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ON TRANSFER OPERATORS FOR C^*-DYNAMICAL SYSTEMS

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ABSTRACT. The theme of the paper is the question of existence and basic structure of transfer operators for endomorphisms of a unital C^*-algebra. We establish a complete description of non-degenerate transfer operators, characterize complete transfer operators and clarify their role in crossed product constructions. Also, we give necessary and sufficient conditions for existence of transfer operators for commutative systems, and discuss their form for endomorphisms of $\mathcal{B}(H)$ which is relevant to the Kadison-Singer problem.

Introduction. In [**10**] Exel introduced a notion of *transfer operator* for C^*-dynamical systems as a natural generalization of the corresponding notion from classical dynamics–the Ruelle-Perron-Frobenius operator. His aim was to use transfer operators in a construction of crossed-products associated with irreversible C^*-dynamical systems, and one of the problematic issues was the dependence of his construction on the choice of a transfer operator which usually is not unique. This problem was to some extent circumvented recently by Bakhtin and Lebedev who introduced in [**6**] a notion of complete transfer operator which, if it exists, is a necessarily unique non-degenerate transfer operator and is a sufficient tool to deal with the most important types of crossed-products, see [**4, 17**]. Thus, in this context, but also for future potential applications in noncommutative dynamics, cf. [**5, 11**], it is essential to understand the structure and the relationship between the non-degenerate and complete transfer operators. So far, however, this topic was not thoroughly investigated and our objective is a response to this deficiency.

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We start with pushing to the limit a relation touched upon in [**10**, Proposition 2.6] which results with a complete description of non-degenerate transfer operators via conditional expectations, Theorem 1.6. We improve investigations of [**6**] by showing that a complete transfer operator for an endomorphism α exists if and only if α has a unital kernel and a hereditary range. This allows us to give an explicit definition of complete transfer operators (Definition 1.7), characterize them in terms of Hilbert C^*-modules (Proposition 1.9), and clarify their predominant role in the theory of crossed-products by endomorphisms (Remark 1.10).

In Section 2 we illustrate the problem of uniqueness and existence of transfer operators including a brief survey of commutative systems where transfer operators are related to the so-called averaging operators [**8, 21**]. In particular, Theorem 2.5 presents necessary and sufficient conditions for the existence, respectively, of non-zero and non-degenerate transfer operators.

In the closing section we investigate C^*-dynamical systems on the algebra $\mathcal{B}(H)$ of all bounded operators on a separable Hilbert space H. We incline toward the point of view that an endomorphism $\alpha : \mathcal{B}(H) \to \mathcal{B}(H)$ of index n may be considered as a non-commutative n-to-one mapping, and thus we represent transfer operators for $(\mathcal{B}(H), \alpha)$ as integrals on the space consisting of n-elements. This description includes all possible transfer operators in the case n is finite, and all normal and certain singular operators in the case $n = \infty$, Theorems 3.3 and 3.6. The complete description when $n = \infty$ is challenging and is related to the celebrated Kadison-Singer problem [**13**, page 10].

1. Transfer operators and their characterizations. Throughout the paper we let $\mathcal{A}$ be a C^*-algebra with an identity 1 and $\alpha : \mathcal{A} \to \mathcal{A}$ an endomorphism of this C^*-algebra, referring to the pair $(\mathcal{A}, \alpha)$ as to a C^*-dynamical system. We start with recalling basic definitions and facts concerning transfer operators, cf. [**6, 10**].

Definition 1.1. A linear map $\mathcal{L} : \mathcal{A} \to \mathcal{A}$ is called a transfer operator for $(\mathcal{A}, \alpha)$ if it is continuous, positive and such that

$$\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b), \quad a, b \in \mathcal{A}. \tag{1}$$

If $\mathcal{L}$ is a transfer operator for $(\mathcal{A},\alpha)$, by passing to adjoints one gets the symmetrized version of (1):

$$\mathcal{L}(b\alpha(a)) = \mathcal{L}(b)a, \quad a, b \in \mathcal{A}.$$

This implies that the range of $\mathcal{L}$ is a two-sided ideal and $\mathcal{L}(1)$ is a positive central element in $\mathcal{A}$. Clearly, every pair $(\mathcal{A},\alpha)$ admits a transfer operator, namely, the zero operator will do. In fact, as we shall see, there are C^*-dynamical systems for which the only transfer operator is the zero one. In order to avoid such degenerated cases, it is natural to impose additional conditions such as presented in the following

Proposition 1.2 ([**10**, Proposition 2.3]). *Let $\mathcal{L}$ be a transfer operator for the pair $(\mathcal{A},\alpha)$. Then the following are equivalent*:

i) $E = \alpha \circ \mathcal{L}$ *is a conditional expectation from $\mathcal{A}$ onto $\alpha(\mathcal{A})$*,

ii) $\alpha \circ \mathcal{L} \circ \alpha = \alpha$,

iii) $\alpha(\mathcal{L}(1)) = \alpha(1)$.

Definition 1.3 ([**10**, Definition 2.3]). A transfer operator $\mathcal{L}$ is said to be *non-degenerate* if the equivalent conditions of Proposition 1.2 hold.

In connection with condition i) of Proposition 1.2, one easily sees that a conditional expectation from $\mathcal{A}$ onto $\alpha(\mathcal{A})$ is uniquely determined by its restriction to $\alpha(1)\mathcal{A}\alpha(1)$ which yields a conditional expectation from $\alpha(1)\mathcal{A}\alpha(1)$ onto $\alpha(\mathcal{A})$. This observation implies

Proposition 1.4. *A transfer operator $\mathcal{L}$ for $(\mathcal{A},\alpha)$ is non-degenerate if and only if $E = \alpha \circ \mathcal{L}$ is a conditional expectation from $\alpha(1)\mathcal{A}\alpha(1)$ onto $\alpha(\mathcal{A})$.*

An essential part of structure and a necessary condition for the existence of non-degenerate transfer operators is presented in the following proposition which is an improvement of [**10**, Proposition 2.5] and the corresponding results in [**6**]. For an ideal I in $\mathcal{A}$ we denote by $I^{\perp} = \{a \in \mathcal{A} : aI = 0\}$ its annihilator.

