The $q-$extended hyperbolic functions of n-th order $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ which are Z_n-components of $\exp_q$ function form the set fundamental solutions of a simple $q-$difference equation. Against the background of $q-$deformed hyperbolic functions of n-th order other natural extensions and related topics are considered. Apart from easy general solution of homogeneous ordinary $q-$difference equations of n-th order main trigonometric-like identity known for hyperbolic functions of n-th order is given its $q-$commutative counterpart. Hint how to arrive at other identities is implicit in the $q-$treatment proposed. The paper constitutes an example of the application of the method of projections presented in Advances in Applied Clifford Algebras publication [19] ; see also references to Ben Cheikh's papers.

1. Introduction

Decomposition of functions with respect to the cyclic group of order n has appeared very fruitful in many aspects (see Ben Cheikh and Duak papers [4-13] and references therein). The method of Z_n decompositions of functions [5] i.e. the method of introducing Z_n grading into the linear spaces of functions or algebras of endomorphisms dates back at least to works of Srivastava (1979) , Ricci (1978) , Osler (1975), Ismail (1986) (see [28]) and explicitly or implicitly

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was put to good account by others earlier (see for example [23-25]). Nowadays this approach of Z_n decompositions is being explicitly or implicitly applied advantageously by other authors apart from the already mentioned ones (see for example [14, 18-20 , 1-3] and references therein).

As a matter of fact the idea of n-cyclicity ($U^n = I$; $U^k \neq I$ for $k < n$) hence Z_n grading appearance may be traced back [18] perhaps to Weierstrass [31].

For extensive applications based on Z_n cyclic group of order n i.e. applications of algebra gradings with grading group $\Gamma = Z_k \bigotimes Z_k \ldots \bigotimes Z_k$ see [21] and a lot of references therein. In the present article we shall follow Ben Cheikh's papers [8], [4] where a methodological attitude to the Z_n decomposition was formulated in the form of the following problem:

"(**P**) : *If special function f satisfies a property $\mathcal{P}$ what corresponds to the property of the corresponding* $f_{[n,k]}$?"

Here ([4-7], [19,1,2]) $f_{[n,k]} \equiv f_k = \Pi_k\, f$ where projection operators $\{\Pi_k\}_{k \in Z_n}$ are defined according to:

$$\Pi_k := \frac{1}{n} \sum_{s \in \mathbf{Z}_n} \omega^{-ks} \Omega^s;\ (\Omega f)(z) := f(\omega z).\ Z_n = \{0, 1, \ldots, n-1\}$$

denotes cyclic group under the addition i.e. for $k, l \in Z_n$: $k \dot{+} l$ means addition mod n ; $k \dot{-} l$ stays for subtraction mod n and $\omega = \exp\left(i\frac{2\pi}{n}\right)$; $n > 1$. In the sequel we shall use notation $\{\Pi_k\}_{k \in Z_n} \equiv \{\Pi_{[n,k]}\}_{k \in Z_n} \equiv \{\Delta_k\}_{k \in Z_n}$ in conformity with the papers [4-7] , [19] mentioned. Of course Z_n labeled projection operators $\{\Pi_k\}_{k \in Z_n}$ do satisfy $\Pi_l \Pi_m = \delta_{lm} \Pi_l$ and $\sum_{k \in Z_n} \Pi_k = id$. Hence $\sum_{k \in Z_n} f_k = f$ and this very formula shall be called the decomposition of function f with respect to the cyclic group of order n or in short: Z_n decomposition of f.

As for the examples of the property $\mathcal{P}$ declared above these are the following: integral representation existence, type of recurrence equation being satisfied (in the case of function sequences), generating function description (in the case of function sequences), hypergeometric representation relations, orthogonality relations, type of differential equation being satisfied or type of difference equation being satisfied etc.

The latter means: find differential/difference equation being satisfied by $\{f_k\}_{k \in Z_n}$ components of $f \in \mathcal{V}$ when the differential/difference equation determining f is known. This question was systematically and with full particulars treated in [8] - see there Theorem 3.1 and applications that follow it.

In this note we are concerned mostly with $q-$extended hyperbolic functions of n-th order $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ [20]. Functions $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ are fundamental solutions of $q-$difference equation

$$\partial_q^n h_{q,k} = h_{q,k}(x) \; ; \; k \in \mathbf{Z}_n \, , \tag{1}$$

resulting from

$$\partial_q^k h_{q,l} = h_{q,l\dot{-}k}(x) \; ; \; k,l \in \mathbf{Z}_n \, , \tag{2}$$

The $q-$extended hyperbolic functions $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ of n-th order are Z_n-components of $\exp_q$ function so that

$$\exp_q(z) = \sum_{k\in\mathbf{Z}_n} h_{q,k}(z) \; . \tag{3}$$

The difference operator ∂_q called the Jackson's derivative [16,15] and $\exp_q$ function are defined below in the preliminaries.

Against a background of $q-$deformed hyperbolic functions $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ of n-th order and their natural extensions other related topics are discussed in this note. Apart from easy general solution of homogeneous ordinary $q-$difference equations ($O\ \partial_q E$) of n-th order (with constant coefficients) main trigonometric-like identity known for hyperbolic functions $\{h_s(z)\}_{s\in\mathbf{Z}_n}$ of n-th order [18-20, 23-25] is given its $q-$noncommutative counterpart. Hint how to arrive at others is implicit in the $q-$treatment proposed. The idea of such treatment goes back at least to Cigler [32] (see formula (7), (11) and other that follow there - see also Kirchenhofer [32] for further systematic development). As for the matrix form of $q-$binomiality property (for q =1 see : Ungar, Mooldon, Kalman [20,23]) - this is only formulated here. More detailed investigation is postponed to a subsequent article. Before proceeding further we introduce basic notation in the setting of the above outlined motivation.

