C^*-rigidity of topological dynamical systems

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Cartan C^*-subalgebras and noncommutative dynamics

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C^*-algebras of topological dynamical systems

There is a long tradition for constructing C^*-algebras from dynamical systems. Motivations for doing this include:

1. constructing new examples of C^*-algebras which can be studied via dynamical systems,
2. use operator algebra theory to study dynamical systems.

C^*-rigidity of dynamical systems is the principle that dynamical systems can be recovered, up to a suitable notion of equivalence, from C^*-algebraic data associated to them.

Cantor minimal systems

- A *Cantor minimal system* is a pair (X, ϕ) where X is a totally disconnect compact metric space with no isolated points and $\phi : X \to X$ is a homeomorphism such that there is no non-trivial closed subspace $C \subsetneq X$ such that $\phi(C) = C$. The latter condition is equivalent to the condition that the *orbit* $\mathrm{orb}(x) := \{\phi^n(x) : x \in \mathbb{Z}\}$ of any $x \in X$ is dense in X.
- Two Cantor minimal systems (X, ϕ) and (Y, ψ) are *strong orbit equivalent* if there is a homeomorphism $h : X \to Y$ and maps $m, n : X \to \mathbb{Z}$ such that $h(\phi(x)) = \psi^{m(x)}(h(x))$ and $h(\phi^{n(x)}(x)) = \psi(h(x))$ for $x \in X$, and m and n each have at most one point of discontinuity.

Cantor minimal systems

Theorem (Elliott 1993 and Giordano+Putnam+Skau 1995)

Let (X, ϕ) and (Y, ψ) be Cantor minimal systems. TFAE:

1. *$C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic.*
2. *$K_0(C(X) \rtimes_\phi \mathbb{Z})$ and $K_0(C(Y) \rtimes_\psi \mathbb{Z})$ are isomorphic by an order preserving isomorphism that maps the class of the unit to the class of the unit.*
3. *(X, ϕ) and (Y, ψ) are strong orbit equivalent.*

Cantor minimal systems

- Two Cantor minimal systems (X, ϕ) and (Y, ψ) are *continuously orbit equivalent* if there is a homeomorphism $h : X \to Y$ and continuous maps $m, n : X \to \mathbb{Z}$ such that $h(\phi(x)) = \psi^{m(x)}(h(x))$ and $h(\phi^{n(x)}(x)) = \psi(h(x))$ for $x \in X$;
- and they are flip conjugate if there is a homeomorphism $h : X \to Y$ such that either $h(\phi(x)) = \psi(h(x))$ for all $x \in X$, or $h(\phi(x)) = \psi^{-1}(h(x))$ for all $x \in X$.

Theorem (Boyle 1983 and Giordano+Putnam+Skau 1995)

Let (X, ϕ) and (Y, ψ) be Cantor minimal systems. TFAE:

1. *$C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.*
2. *(X, ϕ) and (Y, ψ) are continuously orbit equivalent.*
3. *(X, ϕ) and (Y, ψ) are flip conjugate.*

Topologically transitive dynamical systems on compact spaces

A topologically dynamical system (X, ϕ) consisting of a topological space X and a homeomorphism $\phi : X \to X$ is *topologically transitive* if there is an $x \in X$ such that $\mathrm{orb}(x)$ is dense in X.

Theorem (Boyle 1983 and Tomiyama 1996)

Let (X, ϕ) and (Y, ψ) be topologically transitive dynamical systems on compact metric spaces X and Y. TFAE:

1. *$C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.*
2. *(X, ϕ) and (Y, ψ) are continuously orbit equivalent.*
3. *(X, ϕ) and (Y, ψ) are flip conjugate.*

Topologically free dynamical systems on compact spaces

A topologically dynamical system (X, ϕ) consisting of a topological space X and a homeomorphism $\phi : X \to X$ is *topologically free* if the set $\{x \in X : \phi^n(x) \neq x \text{ for all } n \neq 0\}$ is dense in X.

Theorem (Boyle and Tomiyama 1998)

Let (X, ϕ) and (Y, ψ) be topologically free dynamical systems on compact Hausdorff spaces X and Y. TFAE:

1. *$C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.*
2. *(X, ϕ) and (Y, ψ) are continuously orbit equivalent.*
3. *There exist decompositions $X = X_1 \sqcup X_2$ and $Y = Y_1 \sqcup Y_2$ such that X_1, X_2, Y_1, Y_2 are clopen and invariant, $\phi|_{X_1}$ is conjugate to $\psi|_{Y_1}$, and $\phi|_{X_2}$ is conjugate to $\psi^{-1}|_{Y_2}$.*