Proposition 1.5. *Suppose that there exists a non-degenerate transfer operator for* $(\mathcal{A},\alpha)$*. Then* $\ker\alpha$ *is unital and hence* $\mathcal{A}$ *admits the decomposition*

$$\mathcal{A} = \ker\alpha \oplus (\ker\alpha)^{\perp}.$$

Moreover, if $\mathcal{L}$ *is a non-degenerate transfer operator for* $(\mathcal{A},\alpha)$*, then*

i) $\mathcal{L}(\mathcal{A}) = (\ker\alpha)^{\perp}$*, in particular,* $1-\mathcal{L}(1)$ *is the unit in* $\ker\alpha$*,*

ii) $\mathcal{L}:\alpha(\mathcal{A})\to\mathcal{L}(\mathcal{A})$ *is a* $*$*-isomorphism uniquely determined by* α*. Namely, it is the inverse to the* $*$*-isomorphism* $\alpha:\mathcal{L}(\mathcal{A}) = (\ker\alpha)^{\perp}\to\alpha(\mathcal{A})$*.*

Proof. That $\mathcal{A} = \ker\alpha\oplus\mathcal{L}(A)$ follows from [**10**, Proposition 2.5]. Since however $\alpha(1)$ is a projection, $\alpha(\mathcal{L}(1)) = \alpha(1)$ and $\alpha:(\ker\alpha)^{\perp}\to\alpha(\mathcal{A})$ is a $*$-isomorphism, we get that $\mathcal{L}(1) = (\alpha|_{\mathcal{L}(\mathcal{A})})^{-1}(\alpha(1))$ is a (central) projection in $\mathcal{A}$. Thus, $1-\mathcal{L}(1)$ is the unit in $\ker\alpha$ and $\mathcal{L}(\mathcal{A}) = (\ker\alpha)^{\perp}$. Using $\alpha\circ\mathcal{L}\circ\alpha = \alpha$ one sees that $\mathcal{L}:\alpha(\mathcal{A})\to\mathcal{L}(\mathcal{A})$ coincides with $(\alpha|_{(\ker\alpha)^{\perp}})^{-1}$. □

Combining Propositions 1.4 and 1.5 we get

Theorem 1.6. *There exists a non-degenerate transfer operator for* $(\mathcal{A},\alpha)$ *if and only if* $\ker\alpha$ *is unital, and there exists a conditional expectation* $E:\alpha(1)\mathcal{A}\alpha(1)\to\alpha(\mathcal{A})$*.*

More precisely, if $\ker\alpha$ *is unital we have a one-to-one correspondence between non-degenerate transfer operators* $\mathcal{L}$ *for* $(\mathcal{A},\alpha)$ *and conditional expectations* E *from* $\alpha(1)\mathcal{A}\alpha(1)$ *onto* $\alpha(\mathcal{A})$ *established via the formulae*

$$E = \alpha\circ\mathcal{L}|_{\alpha(1)\mathcal{A}\alpha(1)},\qquad \mathcal{L}(a) = \alpha^{-1}(E(\alpha(1)a\alpha(1))),\quad \alpha\in\mathcal{A},$$

where α^{-1} *is the inverse to the isomorphism* $\alpha:(\ker\alpha)^{\perp}\to\alpha(\mathcal{A})$*.*

Proof. In view of Propositions 1.4 and 1.5, it suffices to check that $\mathcal{L}(a) = \alpha^{-1}(E(\alpha(1)a\alpha(1)))$ is a non-degenerate transfer operator which is straightforward. □

In view of the above the nicest situation one may imagine is that when $\alpha(\mathcal{A}) = \alpha(1)\mathcal{A}\alpha(1)$ which holds if and only if $\alpha(\mathcal{A})$ is a hereditary subalgebra of $\mathcal{A}$, cf. [**10**, Proposition 4.1]. Then the only conditional expectation we may consider is the identity, and thereby (if $\ker\alpha$ is unital) there is a unique non-degenerate transfer operator for α. The operator arising in this manner obviously satisfies

$$\alpha(\mathcal{L}(a)) = \alpha(1)a\alpha(1), \quad a \in \mathcal{A},$$

and hence it coincides with the one called by Bakhtin and Lebedev [**6, 17**, subsection 2.3] a *complete transfer operator*. Thus, contrary to [**6, 17**], we may give a definition of this notion in terms intrinsic to the system $(\mathcal{A}, \alpha)$.

Definition 1.7. We shall say that $(\mathcal{A}, \alpha)$ admits a complete transfer operator if $\alpha : \mathcal{A} \to \mathcal{A}$ has a unital kernel and a hereditary range. Then, by Theorem 1.6, there is a unique non-degenerate transfer operator for $(\mathcal{A}, \alpha)$ given by

$$\mathcal{L}(a) = \alpha^{-1}(\alpha(1)a\alpha(1)), \quad a \in \mathcal{A},$$

where α^{-1} is the inverse to the isomorphism $\alpha : (\ker\alpha)^\perp \to \alpha(\mathcal{A})$. We shall call $\mathcal{L}$ a complete transfer operator for $(\mathcal{A}, \alpha)$.