2. Preliminaries

The easy eigenvalue and eigenspaces problem for ω-scaling operator $\Omega : \mathcal{V} \rightarrow \mathcal{V}$

$$(\Omega f)(z) := f(\omega z) \tag{4}$$

where $\mathcal{V}$ stands for chosen function vector space - leads to natural Z_n grading of this very linear space as well as the algebra $End(\mathcal{V})$ of its endomorphisms:

$$\mathcal{V} = \bigotimes_{k\in Z_n} \mathcal{V}_k\,; \tag{5}$$

$$End\,(\mathcal{V}) = \bigotimes_{k\in Z_n} [End\,(\mathcal{V})]_k\,. \tag{6}$$

Here the vector subspace $\mathcal{V}_k$ designates the eigenspace of ω-scaling operator Ω appointed to its eigenvalue ω^k ; $k \in Z_n$:

$$\mathcal{V}_k \ni f_k\,; \quad \Omega\ f_k = \omega^k\ f_k\ ;\ k \in Z_n\ . \tag{7}$$

In turn the vector subspace $[End(\mathcal{V})]_k$ stands for linear space of homogeneous endomorphisms of degree k [4,8]

$$[End(\mathcal{V})]_k \ni \Phi_k \quad \text{iff} \quad \Phi_k : \mathcal{V}_l \to \mathcal{V}_{l\dot{+}k}; \quad l,k \in Z_n \tag{8}$$

The subspaces $[End(\mathcal{V})]_k$; $k \in Z_n$ are also eigenspaces of a certain operator and are obtained as a result of projection $\mathcal{P}_k : End(\mathcal{V}) \to [End(\mathcal{V})]_k$. Namely - it is obvious that similarly to projection operators $\{\Pi_k\}_{k\in Z_n}$ being defined according to:

$$\Pi_k := \frac{1}{n}\sum_{s\in\mathbf{Z}_n} \omega^{-ks}\Omega^s; \quad k \in Z_n \tag{9}$$

also $\mathcal{P}_k; k \in Z_n$ (due to $\beta^n = id,\ \beta^r \neq id; r < n$)

$$\mathcal{P}_k := \frac{1}{n}\sum_{s\in\mathbf{Z}_n} \omega^{-ks}\beta^s; \quad k \in Z_n \tag{9b}$$

constitute a family of projection operators where

$$\beta : End(\mathcal{V}) \to [End(\mathcal{V})];\ End(\mathcal{V}) \ni \Phi \to \beta(\Phi) = \Omega{\circ}\Phi{\circ}\Omega^{-1} \in End(\mathcal{V}). \tag{10}$$

Naturally $\sum_{k\in Z_n} \mathcal{P}_k = id$ and $\mathcal{P}_k\mathcal{P}_l = \mathcal{P}_k\delta_{kl}\ ;\ l,k \in Z_n$ - therefore

$$\mathcal{P}_k[End(\mathcal{V})] = [End(\mathcal{V})]_k\ ; \quad k \in Z_n \tag{11}$$

and of course (6) holds i.e. $End(\mathcal{V}) = \bigotimes_{k\in Z_n} [End(\mathcal{V})]_k$.

Example 1.

Here come few examples [4,8,9] of homogeneous mappings: $l, k \in Z_n$, $[End(\mathcal{V})]_k \ni \Phi_k \ \Phi_k : \mathcal{V}_l \to \mathcal{V}_{l \dot{+} k} \cdot \Omega$ and $\{\Pi_k\}_{k \in Z_n}$ mappings are of course homogeneous of degree zero mappings in the terminology of graded vector spaces or modules while at a point multiplication by f_k ; $k \in Z_n$ is the example of k-th degree homogeneous mappings. The so called [4] "n-translation operator" ${}_n\tau_z$

$$({}_n\tau_z)(\alpha) := \frac{1}{n} \sum_{k \in \mathbf{Z}_n} f(\alpha + \omega^k z) = \Pi_0 f(\alpha + z) \tag{12}$$

(Π_0 acts on g_α function $g_\alpha(z) \equiv f(\alpha + z)$ - is the homogeneous operator of degree zero.

Finally and for the future use (see α-Example 4): α-projection operators $\{\Pi_l^{(\alpha)}\}_{l \in Z_n}$ introduced in [20] are homogeneous of degree zero.

These are defined according to $\Pi_k^{(\alpha)} := \frac{1}{n}\alpha^{-\frac{k}{n}} \sum_{s \in \mathbf{Z}_n} \omega^{-ks}\Omega^s S(\sqrt[n]{\alpha})$. Here $\sqrt[n]{\alpha}$ is an arbitrarily specified n-th root of α and $S(\lambda)$ denotes scaling operator $(S(\lambda)f)(z) := f(\lambda z)$. $\{\Pi_l^{(\alpha)}\}_{l \in Z_n}$ is an easy generalization of the family of projection operators used under notation $\{\Pi_{[n,k]}\}_{k \in Z_n}$ in [4,5,6] ; namely $\Pi_k^{(\alpha)} = \alpha^{\frac{-k}{n}} S\left(\sqrt[n]{\alpha}\right) \Pi_k$. Hence we infer that

$$\Pi_l^{(\alpha)} \Pi_m^{(\alpha)} = \delta_{lm} \Pi_l^{(\alpha)} \alpha^{\frac{-m}{n}} S\left(\sqrt[n]{\alpha}\right) \text{ and } \sum_{k \in Z_n} \alpha^{\frac{k}{n}} \Pi_k^{(\alpha)} = S\left(\sqrt[n]{\alpha}\right).$$

Differentiation $D \equiv \frac{d}{dz}$ mapping is of degree $\dot{-}1 = n-1$; $[End(\mathcal{V})]_{n-l} \ni D$. Indeed; due to Leibnitz rule and differentiation of superposition rule $D[z^k f(z^n)] = z^{k-l} g(z^n)$.

∂_q - Example 2.

$q-$Difference ∂_q homogeneous mapping is of degree

$$\dot{-} = n - 1; [End(\mathcal{V})]_{n-l} \ni \partial_q.$$

Indeed; to see that let us at first define (see [20,21,15,16] and references therein) what is known since a long time; see [16] from 1910 year and [15] for Heine and Gauss contribution and also [17] for may be application to quantum processes description and overall theory of the so called non-commutative geometry.

We perform after Heine and Gauss [15] a replacement $x \mapsto x_q$ thus arriving at the standard by now $q-$extension of the variable $x \in C$ according to the prescription:

$$x \mapsto x_q \equiv \frac{1-q^x}{1-q} \xrightarrow[q\to 1]{} x.$$

Then consequently we have for n_q and $q-$factorial

$$n \mapsto n_q \equiv \frac{1-q^n}{1-q} = 1 + q + \ldots + q^{n-1} \xrightarrow[q\to 1]{} n$$

$$n_q! = n_q(n-1)_q!; \quad 1_q! = 0_q! = 1; \; n_q! \xrightarrow[q\to 1]{} n!$$

Also integration and derivation [15,16] might be $q-$extended. Here we introduce only - what is called - Jackson's derivative ∂_q -a kind of difference operator. It is defined as follows :

$$\left(\partial_q \varphi\right)(x) = \frac{\varphi(x) - \varphi(qx)}{(1-q)x}. \tag{13}$$

Exercise 1. Show that $\partial_q x^n = n_q x^{n-1}$

Exercise 2. Prove that the famous [15-17] $q-$exp function $\exp_q[z] := \sum_{k=0}^{\infty} \frac{z^k}{k_q!}$ satisfies:

$$\partial_q \exp_q = \exp_q; \quad \exp_q[z]_{z=0} = 1.$$

Note: $\exp_q[z] := \sum_{k=0}^{\infty} \frac{z^k}{k_q!}$ is well defined only for $q; q^n \neq 1, \; n \in N$