Homeomorphisms of compact Hausdorff spaces

Theorem (Carlsen+Ruiz+Sims+Tomforde 2017)

Let X and Y be second-countable compact Hausdorff spaces and $\phi : X \to X$ and $\psi : Y \to Y$ homeomorphisms. TFAE:

1. *$C(X) \rtimes_\phi \mathbb{Z}$ and $C(Y) \rtimes_\psi \mathbb{Z}$ are isomorphic by an isomorphism that maps $C(X)$ onto $C(Y)$.*
2. *There exist decompositions $X = X_1 \sqcup X_2$ and $Y = Y_1 \sqcup Y_2$ such that X_1, X_2, Y_1, Y_2 are clopen and invariant, $\phi|_{X_1}$ is conjugate to $\psi|_{Y_1}$ and $\phi|_{X_2}$ is conjugate to $\psi^{-1}|_{Y_2}$.*

Two actions (A, α) and (B, β) of a locally compact group G on two C^*-algebras A and B are *conjugate* if there is an isomorphism $\psi : A \to B$ such that $\psi \circ \alpha_\gamma = \beta_\gamma \circ \psi$ for each $\gamma \in G$, and they are *outer conjugate* if (A, α) is conjugate to an action β' on B such that there is a strictly continuous unitary map $u : G \to M(B)$ such that $u_{\gamma_1\gamma_2} = u_{\gamma_1}\beta_{\gamma_1}(u_{\gamma_2})$ for $\gamma_1, \gamma_2 \in G$, and $\beta'_\gamma = \mathrm{Ad} \circ \beta_\gamma$ for $\gamma \in G$.

Theorem (Pedersen 1982, Kaliszewski+Omland+Quigg 2018)

Let G be a locally compact group, let α be an action of G on a C^*-algebra A, and let β be an action of G on a C^*-algebra B. TFAE:

1. *$\phi : A \rtimes_\alpha G$ and $\phi : B \rtimes_\beta G$ are isomorphic by an isomorphism that maps A onto B and intertwines the dual coactions $\widehat{\alpha}$ and $\widehat{\beta}$.*
2. *(A, α) and (B, β) are outer conjugate.*

Imai—Takai—Takesaki duality

Theorem (Takesaki 1972, Imai+Takai 1978)

Let G be a locally compact group, let α be an action of G on a C^*-algebra A, and let β be an action of G on a C^*-algebra B. TFAE:

1. *$\phi : A \rtimes_\alpha G$ and $\phi : B \rtimes_\beta G$ are isomorphic by an isomorphism that intertwines the dual coactions $\widehat{\alpha}$ and $\widehat{\beta}$.*
2. *$(A \otimes \mathcal{K}(L^2(G)), \alpha \otimes \mathrm{Ad}\,\rho)$ and $(B \otimes \mathcal{K}(L^2(G)), \beta \otimes \mathrm{Ad}\,\rho)$ are conjugate (here ρ is right regular representation of G on $\mathcal{K}(L^2(G))$).*

Actions of discrete groups

Theorem (Kaliszewski+Omland+Quigg 2019)

Let G be a discrete group, let α be an action of G on a C^*-algebra A, and let β be an action of G on a C^*-algebra B. TFAE:

1. $\phi : A \rtimes_\alpha G$ *and* $\phi : B \rtimes_\beta G$ *are isomorphic by an isomorphism that intertwines the dual coactions* $\widehat{\alpha}$ *and* $\widehat{\beta}$.
2. (A, α) *and* (B, β) *are outer conjugate.*

Theorem (Kaliszewski+Omland+Quigg 2019)

Let $G \curvearrowright X$ and $G \curvearrowright Y$ be actions of a locally compact group on locally compact Hausdorff spaces. TFAE:

1. $C_0(X) \rtimes G \to C_0(Y) \rtimes G$ *are isomorphic by an isomorphism that intertwines the dual coactions.*
2. *The actions* $G \curvearrowright X$ *and* $G \curvearrowright Y$ *are conjugate.*