We now characterize complete transfer operators via Hilbert C^*-modules which will allow us to reveal the relationship between the various crossed products. We recall, [**12**, Example 16] and [**15**], that there is a natural structure of a C^*-correspondence over $\mathcal{A}$ on the space $E := \alpha(1)\mathcal{A}$ given by

$$a \cdot x := \alpha(a)x, \qquad x \cdot a := xa,$$

and

$$\langle x, y\rangle_{\mathcal{A}} := x^*y, \qquad x, y \in E, \quad a \in \mathcal{A}.$$

So, in particular, E is a right Hilbert $\mathcal{A}$-module equipped with a left action by adjointable maps, cf. [**20**]. Such objects are sometimes called Hilbert bimodules [**12**]. However, there are reasons to reserve this term for an object with an additional structure. Namely, as in [**1, 7**], we adopt the following

Definition 1.8. A Hilbert $\mathcal{A}$-bimodule is a Banach space E which is both right Hilbert $\mathcal{A}$-module and left Hilbert $\mathcal{A}$-module with $\mathcal{A}$-valued inner products $\langle\cdot,\cdot\rangle_{\mathcal{A}}$ and ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ connected via the so-called imprimitivity condition

$$x \cdot \langle y, z\rangle_{\mathcal{A}} = {}_{\mathcal{A}}\langle x, y\rangle \cdot z, \quad \text{for all } x, y, z \in E.$$

Now, let E be a C^*-correspondence of a C^*-dynamical system $(\mathcal{A}, \alpha)$.

Proposition 1.9. *There exists a left $\mathcal{A}$-valued inner product ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ making E (with its predefined left action) into a Hilbert bimodule if and only if $(\mathcal{A}, \alpha)$ admits a complete transfer operator. Moreover, if $\mathcal{L}$ is a complete transfer operator, then*

$${}_{\mathcal{A}}\langle x, y\rangle = \mathcal{L}(xy^*), \qquad \mathcal{L}(a) = {}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1)\rangle, \quad x, y \in E,\ a \in \mathcal{A}.$$

Proof. If $\mathcal{L}$ is a complete transfer operator, then ${}_{\mathcal{A}}\langle x, y\rangle := \mathcal{L}(xy^*)$ makes E a left semi-inner $\mathcal{A}$-module by (1) and positivity of $\mathcal{L}$. Since, for $x \in E = \alpha(1)\mathcal{A}$, we have $xx^* \in \alpha(1)\mathcal{A}\alpha(1) = \alpha(\mathcal{A})$ and $\mathcal{L}$ is faithful on $\alpha(\mathcal{A})$, ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$ is non-degenerate. Furthermore, for $x, y, z \in E$,

$${}_{\mathcal{A}}\langle x, y\rangle \cdot z = \alpha(\mathcal{L}(xy^*))z = \alpha(1)xy^*\alpha(1)z = xy^*z = x \cdot \langle y, z\rangle_{\mathcal{A}}.$$

Hence, E is a Hilbert $\mathcal{A}$-bimodule. Conversely, if E is a Hilbert $\mathcal{A}$-bimodule and $\mathcal{L}(a) := {}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1)\rangle$, $a \in \mathcal{A}$, then

$$\begin{aligned}\mathcal{L}(\alpha(b)a) &= {}_{\mathcal{A}}\langle \alpha(1)\alpha(b)a, \alpha(1)\rangle = {}_{\mathcal{A}}\langle b \cdot \alpha(1)a, \alpha(1)\rangle \\ &= b\,{}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1)\rangle = b\mathcal{L}(a)\end{aligned}$$

and

$$\begin{aligned}\alpha(\mathcal{L}(a)) &= \alpha({}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1)\rangle) = \alpha({}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1)\rangle)\alpha(1) \\ &= {}_{\mathcal{A}}\langle \alpha(1)a, a(1)\rangle \cdot \alpha(1) = \alpha(1)a \cdot \langle \alpha(1), \alpha(1)\rangle_{\mathcal{A}} \\ &= \alpha(1)a\alpha(1).\end{aligned}$$

Moreover, using the fact that $\mathcal{A}$ acts on E from the right via operators adjointable with respect to ${}_{\mathcal{A}}\langle\cdot,\cdot\rangle$, cf. [**7**, Remark 1.9], we get

$$\mathcal{L}(aa^*) = {}_{\mathcal{A}}\langle \alpha(1)aa^*, \alpha(1)\rangle = {}_{\mathcal{A}}\langle \alpha(1)a, \alpha(1)a\rangle \geq 0.$$

Hence, $\mathcal{L}$ is the complete transfer operator for $(\mathcal{A}, \alpha)$. ◻

Remark 1.10. If $(\mathcal{A}, \alpha)$ admits a complete transfer operator and E is the corresponding bimodule, then similarly as in [**12**, Example 16], cf. [**15, 18**], one may see that there is a one-to-one correspondence between covariant representations of $(\mathcal{A}, \alpha)$, see [**4**, subsection 2.4], [**17**, subsection 3.1] and covariant representations of the bimodule E, see [**1**, subsection 2.1]. Thus the *crossed product* $\mathcal{A}\rtimes\mathbf{Z}$ considered in [**4**] (and as a particular case in [**17**]) coincides with the *crossed product* $\mathcal{A}\rtimes_E\mathbf{Z}$ by the Hilbert bimodule E introduced in [**1**]. Furthermore, by [**4**, Theorem 4.15], these algebras coincide with *Exel's crossed product* $\mathcal{A}\rtimes_{\alpha,\mathcal{L}}\mathbf{Z}$, and by [**14**, Proposition 3.7], with Katsura's version of the Cuntz-Pimsner algebra $\mathcal{O}_E$. The characteristic feature of the algebra considered is that $\mathcal{A}$ embeds in it as the core–the fixed point algebra of the dual action, cf. [**1**, Theorem 3.1]. This means that most (semi-saturated) partial isometric crossed-products may be reduced to the construction based on a C^*-dynamical system with a complete transfer operator, cf. [**4, 15**].

2. Transfer operators for commutative systems. In this section we assume that $\mathcal{A}$ is commutative. Then $\mathcal{A} = C(X)$ where X is a compact Hausdorff space and it is well known, see for instance [**16**, Theorem 2.2], that α is of the form

$$\alpha(a)(x) = \begin{cases} a(\gamma(x)) & x \in \Delta, \\ 0 & x \notin \Delta, \end{cases} \qquad a \in C(X), \tag{2}$$

where Δ is a clopen subset of X and $\gamma : \Delta \to X$ is a continuous map. For any compact Hausdorff space Y we let $\mathrm{Mes}\,(Y)$ denote the space of all finite regular positive Borel measures on Y endowed with the weak* topology.