Naturally $\partial_q \xrightarrow[q\to 1]{} \frac{d}{dx}$ and it is a mere of exercise to prove that $\mathcal{Q}$-Leibnitz rule holds:

$$\partial_q(f \cdot g) = (\partial_q f) \cdot g + (\mathcal{Q} f) \cdot (\partial_q g) \quad \text{where} \quad (\mathcal{Q}\varphi)(z) := \varphi(qz). \tag{14}$$

The "difference-ization" of superposition rule is quite more complicated and it reads $(f' \equiv \partial_q f)$

$$\partial_q f(g(z)) = \frac{g(z)(1-q)f'(g)_{g=g(z)} + f(qg(z)) - f(g(qz))}{g(z) - g(qz)} \partial_q g(z). \tag{15}$$

And so "difference-ization" ∂_q endomorphism is of degree $\dot{-}1 = n - 1$; $[End(\mathcal{V})]_{n-l} \ni \partial_q$ due to the Ω - Leibnitz rule (14) and "difference-ization" of superposition rule (15).
Indeed: $\partial_q[z^k f(z^n)] = z^{k \dot{-} 1} h(z^n)$. One may now conclude that ∂_q^n is homogeneous of order zero endomorphism as $\partial_q^k; k \in \mathbf{Z}_n$ are homogeneous of order $\dot{-}k = n \dot{-} k$ mappings.
Also $\hat{z}D$ and $\hat{z}\partial_q$ are of zero order endomorphisms (concatenation of operator symbols designates their superposition i.e. successive application) where $(\hat{z}f)(z) = zf(z)$.

Remark 1. (Theorem II-1 in [4]) It is to be noted and then used - that ; $[End(\mathcal{V})]_k \ni \Phi$ iff $[\Pi_k, \Phi] \equiv \Pi_k \circ \Phi - \Phi \circ \Pi_k = 0$.

Remark 2. Iteration of $\partial_q \exp_q = \exp_q$ i.e. iteration of the resulting $q-$difference recurrence $\exp_q(x) = \dfrac{\exp_q(x) - \exp_q(qx)}{(1-q)x}$ leads to another equivalent expression for the $\exp_q$ function

$$\exp_q(z) = \prod_{n=0}^{\infty} \frac{1}{1-(1-q)q^n z}. \tag{16}$$

Applying now projection operators $\{\Pi_l\}_{l \in Z_n}$ to $\exp_q$ function we get the family $\{h_{q,s}(z)\}_{s \in \mathbf{Z}_n}$ of $q-$extended hyperbolic functions of order n where $h_{q,s} \equiv \Pi_s \exp_q; s \in \mathbf{Z}_n$ and

$$h_{q,s} = \Pi_s \exp_q;\; s \in Z_n; h_{q,s} \xrightarrow[q \to 1]{} h_s;\; s \in Z_n.$$

Apparently $h_{q,s}(z) = \displaystyle\sum_{k \geq 0} \frac{z^{nk+s}}{(nk+s)_q!}$ named $q - l - \exp_q$ series [19, 20] are eigenfunctions of $\Omega : \Omega h_{q,s} = \omega^s h_{q,s}; s \in \mathbf{Z}_n$ and

$$h_{q,s}(z) = \frac{1}{n} \sum_{k \in \mathbf{Z}_n} \omega^{-ks} \exp_q(\omega^k z); s \in \mathbf{Z}_n. \tag{17}$$

Note also that

$$\exp_q(\omega^l z) = \frac{1}{n} \sum_{k \in \mathbf{Z}_n} \omega^{kl} h_{q,k}(z); l \in \mathbf{Z}_n. \tag{18}$$

At the end let us state the link in-between the most powerful and simple series via $\exp_q$. Namely due to (also see (16) and note that $0^0 = 1$):

$$\frac{\exp_o(z) - 1}{z} = \exp_o(z) \Rightarrow \exp_o(z) = \frac{1}{1-z} = \sum_{k=0}^{\infty} z^k;\ |z| < 1,$$

we ascertain: $\exp_o(z) = \dfrac{1}{1-z}$ and $\exp(z)$ are mutual limit deformations for $|z| < 1$:

$$\sum_{k\geq 0} \frac{z^k}{k!} = \exp(z) \underset{1\leftarrow q}{\longleftarrow} \exp_q(z) = \sum_{n=0}^{\infty} \frac{z^n}{n_q!} \underset{q\rightarrow 0}{\longrightarrow} \exp_{q=0}[z] = \frac{1}{1-z} = \sum_{k\geq 0} z^k; |z| < 1$$

and $h_{0,s}(z) = \sum_{k\geq 0} z^{nk+s}$ $s \in Z_n$. Therefore identities for $\{h_{q,s}(z)\}_{s\in \mathbf{Z}_n}$ including those which mimic hyperbolic-trigonometric ones - comprise also $0-l$-series $h_{0,s}(z) = \sum_{k\geq 0} z^{nk+s}$.

Remark 3. The $q-$difference ∂_q operator called the Jackson's ∂_q-derivative may be represented as a superposition of a differential operator of infinite order and Q scaling operator (recall : $(\mathcal{Q}\varphi)(z) := \varphi(qz)$ where φ a polynomials or formal power series).
Namely the Jackson's ∂_q-derivative is given by [21]

$$(\partial_q \varphi)(x) = \frac{1 - q\mathcal{Q}}{(1-q)} \partial_o \varphi(x).$$

where $\partial_o = \partial_{q=0}$ is the differential operator of infinite order

$$\partial_o = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{z^{n-1}}{n!} \frac{d^n}{dz^n}.$$

3. Linear Homogeneous Constant Coefficient Ordinary $q-$Difference Equations and $q-$hyperbolic Functions of the n-th Order

(The classical references on $q-$difference equations are: Adams [16] and Trjitzinsky [16]).

Linear homogeneous constant coefficient ordinary $q-$difference equations of n-th order $(O\partial_q E)$ or equivalently system of n linear homogeneous constant coefficient ordinary $q-$ difference equations of the first order - may be treated in complete analogy with their differential correspondents and even more $(O\partial_q E) \underset{q\to 1}{\longrightarrow} (ODE)$: i.e. ordinary differential equations may be treated as the $\underset{q\to 1}{\longrightarrow}$ limit of $q-$difference equations.

