C^*-RIGIDITY OF TOPOLOGICAL DYNAMICAL SYSTEMS

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1. Introduction

These are some notes I wrote in connection with a talk I gave at the conference *Cartan C^*-subalgebras and noncommutative dynamics* which took place at IMPAN in Warsaw, Poland, from 25 November to 28 November 2019.

These notes contains a short introduction for what is sometime called C^*-rigidity of topological dynamical systems.

I will give an overview of results about C^*-rigidity and explain how groupoids can be used to prove and generalise some of these results.

2. C^*-algebras of topological dynamical systems

There is a long tradition for constructing C^*-algebras from dynamical systems. Motivations for doing this include:

(1) constructing new examples of C^*-algebras which can be studied via dynamical systems,
(2) use operator algebra theory to study dynamical systems.

When one construct a C^*-algebra from a dynamical system, it is natural to ask how much information about the dynamical system can be obtained from the C^*-algebra. One might also ask if it is possible to recover the dynamical system from the C^*-algebra.

C^*-rigidity of dynamical systems is the principal that dynamical systems can be recovered, up to a suitable notion of equivalence, from C^*-algebraic data associated to them.

3. Cantor minimal systems

A good example of C^*-rigidity of dynamical systems is this theorem by Giordano, Putnam and Skau [9, Theorem 2.1].

Theorem 1 (Elliott 1993 and Giordano+Putnam+Skau 1995)**.** *Let (X, ϕ) and (Y, ψ) be Cantor minimal systems. TFAE:*

(1) $C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic.
(2) $K_0(C(X) \rtimes_\phi \mathbb{Z})$ and $K_0(C(Y) \rtimes_\psi \mathbb{Z})$ are isomorphic by an order preserving isomorphism that maps the class of the unit to the class of the unit.
(3) (X, ϕ) and (Y, ψ) are strong orbit equivalent.

A *Cantor minimal system* is a pair (X, ϕ) where X is a totally disconnect compact metric space with no isolated points and $\phi : X \to X$ is a homeomorphism such that there is no non-trivial closed subspace $C \subseteq X$ such that $\phi(C) = C$. The latter condition is equivalent to the condition that the *orbit* $\mathrm{orb}(x) := \{\phi^n(x) : x \in \mathbb{Z}\}$ of any $x \in X$ is dense in X.

Two Cantor minimal systems (X, ϕ) and (Y, ψ) are *strong orbit equivalent* if there is a homeomorphism $h : X \to Y$ and maps $m, n : X \to \mathbb{Z}$ such that $h(\phi(x)) = \psi^{m(x)}(h(x))$ and $h(\phi^{n(x)}(x)) = \psi(h(x))$ for $x \in X$, and m and n each have at most one point of discontinuity.

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The equivalence of (1) and (2) follows from a classification result proved by George Elliott in 1993 [8]. However, Giordano, Putnam and Skau proved this theorem by showing that if (X, ϕ) and (Y, ψ) are strong orbit equivalent, then the crossed products are isomorphic; and that if the K_0-groups of the crossed products are isomorphic by an order preserving isomorphism that maps the class of the unit to the class of the unit, then the two Cantor minimal systems are strong orbit equivalent. Elliott's classification result is therefore not needed to prove the theorem.

This theorem says that it is possible to recover a Cantor minimal system up to strong orbit equivalence from its C^*-crossed product. However, usually one needs more than just the crossed product to recover a dynamical system.

Giordano, Putnam, and Skau also gaven an example of this. They used a result of Mike Boyle [1] to show the following theorem [9, Theorem 2.4].

Theorem 2 (Boyle 1983 and Giordano+Putnam+Skau 1995)**.** *Let (X, ϕ) and (Y, ψ) be Cantor minimal systems. TFAE:*

(1) $C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.
(2) (X, ϕ) and (Y, ψ) are continuously orbit equivalent.
(3) (X, ϕ) and (Y, ψ) are flip conjugate.