Proposition 2.1. *A mapping $\mathcal{L}$ is a transfer operator for the commutative system $(\mathcal{A}, \alpha)$ if and only if it is of the form*

$$\mathcal{L}(a)(x) = \begin{cases} \int_{\gamma^{-1}(x)} a(y)\, d\mu_x(y) & x \in \mathrm{Int}\,(\gamma(\Delta)), \\ 0 & x \in \overline{X \setminus \gamma(\Delta)}, \end{cases} \qquad a \in \mathcal{A}, \tag{3}$$

where the measure μ_x *is supported on* $\gamma^{-1}(x)$ *and the mapping*

$$\mathrm{Int}\,(\gamma(\Delta)) \ni x \longmapsto \mu_x \in \mathrm{Mes}\,(\Delta)$$

is continuous and vanishing at infinity in the sense that, for every $a \in \mathcal{A}$ *and every* $\varepsilon > 0$ *the set* $\{x : |\int_{\gamma^{-1}(x)} a(y)\, d\mu_x(y)| \geq \varepsilon\}$ *is compact in* $\mathrm{Int}\,(\gamma(\Delta))$. *In particular,*

i) $\mathcal{L}$ *is non-degenerate if and only if* $\gamma(\Delta)$ *is open and* μ_x *is a probability measure for* $x \in \gamma(\Delta)$,

ii) $\mathcal{L}$ *is complete if and only if* $\gamma(\Delta)$ *is open,* $\gamma : \Delta \to \gamma(\Delta)$ *is a homeomorphism, and then*

$$\mathcal{L}(a)(x) = \begin{cases} a(\gamma^{-1}(x)) & x \in \gamma(\Delta), \\ 0 & x \in X \setminus \gamma(\Delta), \end{cases} \qquad a \in \mathcal{A}.$$

Proof. It is well known, cf. [**8, 21**], that a positive operator $\mathcal{L} : C(X) \to C(X)$ is continuous if and only if the mapping $X \ni x \longmapsto \mu_x \in \mathrm{Mes}\,(X)$ where $\mu_x(a) := \mathcal{L}(a)(x)$, $a \in C(X)$, is continuous. Furthermore, if we assume that $\mu_x = 0$ for $x \in \overline{X \setminus \gamma(\Delta)}$, then the mapping $X \ni x \longmapsto \mu_x \in \mathrm{Mes}\,(X)$ is continuous if and only if its restriction to $\mathrm{Int}\,(\gamma(\Delta))$ is continuous and vanishing at infinity.

In view of the above formula (3) defines a bounded positive operator $\mathcal{L}$ which clearly satisfies (1). Conversely, if $\mathcal{L}$ is a transfer operator for $(\mathcal{A}, \alpha)$, then $\mu_x = 0$ for $x \in \overline{X \setminus \gamma(\Delta)}$. Indeed, for every $h \in C(X)$ such that $h(\gamma(\Delta)) = 1$ we have $\alpha(h) = \alpha(1)$, and hence

$$\mu_x(a) = \mathcal{L}(a)(x) = \mathcal{L}(\alpha(1)a)(x) = h(x)\mathcal{L}(a)(x) = h(x)\mu_x(a), \quad a \in \mathcal{A}.$$

This, together with the Urysohn lemma, proves our claim. Let $x \in \mathrm{Int}\,(\gamma(\Delta))$. Equation (1) implies that $\int_X a\, d\mu_x = \int_\Delta a\, d\mu_x$ and $\int_\Delta (a \circ \gamma)(y)\, d\mu_x(y) = \mathcal{L}(1)(x)a(x)$, $a \in C(X)$. Using these one gets $\mathrm{supp}\,\mu_x \subset \gamma^{-1}(x)$, and thus $\mathcal{L}$ is of form (3). Items i) and ii) follow immediately from Proposition 1.5 and Definition 1.7. $\square$

As we shall see in Corollary 2.7 and Proposition 3.1, it often happens that uniqueness of non-degenerate transfer operators is equivalent to completeness; however, in general this is not true.

Example 2.2. Let $X = [0,1] \cup \{2\}$ and $\gamma|_{[0,1]} = id$, $\gamma(2) = 0$. For a continuous mapping $[0,1] \ni x \longmapsto \mu_x \in \mathrm{Mes}\,(X)$ such that μ_x is a probability measure supported on $\gamma^{-1}(x)$, one clearly must have $\mu_x = \delta_x$. Thus, the system $(C(X), \alpha)$, where $\alpha(a) = a \circ \gamma$, has a unique non-degenerate transfer operator, even though it does not admit a complete transfer operator.

It follows from Proposition 2.1 that non-degenerate transfer operators for $(\mathcal{A}, \alpha)$ correspond to norm one left inverses of the composition operator $\alpha : C(\gamma(\Delta)) \to C(\Delta)$, that is, to the so-called *regular averaging operators* for $\gamma : \Delta \to \gamma(\Delta)$, cf. [**8**].

We now adapt the results of [**8**] to obtain criteria for existence of the transfer operators.

Definition 2.3. If X, Y are compact Hausdorff spaces, we denote by $F(X)$ the collection of all closed subsets of X and say that a mapping $\Phi : Y \to F(X)$ is *lower semi-continuous* (l.s.c) if for every open $V \subset X$ the set $\{x \in Y : V \cap \Phi(x) \neq \varnothing\}$ is open in Y.

By a *section* of $\gamma : \Delta \to X$ we mean a mapping $\Phi : \gamma(\Delta) \to F(\Delta)$ such that $\Phi(x) \subset \gamma^{-1}(x)$ for all $x \in \gamma(\Delta)$.