Let then $W_n[z]; degW_n[z] = n$ be a polynomial. Linear homogeneous constant coefficient ordinary $q-$difference equation in the vector space $\mathcal{V}$ is then defined to be :

$$W_n[\partial_q]f = 0; f \in \mathcal{V} \tag{19}$$

When $\mathcal{V}$ is the vector space of entire functions the solution of (17) is easy to obtain after substitution $f(z) = \exp_q[\lambda z]$ is made as then we get the characteristic equation $W_n[\lambda] = 0$. There are now two cases to be considered. 1) Polynomial $W_n[\lambda]$ has n different from each other roots: $W_n[\lambda] = \prod_{k=1}^{n}(\lambda - \lambda_k); \lambda_k \neq \lambda_l$ for $k \neq l$; 2) otherwise. In the first case the general solution of (17) has the familiar form :

$$f(z) = \sum_{k=1}^{n} A_k \exp_p\{\lambda_k z\}. \tag{20}$$

In the second case let λ be the root of $W_n[\lambda]$ with multiplicity k. Then $W_n[\partial_q] = W_{n-k}[\partial_q](\partial_q - \lambda)^k$ and the λ root with multiplicity k contributes to the overall general solution sum via following function of again familiar form:

$$r_\lambda(z) = \exp_p(\lambda z) \sum_{s=0}^{k-1} A_s z^s \tag{21}$$

so that $f(z) = \sum_{\lambda \in \Lambda} r_\lambda(z)$ where Λ = set of different from each other roots of polynomial $W_n[\lambda]$

For that to see it is enough to prove that

$$\forall s \in N; \left(\partial_q - \lambda\right)^s \left[z^{s-1} \exp_q\{\lambda z\}\right] = 0; \tag{22}$$

(use the induction and the property (see: (14)) of ∂_q difference operator : $\partial_q \exp_q(\lambda z) = \lambda \exp_q(\lambda z)$ and also $\partial_q x^n = n_q x^{n-1}$ - of course).

Example 3.

Consider $W_n[\lambda] = \lambda^n - 1$. If in addition to (19) where $\mathcal{V}$ is the vector space of entire functions one adds various initial conditions requirements then the solutions are unique.

For example: $f_k^{(l)}(0) = \omega^{kl}$; $k, l \in \mathbf{Z}_n$ imply $f_k(z) = \exp_q[\omega^k z]$; $k \in \mathbf{Z}_n$.

For example: $f_k^{(l)}(0) = \delta_{0,k \dot{-} l}$; $k, l \in \mathbf{Z}_n$ imply $f_k(z) = h_{q,k}(z)$; $k \in \mathbf{Z}_n$.

Of course both sets of solutions are linear combinations: one of each other and with the above choice - these are (17) and (18) formulas. In the sequel of the *Example 3* we shall be concerned with the family $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$ of $q-$extended hyperbolic functions of order n which are Z_n -components of $\exp_q$. While n is fixed we shall abbreviate: $\{h_{q,s}^{[n]}(z)\}_{s\in\mathbf{Z}_n} \equiv \{h_{q,s}(z)\}_{s\in\mathbf{Z}_n}$. These are - as we have seen - the fundamental solutions of $q-$difference equation

$$\partial_q^n h_{q,k} = h_{q,k}(x) \; ; \; k \in \mathbf{Z}_n, \tag{23}$$

resulting from

$$\partial_q^k h_{q,l} = h_{q,l \dot{-} k}(x) \; ; \; k, l \in \mathbf{Z}_n, \tag{24}$$

for $q-$extended hyperbolic functions $\{h_{q,s}(z)\}_{s\in\mathbf{Z}_n} \equiv \{h_{q,s}^{[n]}(z)\}_{s\in\mathbf{Z}_n}$ of n-th order.

As in (24) also some trigonometric-like identities satisfied by hyperbolic functions of n-th order $\{h_s(z)\}_{s\in\mathbf{Z}_n}$ ([20], [18], [23-25]) $q-$extend almost automatically from the $q = 1$ case after one crucial "revolutionary" observation is made and the way out in new circumstances is accepted. For that to do consider ([19], see also [18, 20]) the convolution identity

$$\sum_{k\in\mathbf{Z}_n} h_k(\alpha) h_{i-k}(\beta) \equiv h_i(\alpha+\beta); i \in \mathbf{Z}_n. \tag{25}$$

From (25) we see that $\sum_{k\in\mathbf{Z}_n} h_k(\alpha) h_{\dot{-}k}(z) \equiv h_0(\alpha + z)$ and via

$$\Pi_0 \sum_{k\in\mathbf{Z}_3} h_k(\alpha) h_{\dot{-}k}(z) = \Pi_0 h_0(\alpha + z)$$

one arrives at the very founding identity for the special analytical functions of Tchebysheff type [19], [2,3, 22] (from (4.2) in Ricci [25]- to (26) below - one step was missing) i.e.

$$h_0(\alpha)h_0(z) = \frac{1}{n}\sum_{k\in\mathbf{Z}_n} h_0(\alpha+\omega^k z). \tag{26}$$

(Note: Π_0 was acting up there on g_α function of $z; g_\alpha(z) \equiv f(\alpha+z)$; here $f = h_{\dot{-}k}$; $k \in \mathbf{Z}_n$ - compare with (12)).

The convolution property is easily derived from de Moivre formulas in their matrix form [20, 18] due to the fact that $\{H(z) = \exp\{\gamma z\}\}_{z\in C}$ forms what we call de Moivre group because of $H(z)H(w) = H(z+w)$ where $\gamma = (\delta_{i,k\dot{-}1}); k,i \in \mathbf{Z}_n$ and det $H(z) = 1$ (note $\gamma^n = (\delta_{i,k\dot{-}1})^n = I$ and $Tr\gamma = Tr(\delta_{i,k\dot{-}1}) = 0$). As for the property $H(z)H(w) = H(z+w)$ this is no more the case when one replaces exp in $H(z) = \exp\{\gamma z\}$ by $\exp_q$ (with $q \neq 1$) because

$$\exp_q[z]\exp_q[w] \neq \exp_q[z+w] \text{ for } q \neq 1. \tag{27}$$

In order to find a way out and to proceed with maximum analogy to the familiar hyperbolic or α-hyperbolic case [20,18,29] we propose the following. (Put $\alpha = 1 = q$ and you are in with the most familiar hyperbolic case [25, 18]). In full analogy with [20] let us introduce the $q-\alpha-$de Moivre one parameter family of matrices $\{H_q^\alpha(z) = \exp_q\{\gamma(\alpha)z\}\}_{z\in C}$ where $\gamma(\alpha) = (\delta_{i,k\dot{-}l} + (\alpha - 1)\delta_{n-1,0}); k,i \in \mathbf{Z}_n$ i.e.

$$H_q^\alpha(z) = \begin{pmatrix} \exp_{q,0}^\alpha(z) & \exp_{q,1}^\alpha(z) & \dots & \exp_{q,n-1}^\alpha(z) \\ \alpha\exp_{q,n-1}^\alpha(z) & \exp_{q,0}^\alpha(z) & \dots & \exp_{q,n-2}^\alpha(z) \\ \dots & \dots & \dots & \dots \\ \alpha\exp_{q,1}^\alpha(z) & \alpha\exp_{q,2}^\alpha(z) & \dots & \exp_{q,0}^\alpha(z) \end{pmatrix}.$$

It is obvious that: $\gamma(\alpha)^n = \alpha I$, Tr $\gamma = 0$. However (put $L = \exp_q$ in (α−23) from [20]) $\det H_q^\alpha(z) = \det\exp_q\{\gamma(\alpha)z\} = \prod_{l\in Z_n}\exp_q(\omega^l\ \sqrt[n]{\alpha}z) \neq 1$ for $q \neq 1$ and because of (27) neither the $q-\alpha-$de Moivre one parameter family

$\{H_q^\alpha(z) = \exp_q\{\gamma(\alpha)z\}\}_{z\in C}$ forms the group for $q \neq 1$.