One-sided topological Markov shifts

- Let A be an $n \times n$ matrix with entries in $\{0,1\}$ and with no zero rows and no zero columns.
- We let $X_A := \{(x_i)_{i\in\mathbb{N}} : A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{N}\}$, equip X_A with the product topology, and define $\sigma_A : X_A \to X_A$ by $\sigma_A((x_i)_{i\in\mathbb{N}}) = (x_{i+1})_{i\in\mathbb{N}}$. Then σ_A is a surjective local homeomorphism.
- We say that two one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are *continuously orbit equivalent* if there is a homeomorphism $h : X_A \to X_B$ and continuous maps $k, l : X_A \to \mathbb{N}$ and $k', l' : X_B \to \mathbb{N}$ such that $\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{l(x)}(h(x))$ for $x \in X_A$, and $\sigma_A^{k'(x')}(h^{-1}(\sigma_B(x'))) = \sigma_A^{l'(x')}(h^{-1}(x'))$ for $x' \in X_B$.
- We say that two one-sided topological Markov shifts (X_A, σ_A) and (X_B, σ_B) are *eventually conjugate* if there is a homeomorphism $h : X_A \to X_B$ and continuous maps $k : X_A \to \mathbb{N}$ and $k' : X_B \to \mathbb{N}$ such that $\sigma_B^{k(x)}(h(\sigma_A(x))) = \sigma_B^{k(x)+1}(h(x))$ for $x \in X_A$, and $\sigma_A^{k'(x')}(h^{-1}(\sigma_B(x'))) = \sigma_A^{k'(x')+1}(h^{-1}(x'))$ for $x' \in X_B$.

Cuntz–Krieger algebras

- Let A be an $n \times n$ matrix with entries in $\{0, 1\}$ and with no zero rows and no zero columns.
- We let $\mathcal{O}_A$ be the Cuntz–Krieger algebra of A and $\mathcal{D}_A$ be the C^*-subalgebra $\overline{\text{span}}\{s_{i_1} \dots s_{i_k} s_{i_k}^* \dots s_{i_1}^* : i_1 \dots i_k \in \{0, 1\}^*\}$.
- We let γ^A denote the *gauge action* on $\mathcal{O}_A$. So γ_z^A is for each $z \in \mathbb{T}$ the automorphism of $\mathcal{O}_A$ that satisfies that $\gamma_z^A(s_i) = zs_i$ for each i.

Continuous orbit equivalence and eventual conjugacy of one-sided topological Markov shifts and Cuntz–Krieger algebras

Theorem (Matsumoto 2010, Carlsen+Eilers+Ortega+Restorff 2019)

Let (X_A, σ_A) and (X_B, σ_B) be one-sided topological Markov shifts. TFAE:

1. *There is an isomorphism $\psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\psi(\mathcal{D}_A) = \mathcal{D}_B$.*
2. *(X_A, σ_A) and (X_B, σ_B) are continuously orbit equivalent.*

Theorem (Matsumoto 2017, Carlsen+Rout 2017)

Let (X_A, σ_A) and (X_B, σ_B) be one-sided topological Markov shifts. TFAE:

1. *There is an isomorphism $\psi : \mathcal{O}_A \to \mathcal{O}_B$ such that $\psi(\mathcal{D}_A) = \mathcal{D}_B$ and $\gamma_z^B \circ \psi = \psi \circ \gamma_z^A$ for every $z \in \mathbb{T}$.*
2. *(X_A, σ_A) and (X_B, σ_B) are eventually conjugate.*

Two-sided topological Markov shifts

- Let A be an $n \times n$ matrix with entries in $\{0, 1\}$ and with no zero rows and no zero columns.
- We let $\bar{X}_A := \{(x_i)_{i\in\mathbb{Z}} : A(x_i, x_{i+1}) = 1 \text{ for all } i \in \mathbb{Z}\}$, equip $\bar{X}_A$ with the product topology, and define $\bar{\sigma}_A : \bar{X}_A \to \bar{X}_A$ by $\bar{\sigma}_A((x_i)_{i\in\mathbb{Z}}) = (x_{i+1})_{i\in\mathbb{Z}}$. Then $\bar{\sigma}_A$ is a homeomorphism.
- We say that two two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are *flow equivalent* if there is a homeomorphism $h : (\bar{X}_A \times \mathbb{R})/\sim \to (\bar{X}_B \times \mathbb{R})/\sim$ that maps flow lines onto flow lines in an orientation preserving way, where $\sim$ is the equivalence relation on $\bar{X}_A \times \mathbb{R}$ generated by $(\bar{\sigma}_A(x), t) \sim (x, t + 1)$, and a flow line is a set of the form $\{[x, t] : t \in \mathbb{R}\}$.
- We say that two two-sided topological Markov shifts $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ are *conjugate* if there is a homeomorphism $h : \bar{X}_A \to \bar{X}_B$ such that $h(\bar{\sigma}_A(x)) = \bar{\sigma}_B(h(x))$ for $x \in \bar{X}_A$.
- We let $\mathcal{K}$ denote the C^*-algebra of compact operators on $l^2(\mathbb{N})$ and let $\mathcal{C}$ be the C^*-subalgebra $\overline{\operatorname{span}}\{\theta_{ii} : i \in \mathbb{N}\}$.