Proposition 2.4. *A necessary condition that* $(\mathcal{A}, \alpha)$ *admits*

i) *a non-zero transfer operator is that* $\gamma : \Delta \to \gamma(\Delta)$ *admits an l.s.c. section not identically equal to* $\varnothing$;

ii) *a non-degenerate transfer operator is that* $\gamma(\Delta)$ *is open in* X *and* $\gamma : \Delta \to \gamma(\Delta)$ *admits an l.s.c. section taking values in* $F(\Delta) \setminus \{\varnothing\}$.

Proof. It is known, cf. [**8**, Corollary 3.2], that for any continuous operator $\mathcal{L} : C(X) \to C(X)$ the mapping $X \ni x \mapsto \mathrm{supp}\,\mu_x \in F(X)$ where $\mu_x(a) := \mathcal{L}(a)(x)$, $a \in C(X)$, is l.s.c. Hence, the statement follows from Proposition 2.1. $\square$

As a non-trivial application of the above result we note that the only lower semi-continuous section for the Cantor map $\gamma : [0,1] \to [0,1]$ (which graph is Devil's staircase) is $\Phi \equiv \varnothing$, and hence there are no non-zero transfer operators for $(C([0,1]), \alpha)$ where $\alpha(a) = a \circ \gamma$. For metrizable spaces, the conditions from Proposition 2.4 are not only necessary but also sufficient.

Theorem 2.5. *If X is a metric space, then the system $(\mathcal{A},\alpha)$ admits*

i) *a non-zero transfer operator if and only if $\gamma:\Delta\to\gamma(\Delta)$ admits an l.s.c. section not identically equal to $\varnothing$.*

ii) *a non-degenerate transfer operator if and only if $\gamma(\Delta)$ is open in X and $\gamma:\Delta\to\gamma(\Delta)$ admits an l.s.c. section taking values in $F(\Delta)\setminus\{\varnothing\}$.*

Proof. To show item i), let $\Phi:\gamma(\Delta)\to F(\Delta)$ be an l.s.c. section for γ not identically equal to $\varnothing$. Then the set $V:=\{x\in\gamma(\Delta):\Phi(x)\neq\varnothing\}$ is open and not empty. By the Urysohn lemma there exists a non-zero continuous mapping $h:X\to[0,1]$ with a support contained in a closed subset $K\subset V$. By [**8**, Theorem 3.4], there exists a regular averaging operator $U:C(\gamma^{-1}(K))\to C(K)$ for $\gamma:\gamma^{-1}(K)\to K$, and one easily sees that the formula

$$\mathcal{L}(a)(x):=\begin{cases} h(x)(Ua|_{\gamma^{-1}(K)})(x) & x\in K,\\ 0 & x\in X\setminus K,\end{cases}$$

defines a non-zero transfer operator for $(\mathcal{A},\alpha)$. Item ii) follows from Proposition 2.1 i) and [**8**, Theorem 3.4]. □

We end this section by discussing the case of finite-to-one mappings. To this end for every transfer operator $\mathcal{L}$ we define a function $\rho:\Delta\to[0,\infty)$ by the formula

$$\rho(x):=\mu_{\gamma(x)}(\{x\}),\quad x\in\Delta, \tag{4}$$

where $\mu_x\in\mathrm{Mes}\,(\Delta)$, $x\in\gamma(\Delta)$, is given by $\mu_x(a)=\mathcal{L}(a)(x)$. In general, this function is not continuous, does not determine operator $\mathcal{L}$ uniquely, and not every continuous function might serve as ρ. We have, however, the following

Proposition 2.6. *If the map determined by* (2) *is finite-to-one, then every transfer operator for $(\mathcal{A},\alpha)$ is of the form*

$$\mathcal{L}(a)(x)=\begin{cases}\sum_{y\in\gamma^{-1}(x)}\rho(y)a(y) & x\in\gamma(\Delta),\\ 0 & x\notin\gamma(\Delta),\end{cases}\quad a\in\mathcal{A}, \tag{5}$$

where $\rho:\Delta\to[0,1)$ is a function given by (4). *Moreover,*

i) *if there exists an open cover $\{V_i\}_{i=1}^n$ of Δ such that γ is restricted to V_i, $i=1,\dots,n$, is one-to-one, then ρ is necessarily continuous,*

ii) *if item* i) *holds and* $\gamma : \Delta \to X$ *is an open map* (*that is* $\gamma : \Delta \to \gamma(\Delta)$ *is a local homeomorphism and* $\gamma(\Delta) \subset X$ *is open*), *then every continuous function* $\rho : \Delta \to [0, \infty)$ *defines via* (5) *a transfer operator for* $(\mathcal{A}, \alpha)$.

Proof. As for each $x \in \gamma(\Delta)$ the support $\operatorname{supp} \mu_x \subset \gamma^{-1}(x)$ consists of a finite number of points, formula (5) follows from (3) and (4).

i) Let $\{h_i\}_{i=1}^n \subset C(\Delta)$ be a partition of unity subject to $\{V_i\}_{i=1}^n$. Treating h_i as an element of $C(X)$, we get

$$\alpha(\mathcal{L}(h_i))(x) = \mathcal{L}(h_i)(\gamma(x)) = \rho(x)h_i(x), \quad x \in V_i.$$

Hence, for each $i = 1, \dots, n$, the function $\rho(x)h_i(x)$ is continuous on Δ, and thus $\rho = \sum_{i=1}^n \rho \cdot h_i$ is continuous as well.

ii) Is straightforward since for the local homeomorphism $\gamma : \Delta \to \gamma(\Delta)$, the number of elements in $\gamma^{-1}(x)$ for $x \in \gamma(\Delta)$ is constant on every component of $\gamma(\Delta)$ and $\gamma(\Delta)$ is clopen. □