The identity $\det \exp_q\{\gamma(\alpha)\varphi\} = \prod_{l\in Z_n} \exp_q(\omega^l \sqrt[n]{\alpha}\varphi)$ for $n = 2$ $\alpha = 1$ and $q = 1$ becomes $[\cos h\varphi]^2 - [\sin h\varphi]^2 = 1$ while for $n = 2$ $\alpha = -1$ and $q = 1$ we get $[\cos\varphi]^2 + [\sin\varphi]^2 = 1$.
Nevertheless the identity $\det \exp_q\{\gamma(\alpha)\varphi\} = \prod_{l\in Z_n} \exp_q(\omega^l \sqrt[n]{\alpha}\varphi)$ could hardly be considered hyperbolic-trigonometrical - because no reasonable extension of (25) exists. As for the identity (26) - this may be seen to be the $q = 1$ case of the following quite involved identity (Consult [4] and/or formula (12) in this note for "n-translation operator" ${}_n\tau_z$)

$$\frac{1}{n}\sum_{k\in \mathbf{Z}_n} h_0\big(x +_q \omega^k z\big) = (\sum_{k\in Z} {}_n\tau_{q,\omega^k z}) \exp_q[x] \qquad (q-26)$$

where

$$({}_n\tau_{q,z} f)(\alpha) := \frac{1}{n}\sum_{k\in \mathbf{Z}_n} f(\alpha +_q \omega^k z) = \Pi_0 f(\alpha +_q z) \qquad (q-12)$$

and where

$$f(z +_q w) \equiv E^w(\partial_q) f(z). \qquad (28)$$

Here

$$E^w(\partial_q) \equiv \exp_q\{w\partial_q\} = \sum_{k=0}^{\infty} \frac{w^k \partial_q^k}{k_q!} \qquad (29)$$

plays the role of a generalized translation operator - the most important operator of *extended finite operator calculus of Rota* [21] (compare with $\exp_q(x) = \sum_{k-0}^{\infty} \frac{x^k}{k_q!}$). On that occasion recall: the textbook translation operator is this one: $\exp\{a\frac{d}{dx}\} = \sum_{k=0}^{\infty} \frac{a^k}{k!}\frac{d^k}{dx^k}$ and $\sum_{k=0}^{\infty} \frac{a^k}{k!}\frac{d^k}{dx^k} f(x) = \exp\{a\frac{d}{dx}\} f(x) = f(x+a)$ is just Taylor's expansion formula.
And again the identity $(q-26)$ could hardly be considered hyperbolic-trigonometrical - because - as already stated no reasonable extension of (25) exists.

However.... However there exists an extension of magic intrinsic and intriguing beauty.

Exercise 3 prove the identity $(A+B)^n \equiv \sum_{k\geq 0} \binom{n}{k}_q A^k B^{n-k}$ where $[B,A]_q \equiv BA - qAB = 0$.
Observe now [21] that

$$\exp_q[\hat{z}]\exp_q[\hat{w}] = \exp_q[\hat{z}+\hat{w}] \tag{30}$$

where $[\hat{w},\hat{z}]_q \equiv \hat{w}\hat{z} - q\hat{w}\hat{z} = 0$ - see Proposition 4.2.4 in [17].
The idea to use "$q-$commuting variables" goes back at least to Cigler [32] (see there formula (7), (11) and other that follow - see also Kirchenhofer [32] for further systematic development). We call such variables $q-$ commuting (see [17] Chapter 4 on "quantum planes"). Realization of such pairs (see: Cigler, Kirchenhofer [32] and [21]) may be for our purpose here adopted as follows.

Put $\hat{z} = \hat{x}$ and $\hat{w}(y) = \hat{w} = y\hat{\mathcal{Q}}$ where: $x,\ y \in \mathcal{C}$; $\hat{\mathcal{Q}}\varphi(x) = \varphi(qx); \hat{x}\varphi(x) = x\varphi(x)$. The matrix entries and variables are now *operators* alike $\hat{z}$ and $\hat{w}$. This taken for granted due to (30) we recover (compare with Proposition 4.2.4 in [17]) the extended de Moivre property

$$\exp_q\{\gamma(\alpha)\hat{z}\}\exp_q\{\gamma(\alpha)\hat{w}\} = \exp_q\{\gamma(\alpha)(\hat{z}+\hat{w})\} \tag{31}$$

for the extended de Moivre family $\{H_q^\alpha(\hat{u}) = \exp_q\{\gamma(\alpha)\hat{u}\}\}_{\hat{u}\in\hat{C}},\ \hat{u} = a\hat{z} + b\hat{w}, a, b \in \mathcal{C}$ i.e. $\hat{u} \in \hat{C}$ where $\hat{C}$ designates quantum plane [17]. We also recover the form of intrinsically hyperbolic-trigonometric, distinctive identity (25) - now in $q-$commuting variables ($\alpha = 1$ for simplicity of expression)

$$\sum_{k\in\mathbf{Z}_n} h_{q,k}(\hat{z})h_{q,i-k}(\hat{w}) \equiv h_{q,i}(\hat{z}+\hat{w}) \quad i \in \mathbf{Z}_n. \tag{32}$$

Let us apply now an igneous invention of Cigler [32]

$$(\hat{x}+y\hat{\mathcal{Q}})^n\mathbf{1} \equiv \sum_{k\geq 0}\binom{n}{k}_q x^k y^{n-k} \tag{33}$$

used as in [21] (see Kirchenhofer [32]) for definition of $q-$binomial sequences $\{p_k\}_{k\geq 0}$

$$E^y(\partial_q)p_n(x) \equiv p_n(x+_q y) \equiv \sum_{k\geq 0}\binom{n}{k}_q p_k(x)p_{n-k}(y) = p_n(\hat{x}+y\hat{\mathcal{Q}})\mathbf{1} \equiv$$

$$p_n(\hat{z}+\hat{w})\mathbf{1} \tag{34}$$

hence in particular for $\{p_k\}_{k\geq 0} = \{x^k\}_{k\geq 0}$

$$(\hat{z}+\hat{w})^n\mathbf{1} \equiv (\hat{x}+y\hat{\mathcal{Q}})^n\mathbf{1} \equiv \sum_{k\geq 0}\binom{n}{k}_q x^k y^{n-k} = \mathbf{1} \equiv (x+_q y)^n$$