Two Cantor minimal systems (X, ϕ) and (Y, ψ) are *continuously orbit equivalent* if there is a homeomorphism $h : X \to Y$ and continuous maps $m, n : X \to \mathbb{Z}$ such that $h(\phi(x)) = \psi^{m(x)}(h(x))$ and $h(\phi^{n(x)}(x)) = \psi(h(x))$ for $x \in X$; and they are flip conjugate if there is a homeomorphism $h : X \to Y$ such that either $h(\phi(x)) = \psi(h(x))$ for all $x \in X$, or $h(\phi(x)) = \psi^{-1}(h(x))$ for all $x \in X$.

The result of Boyle gives that (2) implies (3), and it is easy to check that (3) implies (1), and what Giordano, Putnam, and Skau did was to show that (1) implies (2).

4. Topologically transitive dynamical systems on compact spaces

A topologically dynamical system (X, ϕ) consisting of a topological space X and a homeomorphism $\phi : X \to X$ is *topologically transitive* if there is an $x \in X$ such that $\operatorname{orb}(x)$ is dense in X.

Shortly after the Giordano–Putnam–Skau paper, Jun Tomiyama showed in [15] that the previous theorem and its proof can be generalised to topologically transitive dynamical systems on compact metric spaces.

Theorem 3 (Boyle 1983 and Tomiyama 1996)**.** *Let (X, ϕ) and (Y, ψ) be topologically transitive dynamical systems on compact metric spaces X and Y. TFAE:*

(1) $C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.
(2) (X, ϕ) and (Y, ψ) are continuously orbit equivalent.
(3) (X, ϕ) and (Y, ψ) are flip conjugate.

A Cantor minimal system is topologically transitive, so this theorem is indeed a generalisation of the previous.

5. Topologically free dynamical systems on compact spaces

A topologically dynamical system (X, ϕ) consisting of a topological space X and a homeomorphism $\phi : X \to X$ is *topologically free* if the set $\{x \in X : \phi^n(x) \neq x \text{ for all } n \neq 0\}$ is dense in X.

A few years after the previous theorem, Boyle and Tomiyama together showed in [2] the following theorem.

Theorem 4 (Boyle and Tomiyama 1998)**.** *Let (X, ϕ) and (Y, ψ) be topologically free dynamical systems on compact Hausdorff spaces X and Y. TFAE:*

(1) $C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.
(2) (X, ϕ) and (Y, ψ) are continuously orbit equivalent.
(3) There exist decompositions $X = X_1 \sqcup X_2$ and $Y = Y_1 \sqcup Y_2$ such that X_1, X_2, Y_1, Y_2 are clopen and invariant, $\phi|_{X_1}$ is conjugate to $\psi|_{Y_1}$, and $\phi|_{X_2}$ is conjugate to $\psi^{-1}|_{Y_2}$.

A topologically transitive dynamical systems on an infinite space is topologically free. Moreover, if either X is connected or ϕ is transitive, then either X_1 or X_2 in (3) would have to be empty, and so condition (3) says in that case that ϕ and ψ are flip conjugate. This theorem is therefore a generalistion of the previous.

6. Homeomorphisms of compact Hausdorff spaces

Recently, Carlsen, Ruiz, Sims, and Tomforde partially generalised the previous theorem by proving the following theorem [6, Theorem 9.1].

Theorem 5 (Carlsen+Ruiz+Sims+Tomforde 2017)**.** *Let X and Y be second-countable compact Hausdorff spaces and $\phi : X \to X$ and $\psi : Y \to Y$ homeomorphisms. TFAE:*

(1) $C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.
(2) There exist decompositions $X = X_1 \sqcup X_2$ and $Y = Y_1 \sqcup Y_2$ such that X_1, X_2, Y_1, Y_2 are clopen and invariant, $\phi|_{X_1}$ is conjugate to $\psi|_{Y_1}$ and $\phi|_{X_2}$ is conjugate to $\psi^{-1}|_{Y_2}$.