Corollary 2.7. *If* $\gamma : \Delta \to \gamma(\Delta)$ *is a local homeomorphism and* $\gamma(\Delta) \subset X$ *is open, then* (5) *establishes a one-to-one correspondence between the non-degenerate transfer operators for* $(\mathcal{A}, \alpha)$ *and continuous functions* $\rho : \Delta \to [0, 1)$ *such that* $\sum_{\gamma(x)=\gamma(y)} \rho(y) = 1$ *for all* $x \in \Delta$. *In particular, there is a unique non-degenerate transfer operator for* $(\mathcal{A}, \alpha)$ *if and only if it is a complete transfer operator.*

3. Transfer operators for systems on $\mathcal{B}(H)$.

Let H be a separable Hilbert space. We recall that, due to [**23**, Theorem 1], every bounded positive linear map $T : \mathcal{B}(H) \to \mathcal{B}(H)$ has a unique decomposition into its normal and singular part:

$$T = T_n + T_s$$

where T_n, T_s are bounded positive linear operators, T_n is normal and T_s is singular (T_s annihilates the algebra $\mathcal{K}(H)$ of compact operators). Moreover, taking into account the explicit form of Tomiyama's decomposition, see [**23**], one sees that if T is an endomorphism (respectively a transfer operator, or a conditional expectation) T_n, T_s are again endomorphisms (respectively transfer operators, conditional expectations).

In particular, since there is no non-zero representation of the Calkin algebra on a separable Hilbert space, every endomorphism α of $\mathcal{B}(H)$ is normal, and the classification of transfer operators for $(\mathcal{B}(H),\alpha)$ splits into classification of all normal and all singular ones.

We fix an endomorphism $\alpha:\mathcal{B}(H)\to\mathcal{B}(H)$. Normality of α implies that there exists a number $n=1,2,\dots,\infty$ called an *index* for α, and a family of isometries U_i, $i=1,\dots,n$ with orthogonal ranges such that

$$\alpha(a)=\sum_{i=1}^{n}U_iaU_i^*,\quad a\in\mathcal{B}(H), \tag{6}$$

where (if $n=\infty$) the sum is weakly convergent, see, for instance, [**19**]. As transfer operators for $(\mathcal{B}(H),\alpha)$ are completely determined by their action on the algebra $\alpha(1)\mathcal{B}(H)\alpha(1)$ we note that $\alpha(1)\mathcal{B}(H)\alpha(1)$ may be presented as the following W^*-tensor product

$$\alpha(1)\mathcal{B}(H)\alpha(1)=\mathcal{B}(H_0)\otimes\mathcal{B}(H_n)=\mathcal{B}(H_0\otimes H_n)$$

where $\dim H_0=\infty$ and $\dim H_n=n$. To be more precise, this decomposition is obtained assuming the following identification

$$a\otimes b=U^*aU\sum_{i,j=1}^{n}b(i,j)U_iU_j^* \tag{7}$$

where the sum is weakly convergent, $\{b(i,j)\}_{i,j=1}^n$ is a matrix of $b\in\mathcal{B}(H_n)$ with respect to a fixed orthonormal basis $\{e_i\}_{i=1}^n$ for H_n, and U is the isometry U_{i_0} for an arbitrarily chosen but fixed index $i_0=1,\dots,n$ (then $H_0=UU^*H$), cf. [**9, 19**]. Under these identifications, the homomorphism $\alpha:\mathcal{B}(H)\to\mathcal{B}(H_0\otimes H_n)\subset\mathcal{B}(H)$ has an especially simple form: $\alpha(a)=UaU^*\otimes 1$, $a\in\mathcal{B}(H)$, and in particular, $\alpha(\mathcal{B}(H))=\mathcal{B}(H_0)\otimes 1$.

Proposition 3.1. *For every transfer operator $\mathcal{L}$ for $(\mathcal{B}(H),\alpha)$, there exists a positive linear functional ω on $\mathcal{B}(H_n)$ such that*

$$\mathcal{L}(a\otimes b)=U^*aU\omega(b),\qquad a\otimes b\in\mathcal{B}(H_0)\otimes\mathcal{B}(H_n). \tag{8}$$

In particular, $\mathcal{L}$ is non-degenerate if and only if ω is a state. Moreover,

i) *relation* (8) *establishes an isometric isomorphism between the cone of normal transfer operators* $\mathcal{L}$ *for* $(\mathcal{B}(H), \alpha)$ *and the cone of normal positive linear functionals* ω *on* $\mathcal{B}(H_n)$.

ii) $\mathcal{L}$ *is singular if and only if* ω *is singular* (*i.e., annihilates* $\mathcal{K}(H_n)$), *in particular, if* n *is finite then all transfer operators are normal.*

iii) *there is a unique non-degenerate transfer operator* $\mathcal{L}$ *for* $(\mathcal{B}(H), \alpha)$ *if and only if* $n = 1$, *and then* $\alpha(a) = UaU^*$ *and* $\mathcal{L}(a) = U^*aU$ *is a complete transfer operator.*

Proof. For any $b \in \mathcal{B}(H_n)$, $\mathcal{L}(1 \otimes b)$ commutes with every $a \in \mathcal{B}(H)$:

$$\begin{aligned} a\mathcal{L}(1 \otimes b) &= \mathcal{L}(\alpha(a) \cdot 1 \otimes b) = \mathcal{L}((UaU^* \otimes 1)(1 \otimes b)) \\ &= \mathcal{L}(UaU^* \otimes b) = \mathcal{L}((1 \otimes b)(UaU^* \otimes 1)) \\ &= \mathcal{L}((1 \otimes b)\alpha(a)) = \mathcal{L}(1 \otimes b)a. \end{aligned}$$

Thus we have $\mathcal{L}(1 \otimes b) = \omega(b) \cdot 1$ for certain $\omega(b) \in \mathbf{C}$, and this relation defines a positive linear functional on $\mathcal{B}(H_n)$ for which (8) holds.