Flow equivalence and conjugacy of two-sided topological Markov shifts and Cuntz–Krieger algebras

Theorem (Cuntz+Krieger 1980, Matsumoto+Matui 2014, Carlsen+Eilers+Ortega+Restorff 2019)

Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ be two-sided topological Markov shifts. TFAE:

1. $(\bar{X}_A, \bar{\sigma}_A)$ *and* $(\bar{X}_B, \bar{\sigma}_B)$ *are flow equivalent.*
2. *There is an isomorphism* $\psi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ *such that* $\psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$.

Theorem (Cuntz+Krieger 1980, Cuntz 1981, Carlsen+Rout 2017)

Let $(\bar{X}_A, \bar{\sigma}_A)$ and $(\bar{X}_B, \bar{\sigma}_B)$ be two-sided topological Markov shifts. TFAE:

1. $(\bar{X}_A, \bar{\sigma}_A)$ *and* $(\bar{X}_B, \bar{\sigma}_B)$ *are conjugate.*
2. *There is an isomorphism* $\psi : \mathcal{O}_A \otimes \mathcal{K} \to \mathcal{O}_B \otimes \mathcal{K}$ *such that* $\psi(\mathcal{D}_A \otimes \mathcal{C}) = \mathcal{D}_B \otimes \mathcal{C}$ *and* $(\gamma_z^B \otimes \mathrm{id}) \circ \psi = \psi \circ (\gamma_z^A \otimes \mathrm{id})$ *for every* $z \in \mathbb{T}$.

C^*-rigidity of étale groupoids

- A *groupoid* is a small category in which every morphism has an inverse.
- A topological groupoid is *étale* if $r : G^{(1)} \to G^{(0)}$ (equivalently $s : G^{(1)} \to G^{(0)}$) is a local homeomorphism.
- A second-countable locally compact Hausdorff étale groupoid G is *topologically principal* (or *effective*) if the interior of $\{\eta \in G^{(1)} : r(\eta) = s(\eta)\}$ is $\{1_x : x \in G^{(0)}\}$.

Theorem (Renault 2008)

Let G_1 and G_2 be topologically principal second-countable locally compact Hausdorff étale groupoids. TFAE:

1. *There is an isomorphism $\psi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\psi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$.*
2. *G_1 and G_2 are topologically isomorphic.*

Graded groupoids

- Let Γ be a topological group. A *cocycle* from G to Γ is a map $c : G^{(1)} \to \Gamma$ such that $c(\eta^{-1}) = c(\eta)^{-1}$ for $\eta \in G^{(1)}$, and $c(\eta_1\eta_2) = c(\eta_1)c(\eta_2)$ for $(\eta_1, \eta_2) \in G^{(2)}$.
- A continuous cocycle $c : G^{(1)} \to \Gamma$ induces a Γ-*grading* $\{c^{-1}(\gamma)\}_{\gamma\in\Gamma}$ of $G^{(1)}$ (i.e., $\bigcup_{\gamma\in\Gamma} c^{-1}(\gamma) = G^{(1)}$, $c^{-1}(\gamma_1) \cap c^{-1}(\gamma_2) = \varnothing$ for $\gamma_1 \neq \gamma_2$, and $\eta_1\eta_2 \in c^{-1}(\gamma_1\gamma_2)$ if $(\eta_1, \eta_2) \in G^{(2)}$, $\eta_1 \in c^{-1}(\gamma_1)$, and $\eta_2 \in c^{-1}(\gamma_2)$).
- It also induces a coaction $\delta_c : C_r^*(G) \to C_r^*(G) \otimes C_r^*(\Gamma)$ such that $\delta_c(f) = f \otimes \lambda_g$ whenever $g \in \Gamma$ and $f \in C_c(G^{(1)})$ with $\operatorname{supp}(f) \subseteq c^{-1}(g)$ (here λ is the left-regular representation of Γ on $C_r^*(\Gamma)$).

C^*-rigidity of étale groupoids

Theorem (Carlsen+Ruiz+Sims+Tomforde 2017)

Let Γ be a discrete group and let (G_1, c_1) and (G_2, c_2) be Γ-graded second-countable locally compact Hausdorff étale groupoids such that the interior of $\{\eta \in c^{-1}(e) : r(\eta) = s(\eta)\}$ is torsion-free and abelian. TFAE:

1. *There is an isomorphism $\psi : C_r^*(G_1) \to C_r^*(G_2)$ such that $\psi(C_0(G_1^{(0)})) = C_0(G_2^{(0)})$ and $\delta_{c_2} \circ \psi = (\psi \otimes \mathrm{id}) \circ \delta_{c_1}$.*
2. *There is a topological isomorphism $\phi : G_1 \to G_2$ such that $c_2 \circ \phi = c_1$.*

Thank you for your attention.