As one can see - due to (32) with (34) being taken into account - the family of $q-$hyperbolic mappings $\{h^{[n+1]}_{q,n-1}(\hat{z})\}_{n\geq 0}$ constitutes the nontrivial example of the $q-$binomial sequence of functions mapping groupoids into rings - which are not polynomials - see Remark 4 - below.
From (31) and (33) we infer then that

$$[\exp_q\{\gamma(\alpha)\hat{z}\}\exp_q\{\gamma(\alpha)\hat{w}\}]\mathbf{1} = \exp_q\{\gamma(\alpha)(\hat{z}+\hat{w})\}\mathbf{1} = \exp_q\{\gamma(\alpha)(x+y)\}. \tag{35}$$

From (32) alike in (35) (see (34) and consult eventually [21] and [32]) we arrive at

$$\left[\sum_{k\in\mathbf{Z}_n} h_{q,k}(\hat{z})h_{q,i-k}(\hat{w})\right]\mathbf{1} \equiv h_{q,i}(\hat{z}+\hat{w})\mathbf{1} \quad i\in\mathbf{Z}_n$$

from which we have

$$E^y(\partial_q)h_{q,i}(x) \equiv h_{q,i}(\hat{z}+\hat{w})\mathbf{1} = \sum_{k\in\mathbf{Z}_n} h_{q,k}(x)h_{q,i-k}(y)\ ,\ i\in\mathbf{Z}_n. \tag{36}$$

The equation (36) may be considered as the characterization of the extended de Moivre family $\{H^\alpha_q(\hat{u}) = \exp_q\{\gamma(\alpha)\hat{u}\}\}_{\hat{u}\in\hat{C}}$ and it is pertinent generalization of $q=1$ de Moivre group characterization via convolution identity (32) with specification : $\hat{z}=x$ and $\hat{w}=y$. Note $[\hat{z},\hat{z}]_q \neq 0$ therefore $\sum_{k\in\mathbf{Z}_n} h_k(\hat{z})h_{i-k}(\hat{z}) \neq h_i(2\hat{z})$ $i\in\mathbf{Z}_n$ although there is a reason for complacency - namely:

$$\det\ \exp_q\{\gamma(\alpha)\hat{z}\} = \mathbf{I} \Leftrightarrow\ \det\ H^\alpha_q(\hat{z})\mathbf{1} = 1. \tag{37}$$

Indeed: det $H^\alpha_q(\hat{z}) = \det\exp_q\{\gamma(\alpha)\hat{z}\} = \prod_{k\in Z_n}\exp_q(\omega^k\ \sqrt[n]{\alpha}\hat{z}) = \exp_q$ $\left\{\hat{z}\ \sqrt[n]{\alpha}\sum_{k\in Z_n}\omega^k\right\} = \mathbf{I}$, where we define associative ring valued det mapping for commuting entries a_{ij} of matrix $A = (a_{ij})$ formally as in a standard case of the field entries:

$$\det \text{A} \ = \sum_{\sigma \in S_n} \ sign\ \sigma\ a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3} \dots a_{\sigma(n)n}.$$

On this occasion note that one may keep (37) etc. survive while remedy the $[\hat{z}, \hat{z}]_q \neq 0$ and $\sum_{k \in \mathbf{Z}_n} h_k(\hat{z}) h_{i-k}(\hat{z}) \neq h_i(2\hat{z})$ so as to get equalities instead inequalities. For that to arrive chose Grassmann$n$-like representation: $\hat{z} = \theta \hat{x}$ and $\hat{w} = y\theta\hat{\mathcal{Q}}$ where $\theta^2 = 0$. The price to pay for it is kind of triviality as then $\exp_q\{\gamma(\alpha)\hat{z}\} = 1 + \gamma(\alpha)\hat{z}, [\exp_q\{\gamma(\alpha)\hat{z}\}]^n = 1 + n\gamma(\alpha)\hat{z}$ as well as $[1 + \hat{w}]^n = 1 + n\hat{w}$.
To this end let us note that among others the $q-$extension of considerations of Ungar ((1983), [29]) and Kalman and Ungar [29] seams to be at hand.

- Do we then have now the permit of driving license to keep straight to the roots of "$q-$commutative hyperbolic-trigonometric routs" ?

For any case and to this end let us return to the *related examples* of $q-$difference equations.

$\alpha-$ **Example 4.**

Consult [20], *Example 1* and compare with A. A. Ungar in [29]. Consider now $W_n[\lambda] = \lambda^n - \alpha$. If in addition to (19) where $\mathcal{V}$ is the vector space of entire functions one adds various initial conditions requirements then the solutions are unique.

For example: $f_k^{(l)}(0) = \alpha^{\frac{l}{n}} \omega^{kl}$; $k, l \in \mathbf{Z}_n$ imply $f_k(z) = \exp_q[\sqrt[n]{\alpha}\omega^k z]$; $k \in \mathbf{Z}_n$.

For example: $f_k^{(l)}(0) = (1+(\alpha-1)\delta_{k,l})$; $k, l \in \mathbf{Z}_n$ imply $f_k(z) = h^{\alpha}_{q,k}(z)$; $k \in \mathbf{Z}_n$.

Indeed: $\partial_q h^{\alpha}_{q,s}(z) = (1 + (\alpha - 1)\delta_{0,s})\ h^{\alpha}_{q,s\dot{-}1}(z)$; $s \in \mathbf{Z}_n$ from which [20]

$$\partial_q^k h^{\alpha}_{k,l} = \prod_{s=0}^{k\dot{-}1} (1 + (\alpha - 1)\delta_{0,l\dot{-}s}) h^{\alpha}_{q,l\dot{-}k} \ ; \ k, l \in \mathbf{Z}_n, \tag{39}$$

follows where (see [20])

$$h^{\alpha}_{q,s}(z) = \frac{1}{n}\alpha^{\frac{-s}{n}} \sum_{k\in\mathbf{Z}_n} \omega^{-ks} \exp_q(\omega^k \sqrt[n]{\alpha} z); \quad s \in \mathbf{Z}_n \tag{40}$$

and naturally

$$\partial^n_q h^{\alpha}_{q,l} = \alpha h^{\alpha}_{q,l}; \quad l \in \mathbf{Z}_n \tag{41}$$

Using (α - 19) from [20] with $L = \exp_q$ and recalling [20] that $h^{\alpha}_{q,s} = \Pi^{\alpha}_s \exp_q$ where