i) That normality of $\mathcal{L}$ implies normality of ω is straightforward. Conversely, as tensors $a \otimes b$ are linearly strongly dense in $\mathcal{B}(H_0) \otimes \mathcal{B}(H_n)$, any normal ω determines uniquely via (8) a normal operator $\mathcal{L}$. Moreover, since U is an isometry, $\mathcal{L}$ is a transfer operator for $\alpha(a) = UaU^* \otimes 1$ and $\|\mathcal{L}\| = \|\omega\|$. Item ii) follows from the equality $\mathcal{K}(H) = \mathcal{K}(H_0) \otimes \mathcal{K}(H_n)$ and formula (8). Item iii) follows from item i). □

Combining the above statement with Proposition 1.6 one gets

Proposition 3.2 ([**22**, Proposition 2.4]). *There is a one-to-one correspondence between the normal conditional expectations* $E : \mathcal{B}(H_0 \otimes H_n) \to \mathcal{B}(H_0) \otimes 1$ *and normal states* ω *on* $\mathcal{B}(H_n)$, *established by the relation*

$$E(a \otimes b) = a \otimes \omega(b)1, \qquad a \otimes b \in \mathcal{B}(H_0) \otimes \mathcal{B}(H_n). \tag{9}$$

Moreover, if n *is finite every conditional expectation* $E : \mathcal{B}(H_0 \otimes H_n) \to \mathcal{B}(H_0) \otimes 1$ *is normal.*

Since every normal positive functional ω on $\mathcal{B}(H_n)$ is of the form $\omega(a) = \mathrm{Tr}\,(\rho a)$ where $\rho \in \mathcal{K}(H_n)$ is a positive trace class operator, one may deduce the following explicit form of normal transfer operators.

Theorem 3.3. *An operator $\mathcal{L} : \mathcal{B}(H) \to \mathcal{B}(H)$ is a normal transfer operator for $(\mathcal{B}(H), \alpha)$ if and only if it is of the form*

$$\mathcal{L}(a) = \sum_{i,j=1}^{n} \rho(i,j) U_i^* a U_j, \quad a \in \mathcal{B}(H), \tag{10}$$

where $\{U_i\}_{i=1}^n$ is a family of isometries satisfying (6) *and $\{\rho(i,j)\}_{i,j=1}^n$ is a matrix of a positive trace class operator $\rho \in \mathcal{K}(H_n)$, that is,*

$$\sum_{i,j} \rho(i,j) z_i \overline{z}_j \geq 0 \quad \textit{for all } z_i \in \mathbf{C}, \quad i = 1, \dots, n,$$

and

$$\mathrm{Tr}\,\rho = \sum_{i=1}^{n} \rho(i,i) < \infty.$$

Moreover, $\|\mathcal{L}\| = \mathrm{Tr}\,\rho$ and $\mathcal{L}$ is non-degenerate if and only if $\mathrm{Tr}\,\rho = 1$.

Proof. It suffices to check that (8) and (10) agree on simple tensors $a \otimes b \in \mathcal{B}(H) \otimes \mathcal{B}(H_n)$ where $\omega(a) = \mathrm{Tr}\,(\rho a)$. In view of (7) it is straightforward. □

Corollary 3.4. *If $n < \infty$, then every transfer operator for $(\mathcal{B}(H), \alpha)$ is of the form* (10) *for a positive definite matrix $\{\rho(i,j)\}_{i,j=1}^n$.*

In order to represent not necessarily normal transfer operators (in the case $n = \infty$) we shall replace the sum in (10) with a weak integral. We denote by $\mathrm{Mes}_{f.a}(\mathbf{N})$ the set of all positive finite finitely additive measures on $\mathbf{N}$ and recall that the Lebesgue integral $\int_{\mathbf{N}} f(i)\, d\mu(i)$, $f \in \ell^\infty$, $\mu \in \mathrm{Mes}_{f.a}(\mathbf{N})$ allows us to treat elements of $\mathrm{Mes}_{f.a}(\mathbf{N})$ as positive functionals on the algebra ℓ^∞ of bounded sequences on $\mathbf{N}$. Moreover, identifying ℓ^∞ with the corresponding atomic *masa* of $\mathcal{B}(H_n)$, one may use a form of "integration" (usually expressed via limits along ultrafilters) to extend these functionals from ℓ^∞ to $\mathcal{B}(H_n)$, cf. [**3, 13**]. We shall adopt this method to transfer operators with the help of the following simple consequence of the Riesz lemma.

Lemma 3.5. *For every $\mu \in Mes_{f.a}(\mathbf{N})$ and every bounded function $\phi : \mathbf{N} \to \mathcal{B}(H)$ there exists a unique bounded operator denoted by $\int_{\mathbf{N}} \phi(i)\, d\mu(i)$ satisfying*

$$\left\langle \int_{\mathbf{N}} \phi(i)\, d\mu(i)x, y \right\rangle = \int_{\mathbf{N}} \langle \phi(i)x, y\rangle\, d\mu(i), \quad \text{for all } x, y \in H.$$

Moreover, $\| \int_{\mathbf{N}} \phi(i)\, d\mu(i)\| = \sup_{i\in\mathbf{N}} \|\phi(i)\|$.