Notice that there is no conditions on the homeomorphisms in this theorem.

7. C^*-dynamical systems

Two actions (A, α) and (B, β) of a locally compact group G on two C^*-algebras A and B are *conjugate* if there is an isomorphism $\psi : A \to B$ such that $\psi \circ \alpha_\gamma = \beta_\gamma \circ \psi$ for each $\gamma \in G$, and they are *outer conjugate* if (A, α) is conjugate to an action β' on B such that there is a strictly continuous unitary map $u : G \to M(B)$ such that $u_{\gamma_1\gamma_2} = u_{\gamma_1}\beta_{\gamma_1}(u_{\gamma_2})$ for $\gamma_1, \gamma_2 \in G$, and $\beta'_\gamma = \operatorname{Ad} \circ \beta_\gamma$ for $\gamma \in G$.

Recently, Kaliszewski, Omland, and Quigg have used a result by Gert Pedersen to show the following theorem [10].

Theorem 6 (Pedersen 1982, Kaliszewski+Omland+Quigg 2018)**.** *Let G be a locally compact group, let α be an action of G on a C^*-algebra A, and let β be an action of G on a C^*-algebra B. TFAE:*

(1) $\phi : A \rtimes_\alpha G$ and $\phi : B \rtimes_\beta G$ are isomorphic by an isomorphism that maps A onto B and intertwines the dual coactions $\widehat{\alpha}$ and $\widehat{\beta}$.
(2) (A, α) and (B, β) are outer conjugate.

In some cases, the condition that A is mapped onto B is redundant. For instance, we get from Imai—Takai—Takesaki duality the following result.

Theorem 7 (Takesaki 1972, Imai+Takai 1978)**.** *Let G be a locally compact group, let α be an action of G on a C^*-algebra A, and let β be an action of G on a C^*-algebra B. TFAE:*

(1) $\phi : A \rtimes_\alpha G$ *and* $\phi : B \rtimes_\beta G$ *are isomorphic by an isomorphism that intertwines the dual coactions* $\widehat{\alpha}$ *and* $\widehat{\beta}$.
(2) $(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \operatorname{Ad}\rho)$ *and* $(B \otimes \mathcal{K}(L^2(G)), \beta \otimes \operatorname{Ad}\rho)$ *are conjugate (here* ρ *is right regular representation of* G *on* $\mathcal{K}(L^2(G))$*).*

Kaliszewski, Omland, and Quigg have also proved that if G is a discrete group, then two actions of G on two C^*-algebras A and B are outer conjugate if and only if the corresponding crossed products are isomorphic by an isomorphism that intertwines the dual coactions.

Theorem 8 (Kaliszewski+Omland+Quigg 2019)**.** *Let* G *be a discrete group, let* α *be an action of* G *on a* C^**-algebra* A*, and let* β *be an action of* G *on a* C^**-algebra* B*. TFAE:*
(1) $\phi : A \rtimes_\alpha G$ *and* $\phi : B \rtimes_\beta G$ *are isomorphic by an isomorphism that intertwines the dual coactions* $\widehat{\alpha}$ *and* $\widehat{\beta}$.
(2) (A, α) *and* (B, β) *are outer conjugate.*

8. Actions on commutative C^*-algebras

Kaliszewski, Omland, and Quigg have also shown that if a locally compact group acts on two locally compact Hausdorff spaces, then the two actions are conjugate if and only if the corresponding crossed products are isomorphic by an isomorphism that intertwines the dual coactions.