$\Pi^{(\alpha)}_k := \frac{1}{n}\alpha^{-\frac{k}{n}} \sum_{s\in\mathbf{Z}_n} \omega^{-ks}\Omega^s S(\sqrt[n]{\alpha})$; $(S(\lambda)f)(z) := f(\lambda z)$ one derives

$$\exp_q(\omega^l \sqrt[n]{\alpha} z) = \sum_{k\in\mathbf{Z}_n} \alpha^{\frac{k}{n}} \omega^{kl} h^{\alpha}_{q,k}(x) \tag{42}$$

Of course both sets of solutions are linear combinations one of each other and with the above choice - these are (40) and (42) formulas. Because of the familiar form of $h^{\alpha}_{q,s}, s \in \mathbf{Z}_n$ (compare with [29]) the considered $h^{\alpha}_{q,s}$ entire functions

$$h^{\alpha}_{q,s}(z) = \sum_{k\geq 0} \frac{\alpha^k z^{nk+s}}{(nk+s)_q!} \tag{43}$$

shall be called: the $s-\alpha-q-$hyperbolic series.
It is standard obvious that $H^{\alpha}(z) = \exp\{\gamma(\alpha)z\}$, $\gamma(\alpha) = (\delta_{i,k\dot{-}1} + (\alpha - 1)\delta_{n-1,0}); k, i \in \mathbf{Z}_n$ is the unique solution of the equation

$$\partial_q H^{\alpha}_q(z) = \gamma(\alpha) H^{\alpha}_q(z) \text{ wiht } H^{\alpha}_q(0) = I. \tag{44}$$

Naturally from the above we conclude (compare with Muldoon, Ungar [29]) that

$$\partial^n_q H^{\alpha}_q(z) = \alpha H^{\alpha}_q(z) \tag{45}$$

hence

$$\partial^n_q h^{\alpha}_{q,l} = \alpha h^{\alpha}_{q,l} \ ; \quad l \in \mathbf{Z}_n$$

these follow also by noticing that $\partial_q\ h^{\alpha}_{q,s}(z) = (1+(\alpha-1)\delta_{0,s})\ h^{\alpha}_{q,s\dot{-}l}(z)\ ;\ s \in \mathbf{Z}_n$.
Similarly: introducing $H^{\alpha}(\hat{z}) = \exp\{\gamma(\alpha)\hat{z}\}$ understood - if you like - formally - and after defining on such vector space of series the following derivation $\hat{\partial}_q$: .

$$\left(\hat{\partial}_q \varphi\right)(\hat{z}) = \frac{\varphi(\hat{z}) - \varphi(q\hat{z})}{(1-q)\hat{z}}. \qquad (\hat{\partial}_q - 13)$$

we arrive at $\hat{\partial}_q$-version of (39) and (40): $\hat{\partial}_q H^{\alpha}_q(\hat{z}) = \gamma(\alpha) H^{\alpha}_q(\hat{z})$ with $H^{\alpha}_q(\hat{0}) = I$ and $\hat{\partial}^n_q H^{\alpha}_q(\hat{z}) = \alpha H^{\alpha}_q(\hat{z})$.

On $q-$ binomiality - remark 4

Due to (32) with (34) being taken into account - the family of $q-$hyperbolic mappings $\left\{h^{[n+1]}_{q,n-1}(\hat{z})\right\}_{n\geq 0}$ constitutes the nontrivial example of the $q-$binomial sequence of functions mapping groupoids into rings - which *are not* polynomials - compare with Ungar (1983) [29] (for sequences of functions of binomial type: see Aczel, Brown [32])

The property called by Kalman and Ungar [29] (see (3) and (5) there) "Matrix Binomial Theorem" in the case of $H^{\alpha}_q(x)$ matrices has also its $q-$counterpart - equivalent to (36):

$$H^{\alpha}_q(\hat{z} +_q \hat{w})\mathbf{1} = H^{\alpha}_q(x) H^{\alpha}_q(y). \qquad (46)$$

Using as in [21] generalized shift operator $E^y(\partial_q)$ we observe that under natural action

$$H^{\alpha}_q(\hat{z} +_q \hat{w})\mathbf{1} = E^y(\partial_q) H^{\alpha}_q(x). \qquad (47)$$

Considerations of Sec. 2 in Kalman and Ungar [29] $q-$extend correspondingly.

Namely: Let

$$K = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1_q & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 2_q & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 3_q & 0 & 0 & \dots & 0 & 0 \\ . & . & . & & & \dots & & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & \dots & & . \\ 0 & 0 & 0 & 0 & 0 & & (n-1)_q & 0 \end{pmatrix}. \tag{48}$$

Naturally $K^n = 0$; $K^k \neq 0$ for $0 \leq k < n$. Then for $B \equiv \exp_q[\hat{x}\ K] = \sum_{k=0}^{n-1} \frac{\hat{x}^k K^k}{k_q!}$ we get matrix which for editorial reasons we show up for $n = 3$:

$$M_3(\hat{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \hat{x} & 1 & 0 & 0 \\ \hat{x}^2 & 2_q x & 1 & 0 \\ \hat{x}^3 & 3_q \hat{x}^2 & 3_q x & 1 \end{pmatrix}.$$

Next let

$$N = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ . & . & . & . & & \dots & & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & \dots & . & . \\ 0 & 0 & 0 & 0 & 0 & \cdot & 1 & 0 \end{pmatrix}. \tag{49}$$

Naturally $N^n = 0$; $N^k \neq 0$ for $0 \leq k < n$. Then for $M_n(\hat{x}) \equiv \exp_q[\hat{x}N] = \sum_{k \in Z_n} \frac{\hat{x}^k N^k}{k_q!}$ we get $M_n(\hat{x})$ matrix which - again for editorial reasons write for $n = 3$

$$M_3(\hat{x}) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \hat{x} & 1 & 0 & 0 \\ \hat{x}^2/2_q! & \hat{x} & 1 & 0 \\ \hat{x}^3/3_q! & \hat{x}^2/2_q! & \hat{x} & 1 \end{pmatrix}.$$