Theorem 3.6. *Let the index of α be $n = \infty$. For every $\mu \in \mathrm{Mes}_{f.a.}(\mathbf{N})$, the formula*

$$\mathcal{L}(a) = \int_{\mathbf{N}} U_i^* a U_i\, d\mu(i), \quad a \in \mathcal{B}(H), \tag{11}$$

defines a transfer operator for $(\mathcal{B}(H), \alpha)$. Moreover, $\|\mathcal{L}\| = \mu(\mathbf{N})$ and

i) *$\mathcal{L}$ is singular if and only if μ has no atoms.*

ii) *the positive linear functional ω associated with $\mathcal{L}$ via* (8) *is diagonalizable, that is, $\omega(a) = \int_{\mathbf{N}} \langle a e_i, e_i\rangle\, d\mu(i)$, $a \in \mathcal{B}(H_n)$, ($\{e_i\}_{i\in\mathbf{N}}$ is the fixed basis).*

iii) *$\mathcal{L}$ is non-degenerate if and only if $\mu(\mathbf{N}) = 1$, and then the conditional expectation $E = \alpha \circ \mathcal{L}$ is given by*

$$E(a) = \sum_{j=1}^{\infty} \int_{\mathbf{N}} U_j U_i^* a U_i U_j^*\, d\mu(i).$$

Proof. Plainly, $\mathcal{L}(a) = \int_{\mathbf{N}} U_i^* a U_i d\mu(i)$ is positive. It satisfies (1) as the ranges of isometries $\{U_i\}_{i\in\mathbf{N}}$ are orthogonal and hence, for $a, b \in \mathcal{B}(H)$, we have

$$\begin{aligned}\mathcal{L}(\alpha(a)b) &= \int_{\mathbf{N}} U_i^* \left(\sum_{j=1}^{n} U_j a U_j^* \right) b U_i d\mu(i) \\ &= \int_{\mathbf{N}} a U_i^* b U_i d\mu(i) = a\mathcal{L}(b).\end{aligned}$$

For $x, y \in H$, $\|x\| = \|y\| = 1$, we have $\langle \mathcal{L}(a)x, y\rangle| \leq \int_{\mathbf{N}} |\langle U_i^* a U_i x, y\rangle| \, d\mu(i) \leq \|a\|\mu(\mathbf{N})$, and as $\mathcal{L}(1) = 1 \cdot \mu(\mathbf{N})$, we get $\|\mathcal{L}\| = \mu(\mathbf{N})$.

i) It will follow from ii) and Proposition 3.1 ii).

ii) By (7), for $a \in \mathcal{B}(H_n)$ we have $1 \otimes a = \sum_{i,j=1}^{\infty} a(i,j) U_i U_j^* \in \mathcal{B}(H)$ where $a(i,j) = \langle a e_j, e_i\rangle$. Thus using orthogonality of the ranges of $\{U_i\}_{i \in \mathbf{N}}$ we get

$$\begin{aligned} 1 \cdot \omega(a) = \mathcal{L}(1 \otimes a) &= \int_{\mathbf{N}} U_i^* \Big(\sum_{k,j=1}^{\infty} a(k,j) U_k U_j^* \Big) U_i d\mu(i) \\ &= \int_{\mathbf{N}} 1 \cdot a(i,i) \, d\mu(i) = 1 \cdot \int_{\mathbf{N}} a(i,i) \, d\mu(i) \\ &= 1 \cdot \int_{\mathbf{N}} \langle a e_i, e_i \rangle \, d\mu(i). \end{aligned}$$

iii) Follows from the equality $\alpha(\mathcal{L}(1)) = \mu(\mathbf{N})\alpha(1)$. □

Corollary 3.7. *For every diagonalizable singular state ω on $\mathcal{B}(H_n)$, there exists a singular conditional expectation $E : \mathcal{B}(H_0 \otimes H_n) \to \mathcal{B}(H_0) \otimes 1$ satisfying* (9)*.*

Formula (11) describes transfer operators "diagonalizable" with respect to the basis $\{e_i\}_{i \in \mathbf{N}}$ in H_n. Changing this basis one gets more transfer operators. Namely, for every matrix $\{u(i,j)\}_{i,j=1}^{\infty}$ of a unitary operator $u \in \mathcal{B}(H_n)$; $u(i,j) = \langle u e_j, e_i \rangle$, $i, j \in \mathbf{N}$, the formula

$$\mathcal{L}(a) = \sum_{k,j=1}^{\infty} \int_{\mathbf{N}} u(k,i)\overline{u(k,j)} U_j^* a U_i \, d\mu(i), \quad a \in \mathcal{B}(H), \tag{12}$$

defines a transfer operator for $(\mathcal{B}(H), \alpha)$. In particular, taking into account all unitary matrices and all σ-additive measure on $\mathbf{N}$, (12) is equivalent to (10), that is, it describes all normal transfer operators. However, it is almost certain that (12) does not describe all transfer operators. The functionals associated with transfer operators from Theorem 3.6 are specific extensions of functionals on the algebra of all diagonalizable operators $\ell^{\infty} \subset \mathcal{B}(H_n)$, and if the Kadison-Singer conjecture is false such extensions even for pure states might not be unique. Moreover, in the light of the recent result of Akemann and Weaver [**2**] pure states exist which are not diagonalizable, i.e., states not having the form appearing in Theorem 3.6 ii). In order to clarify the role of the Kadison-Singer conjecture in such considerations we rephrase Theorem 3.6.

Proposition 3.8. *Let $n = \infty$. For every state $\omega \in (\ell^\infty)^*$ of the atomic masa $\ell^\infty \subset \mathcal{B}(H_n)$, there exists a non-degenerate transfer operator for $(\mathcal{B}(H), \alpha)$ satisfying*

$$\mathcal{L}(a \otimes b) = U^* a U \omega(b), \quad a \in \mathcal{B}(H_0),\ b \in \ell^\infty \subset \mathcal{B}(H_n), \tag{13}$$

and such that, if ω is a pure state on ℓ^∞, then the functional associated with $\mathcal{L}$ via (8) *is a pure extension of ω onto $\mathcal{B}(H_n)$.*

We have two relevant problems:

Problem 1. Does equation (8), where ω is a pure state of $\mathcal{B}(H_n)$ whose restriction to ℓ^∞ is pure, uniquely determine a non-degenerate transfer operator?

Problem 2. Does equation (13), where ω is a pure state of ℓ^∞, uniquely determine a non-degenerate transfer operator?

The positive answer to Problem 2 implies the positive answer to Problem 1. If the Kadison-Singer conjecture is true, then Problems 1 and 2 are equivalent, and finally if the answer to Problem 1 is positive then Problem 2 and the Kadison-Singer problem are equivalent.

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