Theorem 9 (Kaliszewski+Omland+Quigg 2019)**.** *Let* $G \curvearrowleft X$ *and* $G \curvearrowleft Y$ *be actions of a locally compact group on locally compact Hausdorff spaces. TFAE:*
(1) $C_0(X) \rtimes G \to C_0(Y) \rtimes G$ *are isomorphic by an isomorphism that intertwines the dual coactions.*
(2) *The actions* $G \curvearrowleft X$ *and* $G \curvearrowleft Y$ *are conjugate.*

9. One-sided topological Markov shifts and Cuntz–Krieger algebras

So far, the C^*-algebra we have been considering are crossed products of group actions, but there are also rigidity results for C^*-algebras constructed from irreversible dynamical systems. For simplicity, we will restric our attention to Cuntz–Krieger algebras, which can be considered as C^*-algebras associated with one-sided topological Markov shifts.

Let A be an $n \times n$ matrix with entries in $\{0, 1\}$ and with no zero rows and no zero columns. We let $\mathcal{O}_A$ be the Cuntz–Krieger algebra of A [7] and $\mathcal{D}_A$ be the C^*-subalgebra

$$\overline{\operatorname{span}}\{s_{i_1} \dots s_{i_k} s_{i_k}^* \dots s_{i_1}^* : i_1 \dots i_k \in \{0, 1\}^*\}.$$

We also let

$$X_A := \{(x_i)_{i \in \mathbb{N}} : A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{N}\},$$

equip X_A with the product topology, and define $\sigma_A : X_A \to X_A$ by $\sigma_A((x_i)_{i\in\mathbb{N}}) = (x_{i+1})_{i\in\mathbb{N}}$. Then σ_A is a surjective local homeomorphism.

We say that two one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are *continuously orbit equivalent* if there is a homeomorphism $h : X_A \to X_B$ and continuous maps $k, l : X_A \to \mathbb{N}$ and $k', l' : X_B \to \mathbb{N}$ such that

$$\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{l(x)}(h(x))$$

for $x \in X_A$, and

$$\sigma_A^{k'(x')}(h^{-1}(\sigma_B(x'))) = \sigma_A^{l'(x')}(h^{-1}(x'))$$

for $x' \in X_B$.

At first, the definition of continuous orbit equivalence might seem a bit strange, but a bijection h from X_A to X_B maps orbits in (X_A, σ_A) to orbits in (X_B, σ_B) if and only if there are maps k and l from X_A to $\mathbb{N}$ such that $\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{l(x)}(h(x))$ for $x \in X_A$. A continuous orbit equivalence is thus a homeomorphism from X_A to X_B that maps orbits to orbits and satisfies the extra condition that the maps k, l, k' and l' can be chosen to be continuous.

10. Continuous orbit equivalence of one-sided topological Markov shifts and diagonal-preserving isomorphism of Cuntz–Krieger algebras

The following results was first proved by Kengo Matsumoto for irreducible topological Markov shifts [11], and then for arbitrary topological Markov shifts by Carlsen, Eilers, Ortega, and Restroff [4].

Theorem 10 (Matsumoto 2010, Carlsen+Eilers+Ortega+Restorff 2019)**.** *Let (X_A, σ_A) and (X_B, σ_B) be one-sided topological Markov shifts. TFAE:*

(1) There is an isomorphism $\psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\psi(\mathcal{D}_A) = \mathcal{D}_B$.
(2) (X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.

11. Eventual conjugacy of one-sided topological Markov shifts and gauge-invariant diagonal preserving isomorphism of Cuntz–Krieger algebras

Let γ^A denote the gauge action on $\mathcal{O}_A$.

Two topological Markov shifts are eventually conjugate if there is a continuous orbit equivalence (h, k, l, k', l') between them such that l is $k + 1$ and l' is $k' + 1$.

The following results was first proved by Kengo Matsumoto for irreducible topological Markov shifts [12], and then for arbitrary topological Markov shifts by Carlsen and Rout [5].

Theorem 11 (Matsumoto 2017, Carlsen+Rout 2017)**.** *Let (X_A, σ_A) and (X_B, σ_B) be one-sided topological Markov shifts. TFAE:*

(1) There is an isomorphism $\psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\psi(\mathcal{D}_A) = \mathcal{D}_B$ and $\gamma_z^B \circ \psi = \psi \circ \gamma_z^A$ for every $z \in \mathbb{T}$.
(2) (X_A, σ_A) and (X_B, σ_B) are eventually conjugate.