As in $H_q^\alpha(\hat{z})$ matrix case - compare with (46) and (36) - the following holds: $M_n(\hat{z} + \hat{w}) = M_n(\hat{z})M_n(\hat{w})$ i.e. $M_n(\hat{x} + y\ \hat{\mathcal{Q}}) = M_n(\hat{x})M_n(y\ \hat{\mathcal{Q}})$ and the relevantly pertinent generalizations read: for $M_n(\hat{z})$ matrix: $M_n(\hat{z}+\hat{w})\mathbf{1} \equiv M_n(x +_q y) = M_n(x)M_n(y)$ as well as for $B_n(\hat{x})$ for which we have: $B_n(\hat{z} + \hat{w})\mathbf{1} \equiv B_n(x +_q y) = B_n(x)B_n(y)$.
In general a characterization of $q-$binomiality of a sequence may be provided in the matrix form. This is again an example of what is called by Kalman and Ungar [29] (see (3) and (5) there) for $q = 1$ "Matrix Binomial Theorem". For that to see observe that $M_n(x)$; $n = 0, 1, 2, ...$ matrices serve to copy the binomiality of the particular binomial sequence $\left\{x^h\right\}_{k\geq 0}$. Similarly the

property $F_n(\hat{z}+\hat{w}) = F_n(\hat{z})F_n(\hat{w})$ of lower triangular matrices $n = 0,1,2,...$: $F_n(\hat{z}) = (f_{ij})_{i,j=0}^{n}$; $f_{i,j} = \frac{p_{i-j}}{(i-j)_q}$ for $i \geq j$ and zero otherwise - *is equivalent* to the q-binomiality of a $\{p_k(x)\}_{k\geq 0}$ polynomial sequence as defined by (34) i.e.

$$E^y(\partial_q)p_n(x) \equiv p_n(x +_q y) \equiv \sum_{k\geq 0} \binom{n}{k}_q p_k(x)p_{n-k}(y).$$

Using B-matrix form of lower triangular matrices: $G_n(\hat{z}) = (g_{ij})_{i,j=0}^{n}$; $g_{i,j} = p_{i-j}$ for $i \geq j$ and zero otherwise - we arrive at equivalent characterization of polynomial sequences of convolution type (called also - divided sequences).

∂_ψ **- Example 6** (ψ-difference equations)**.**

Consult [20,32,21] (see also [26] Chapter 6) and compare with A. A. Ungar in [29]. Similarly as $q-$Difference ∂_q homogeneous mapping from ∂_q - Example 2 the ψ-difference ∂_ψ mapping $\partial_\psi \in End(\mathcal{V})$ may be defined [32, 26,20] as follows. Consider the generalized factorial $n_\psi! \equiv n_\psi(n-1)_\psi\ (n-2)_\psi\ (n-3)_\psi \ldots 2_\psi\ 1_\psi;\ 0_\psi! = 1$ for an arbitrary sequence $\Psi = \{\psi_n\}_{n\geq 1}$ with the condition $\psi_n \neq 0,\ n \in N$. Here n_ψ denotes the ψ-deformed number where in conformity with Viskov [32] notation $n_\psi \equiv \psi_{n-1}(q)\psi_n^{-1}(q)$ or equivalently $n_\psi! \equiv \psi_n^{-1}(q)$. One may now define [21] a difference operator ∂_ψ called ψ-derivative on the entire functions space according to: $\partial_\psi x^n = n_\psi x^{n-1}$; $n > 0,\ \partial_\psi \mathrm{const} = 0$. Then the $\psi-\exp$ function $\exp_\psi[z] := \sum_{k=0}^{\infty} \frac{z^n}{n_\psi!}$ enters the game so that most of all other constructions and statements of this note "Ψ-extend" automatically. Of course $\Omega h_{\psi,s}^{\alpha} = \omega^s h_{\psi,s}^{\alpha}$; $s \in \mathbf{Z}_n$ where $h_{\psi,s}^{\alpha} = \Pi_s^{\alpha}\ \exp_\psi$. All this in mind consider now $W_n[\lambda] = \lambda^n - \alpha$. If in addition to (19) where $\mathcal{V}$ is the vector space of entire functions one adds various initial conditions requirements then the solutions are unique.

For example: $f_k^{(l)}(0) = \alpha^{\frac{l}{n}}\omega^{kl}$; $k,l \in \mathbf{Z}_n$ imply $f_k(z) = \exp_\psi[\sqrt[n]{\alpha}\ \omega^k z]$; $k \in \mathbf{Z}_n$

For example: $= f_k^{(l)}(0) = (1+(\alpha-1)\delta_{k,l})$; $k,l \in \mathbf{Z}_n$ imply $f_k(z) = h_{\psi,k}^{\alpha}(z)$; $k \in \mathbf{Z}_n$ where (see [20] and α - Example 5)

$$h_{\psi,s}^{\alpha}(z) = \frac{1}{n}\alpha^{\frac{-s}{n}} \sum_{k\in\mathbf{Z}_n} \omega^{-ks}\ \exp_\psi\left(\omega^k\ \sqrt[n]{\alpha}\ z\right)\ ;\ s \in \mathbf{Z}_n \qquad ((41), 50)$$

and

$$\exp_{\psi}\left(\omega^{l}\ \sqrt[n]{\alpha}\ z\right) = \sum_{k\in\mathbf{Z}_n} \alpha^{\frac{k}{n}}\omega^{kl}\ h^{\alpha}_{\psi,k}(x) \qquad ((42),51)$$

so that both sets of solutions are linear combinations: one of each other and with the above choice these are the formulas (41) and (42).

Naturally

$$\partial^{n}_{\psi} h^{\alpha}_{\psi,l} = \alpha h^{\alpha}_{\psi,l}\ ;\ l \in Z_n. \qquad ((43),52)$$

Remark 5. Consult the *Remark 3* and for further details - see [21].

Define the operator $\partial_R\ :\ P \to P$ as follows:

$$\partial_R \equiv R(q\mathcal{Q})\partial_o$$

where R is an analytic function on complex plane C or a formal Laurent series. It is not difficult to see that the R-Leibnitz rule has the form

$$\partial_R(f\bullet g)(z) = R(q\mathcal{Q})\{\partial_o(f\bullet g)\}(z) = R(q\mathcal{Q})\{(\partial_o f)(z)\bullet g(z) + f(0)(\partial_o g)(z)\}.$$

Then note: for $\psi_n(q) = \dfrac{1}{R(q^n)!}$ we have $\partial_\psi = \partial_R$ and for $R(x) = \dfrac{1-x}{1-q}$ we get $\partial_R = \partial_q$.

Hystorical ψ-remark:

We quote here Steven Roman (Chapter 6 p.162 in [26]) - with notation and reference changed into the one used in this note. *"Let $n_\psi! \equiv c_n$ be a sequence of nonzero constants. If $n!$ is replaced by c_n throughout the preceding theory, then virtually all of the results remain true, mutatis mutandis. In this way each sequence c_n gives rise to a distinct umbral calculus. Actually, Ward [30] seems to have been the first to suggest such a generalization (of the calculus of finite differences) in 1936, but the idea remained relatively undeveloped until quite recently, perhaps due to a feeling that it was mainly generalization for its own sake. Our purpose here is to indicate that this is not the case."*

Final remark:

Difference equations and functions that were considered here are quite elementary though seemingly new ones. Note that the likeness, resemblance-the similarity of $D = \frac{d}{dz}$ and ∂_q endomorphisms allows the powerful heuristics of ordinary differentiation to be brought into play. We hope that this note has given an idea of the scope of the possible investigation ahead.

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