12. Two-sided topological Markov shifts

Let A be an $n \times n$ matrix with entries in $\{0, 1\}$ and with no zero rows and no zero columns. We let

$$\bar{X}_A := \{(x_i)_{i\in\mathbb{Z}} : A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\},$$

equip $\bar{X}_A$ with the product topology, and define $\bar{\sigma}_A : \bar{X}_A \to \bar{X}_A$ by $\bar{\sigma}_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Then $\bar{\sigma}_A$ is a homeomorphism.

We say that two two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are *flow equivalent* if there is a homeomorphism $h : (\bar{X}_A \times \mathbb{R})/\sim \to (\bar{X}_B \times \mathbb{R})/\sim$ that maps flow lines onto flow lines in an orientation preserving way, where $\sim$ is the equivalence relation on $\bar{X}_A \times \mathbb{R}$ generated by $(\bar{\sigma}_A(x), t) \sim (x, t + 1)$, and a flow line is a set of the form $\{[x, t] : t \in \mathbb{R}\}$.

We say that two two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are *conjugate* if there is a homeomorphism $h : \bar{X}_A \to \bar{X}_B$ such that $h(\bar{\sigma}_A(x)) = \bar{\sigma}_B(h(x))$ for $x \in \bar{X}_A$.

We let $\mathcal{K}$ denote the C^*-algebra of compact operators on $l^2(\mathbb{N})$ and let $\mathcal{C}$ be the C^*-subalgebra $\overline{\operatorname{span}}\{\theta_{ii} : i \in \mathbb{N}\}$.

13. Flow equivalence of two-sided topological Markov shifts and diagonal-preserving stable isomorphism of Cuntz–Krieger algebras

That (1) implies (2) in the following theorem was proved by Cuntz and Krieger [7]. Matsumoto and Matui proved the converse for irreducible topological Markov shifts [13], while the general case was proved by Carlsen, Eilers, Ortega, and Restorff [4].

Theorem 12 (Cuntz+Krieger 1980, Matsumoto+Matui 2014, Carlsen+Eilers+Ortega+Restorff 2019)**.** *Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ be two-sided topological Markov shifts. TFAE:*

(1) $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are flow equivalent.
(2) There is an isomorphism $\psi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ such that $\psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.

14. Conjugacy of two-sided topological Markov shifts and gauge-invariant diagonal-preserving stable isomorphism of Cuntz–Krieger algebras

That (1) implies (2) in the in following theorem was proved by Cuntz and Krieger [7], while the converse was proven by Carlsen and Rout [5].

Theorem 13 (Cuntz+Krieger 1980, Cuntz 1981, Carlsen+Rout 2017)**.** *Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ be two-sided topological Markov shifts. TFAE:*

(1) $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are conjugate.
(2) There is an isomorphism $\psi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ such that $\psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ and $(\gamma_z^B \otimes \mathrm{id}) \circ \psi = \psi \circ (\gamma_z^A \otimes \mathrm{id})$ for every $z \in \mathbb{T}$.

15. Groupoids

Theorem 5, Theorem 10, Theorem 11, Theorem 12, and Theorem 13 have all been proved with the help of groupoids. I will at the end of these notes present a result about how a graded second-countable locally compact Hausdorff étale groupoid can be recoverede from its reduced C^*-algebra. The mentioned results can all, with some extra work, be proved with the help of this result.

First, I will give a short introduction to étale groupoids and their C^*-algebras.

A *groupoid* is a small category in which every morphism has an inverse. If G is a groupoid, then we write $G^{(0)}$ for the set of objects, and $G^{(1)}$ for the set of morphisms. For a morphism $\eta \in G^{(1)}$ we write $s(\eta)$ for its domain or source, $r(\eta)$ for its codomain or range, and η^{-1} for its inverse. The composition or product $\eta_1\eta_2$ of $\eta_1, \eta_2 \in G^{(1)}$ is then defined if and only if $s(\eta_1) = r(\eta_2)$. If $x \in G^{(0)}$, then we denote by $1_x \in G^{(1)}$ the corresponding identity morphism with $r(1_x) = s(1_x) = x$. Then $1_{r(\eta)}\eta = \eta = \eta 1_{s(\eta)}$ for $\eta \in G^{(1)}$, and $\eta^{-1}\eta = 1_{s(\eta)}$ and $\eta\eta^{-1} = 1_{r(\eta)}$ for $\eta \in G^{(1)}$.

Let us now look at two examples of groupoids.

Example 1. Let Γ be a group acting on the right on a set X. We write $x\gamma$ for the action of γ on x. Let $(X \rtimes \Gamma)^{(0)} := X$ and

$$(X \rtimes \Gamma)^{(1)} := X \times \Gamma.$$

Define $r, s : X \rtimes \Gamma \to X$ by $r(x, \gamma) = x$ and $s(x, \gamma) = x\gamma$. Then the product of (x_1, γ_1) and (x_2, γ_2) is defined if and only if $x_2 = x_1\gamma_1$, in which case we let $(x_1, \gamma_1)(x_1\gamma_1, \gamma_2) := (x_1, \gamma_1\gamma_2)$. We also let $(x, \gamma)^{-1} := (x\gamma, \gamma^{-1})$. Then $X \rtimes \Gamma$ is a groupoid.

Example 2. Let (X_A, σ_A) be a one-sided topological Markov shift. Let $G^{(0)}_{(X_A,\sigma_A)} := X_A$,

$$G^{(1)}_{(X_A,\sigma_A)} := \{(x, k - l, y) : \sigma_A^k(x) = \sigma_A^l(y)\},$$

and define $r, s : G^{(1)}_{(X_A,\sigma_A)} \to G^{(0)}_{(X_A,\sigma_A)}$ by $r(x,n,y) = x$ and $s(x,n,y) = y$. Then the product of (x_1,n_1,y_1) and (x_2,n_2,y_2) is defined if and only if $y_1 = x_2$, in which case we let $(x_1,n_1,y_1)(x_2,n_2,y_2) = (x_1,n_1+n_2,y_2)$. We also let $(x,,n,y)^{-1} = (y,-n,x)$. Then $G_{(X_A,\sigma_A)}$ is a groupoid.

16. Étale groupoids

A *topological groupoid* is a groupoid G such that $G^{(1)}$ comes with a topology such that the maps r are s are continuous maps from $G^{(1)}$ to $G^{(0)}$, the map $\eta \mapsto \eta^{-1}$ is a continuous map from $G^{(1)}$ to $G^{(1)}$, and the map $(\eta_1,\eta_2) \mapsto \eta_1\eta_2$ is a continuous map from $G^{(2)} := \{(\eta_1,\eta_2) \in G^{(1)} \times G^{(1)} : s(\eta_1) = r(\eta_2)\}$ to $G^{(1)}$, when $G^{(0)}$ is given the initial topology wrt. the map $x \mapsto 1_x$, and $G^{(2)}$ the product topology.

If G is topological groupoid, then $G^{(1)}$ is Hausdorff if and only if $\{1_x : x \in G^{(0)}\}$ is closed in $G^{(1)}$.

A topological groupoid is *étale* if $r : G^{(1)} \to G^{(0)}$ (equivalently $s : G^{(1)} \to G^{(0)}$) is a local homeomorphism. If G is étale, then $\{1_x : x \in G^{(0)}\}$ is open in $G^{(1)}$.

If Γ is a topological group acting continuously on a topological space X, then $X \rtimes \Gamma$ is a topological groupoid if we equip $(X \rtimes \Gamma)^{(1)} = X \times \Gamma$ with the product topology. Moreover, $X \rtimes \Gamma$ is étale if and only if Γ is discrete.

Let (X_A,σ_A) be a one-sided topological Markov shift. Let $\mathsf{L}(X_A) = \{x_{[0,k]} : x \in X_A,\ k \in \mathbb{N}\}$. For $\mu,\nu \in \mathsf{L}(X_A)$ let $Z(\mu,nu) = \{(\mu x, |\mu| - |\nu|, \nu x) : x, \mu x, \nu x \in X_A\}$. Then $\{Z(\mu,\nu) : \mu,\nu \in \mathsf{L}(X_A)\}$ is a basis for locally compact Hausdorff topology on $G^{(1)}_{(X_A,\sigma_A)}$ such that $G_{(X_A,\sigma_A)}$ is a locally compact Hausdorff étale groupoid when we equip $G^{(1)}_{(X_A,\sigma_A)}$ with this topology.

17. C^*-algebras of étale groupoids

Let G be a locally compact Hausdorff étale groupoid. If $f,g \in C_c(G^{(1)})$ and $\eta \in G^{(1)}$, then the set $\{(\eta_1,\eta_2) \in G^{(2)} : \eta_1\eta_2 = \eta,\ f(\eta_1)g(\eta_2) \neq 0\}$ is finite. We can therefore define a function $f * g : G^{(1)} \to \mathbb{C}$ by

$$(f*g)(\eta) := \sum_{\eta_1\eta_2=\eta} f(\eta_1)g(\eta_2).$$

It is not difficult to check that $f * g \in C_c(G^{(1)})$. The complex vector space $C_c(G^{(1)})$ is a $*$-algebra with multiplication given by $*$ and involution given by $f^*(\eta) = \overline{f(\eta^{-1})}$.

There are two C^*-norms $||\cdot||$ and $||\cdot||_r$ on $C_c(G^{(1)})$. The *full* or *universal* C^*-algebra $C^*(G)$ of G is the completion of $C_c(G^{(1)})$ with respect to $||\cdot||$, and the *reduced* C^*-algebra $C^*_r(G)$ of G is the completion of $C_c(G^{(1)})$ with respect to $||\cdot||_r$.

18. Graded groupoids

Let Γ be a topological group. A *cocycle* from G to Γ is a map $c : G^{(1)} \to \Gamma$ such that $c(\eta^{-1}) = c(\eta)^{-1}$ for $\eta \in G^{(1)}$, and $c(\eta_1\eta_2) = c(\eta_1)c(\eta_2)$ for $(\eta_1,\eta_2) \in G^{(2)}$.

A continuous cocycle $c : G^{(1)} \to \Gamma$ induces a Γ-*grading* $\{c^{-1}(\gamma)\}_{\gamma\in\Gamma}$ of $G^{(1)}$ (i.e., $\bigcup_{\gamma\in\Gamma} c^{-1}(\gamma) = G^{(1)}$, $c^{-1}(\gamma_1)\cap c^{-1}(\gamma_2) = \emptyset$ for $\gamma_1 \neq \gamma_2$, and $\eta_1\eta_2 \in c^{-1}(\gamma_1\gamma_2)$ if $(\eta_1,\eta_2) \in G^{(2)}$, $\eta_1 \in c^{-1}(\gamma_1)$, and $\eta_2 \in c^{-1}(\gamma_2)$). It also induces a coaction $\delta_c : C^*_r(G) \to C^*_r(G) \otimes C^*_r(\Gamma)$ such that $\delta_c(f) = f \otimes \lambda_g$ whenever $g \in \Gamma$ and $f \in C_c(G^{(1)})$ with $\mathrm{supp}(f) \subseteq c^{-1}(g)$ (here λ is the left-regular representation of Γ on $C^*_r(\Gamma)$).

We let $\mathrm{Iso}(c^{-1}(e))^\circ$ denote the interior of $\{\eta \in c^{-1}(e) : r(\eta) = s(\eta)\}$. If G is étale, then $\{1_x : x \in G^{(0)}\} \subseteq \mathrm{Iso}(c^{-1}(e))^\circ$. A second-countable locally compact Hausdorff étale groupoid

G is *topologically principal* if $\{1_x : x \in G^{(0)}\} = \mathrm{Iso}(G^{(1)})^\circ$. (This condition is really called *effective*, but it coincide with G being topologically principal if G is second-countable, locally compact, Hausdorff, and étale).

19. C^*-RIGIDITY OF ÉTALE GROUPOIDS

In 2008, Jean Renault [14] proved, by building on work by Alex Kumjian, that two topological principal second-countable locally compact Hausdorff étale groupoids are topologically isomorphic if and only if there is an isomorphism between their reduced C^*-algebra that sends $C_0(G_1^{(0)})$ onto $C_0(G_2^{(0)})$.

Theorem 14 (Renault 2008)**.** *Let G_1 and G_2 be topological principal second-countable locally compact Hausdorff étale groupoids. TFAE:*

(1) There is an isomorphism $\psi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\psi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.
(2) G_1 and G_2 are topologically isomorphic.

He did that by constructing from a pair consisting of a C^*-algebra and an abelian C^*-subalgebra an étale groupoid called the Weyl groupoid, and then showed that if we begin with a topological principal second-countable locally compact Hausdorff étale groupoid, then the Weyl groupoid of the reduced C^*-algebra and C_0 of the unit space is topologically isomorphic to the original groupoid.

It then follows that if there is an isomorphism between their reduced C^*-algebra of two topological principal, second-countable, locally compact, Hausdorff, étale groupoids that sends $C_0(G_1^{(0)})$ onto $C_0(G_2^{(0)})$, then the corresponding two Weyl groupoids, and thus the two original groupoids, are topological isomorphic.

Since the Weyl groupoid is always topological principal, this only works for topological principal groupoids.

Building on work by Brownlowe, Carlsen, and Whittaker [3], Carlsen, Ruiz, Sims, and Tomforde [6] extended the construction of Weyl groupoids such that the extended extended Weyl groupoid of a Γ-graded second-countable locally compact Hausdorff étale groupoid for which $\mathrm{Iso}(c^{-1}(e))^\circ$ torsion-free and abelian, is graded isomorphic to the original groupoid.

It follows from this that two graded second-countable locally compact Hausdorff étale groupoids are graded isomorphic if and only if there is an isomorphism between their reduced C^*-algebra that sends $C_0(G_1^{(0)})$ onto $C_0(G_2^{(0)})$ and intertwines the actions corresponding to the gradings.

Theorem 15 (Carlsen+Ruiz+Sims+Tomforde 2017)**.** *Let Γ be a discrete group and let (G_1, c_1) and (G_2, c_2) be Γ-graded second-countable locally compact Hausdorff étale groupoids with $\mathrm{Iso}(c_i^{-1}(e))^\circ$ torsion-free and abelian. TFAE:*

(1) There is an isomorphism $\psi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\psi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \psi = (\psi \otimes \mathrm{id}) \circ \delta_{c_1}$.
(2) There is a topological isomorphism $\phi : G_1 \to G_2$ such that $c_2 \circ \phi = c_1$.

$\mathrm{Iso}(c^{-1}(e))^\circ$ is a bundle of groups, so that it is torsion-free and abelian, means that it is a bundle of torsion-free and abelian groups.

It might be possible to get ride of the condition that $\mathrm{Iso}(c^{-1}(e))^\circ$ is abelian by using other techniques, but that $\mathrm{Iso}(c^{-1}(e))^\circ$ is torsion-free is necessary to assume; because we have for example that the reduced C^*-algebras of $\mathbb{Z}_4$ and $\mathbb{Z}_2 \times \mathbb{Z}_2$ are isomorphic.

From this result it is possible, with some extra work, to prove Theorem 5, Theorem 10, Theorem 11, Theorem 12, and Theorem 13.

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