An Algebraic Approach to the Weyl Groupoid and Weyl Bundle

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Étale Groupoids, C*-Algebras and *-Semigroups

- At the beginning of time (1980), there was Renault's thesis:

$$\text{LCH Étale Groupoid} \quad \to \quad \text{C*-Algebra.}$$

- What about $\leftarrow$?

  How do we find groupoid models of C*-algebras?

  1. Exel's (2008) tight groupoid construction:

  $$\text{Inverse Semigroup} \quad \to \quad \text{(ample) LCH Étale Groupoid.}$$

  2. Kumjian (1986) and Renault's (2008) Weyl groupoid:

  $$\text{Cartan C*-Subalgebra} \quad \to \quad \text{(effective) LCH Étale Groupoid.}$$

- Could we unify these two constructions? Yes – *-semigroups!

$$\text{Inverse Semigroups} \quad \subseteq \quad \text{*-Semigroups} \quad \xleftarrow{\text{*-Normalisers}} \quad \text{Cartan C*-algebras.}$$

- First let's go back to basics – commutative C*-algebras.

Ideals

- How do we recover LCH X from $A = C_0(X)$?
- Gelfand's (1943) answer : Consider the characters of A.
- $\approx$ Gelfand's answer: Consider the (proper) closed ideals of A

$$\mathbf{I}^A = \{I \subseteq A : I = I + I = AIA = \mathrm{cl}(I) \neq A\}$$

with the hull-kernel/Jacobson topology, i.e. generated by

$$O_a = \{I \in \mathbf{I}^A : a \notin I\}.$$

- Every $x \in X$ defines a maximal closed ideal

$$M_x = \{a \in A : a(x) = 0\}.$$

- Conversely, every maximal closed ideal is of this form so

$$x \mapsto M_x \text{ is a homeomorphism from } X \text{ onto } \mathbf{M}^A$$

i.e. $\mathbf{M}^A = \{M \in \mathbf{I}^A : \nexists I \in \mathbf{I}^A \ (M \subsetneqq I)\}$ recovers X.

Filters

- How do we recover LCH X from $A^1_+ = C_0(X, [0,1])$?
- Milgram's (1949) answer: Consider the filters $F \subseteq A^1_+$:

$$f, g \in F \qquad \Leftrightarrow \qquad \exists h \in F\ (h \prec f, g) \qquad\qquad \text{(Filter)}$$

where $\prec$ is the domination relation defined by

$$a \prec b \qquad \Leftrightarrow \qquad a = ab \qquad \Leftrightarrow \qquad \operatorname{supp}(a) \subseteq b^{-1}\{1\}$$

- Let $\mathbf{U}^{A^1_+}$ denote the ultrafilters (i.e. maximal proper filters) in A^1_+ again with the topology generated by $(O_a)_{a \in A}$ where

$$O_a = \{U \in \mathbf{U}^{A^1_+} : a \in U\}.$$

- Every $x \in X$ defines an ultrafilter

$$U_x = \{a \in A^1_+ : 1 \in \operatorname{int}(a^{-1}\{1\})\}.$$

- Conversely, every ultrafilter is of this form so $\mathbf{U}^{A^1_+}$ recovers X,

$$x \mapsto U_x \text{ is a homeomorphism from } X \text{ onto } \mathbf{U}^{A^1_+}.$$

- Note: Unlike characters or ideals, ultrafilters depend only in the semigroup structure of (the positive unit ball of) A.

Étale Groupoids

- Given LCH étale groupoid G, consider the *-semigroups

$$S = \{a \in C_0(G, \mathbb{D}) : \mathrm{supp}(a) \text{ is a bisection}\}.$$
$$E = C_0(G^0, \mathbb{D}).$$

$$a^*(g) = \overline{a(g^*)}, \quad ab(gh) = a(g)b(h) \quad \text{for} \quad g \in \mathrm{supp}(a),\ h \in \mathrm{supp}(b).$$

- Normally, we also consider the C*-algebras they generate

$$A = C^*_r(G) = C^*(S).$$
$$B = C_0(G^0) = C^*(E).$$

- Kumjian (1986)/Renault (2008) showed how to recover G from A and B, at least when G is principal/effective:
  1. Recover S as the contractive *-normalisers $n \in A$ of B,
  $$nBn^* \cup n^*Bn \subseteq B. \qquad \text{(*-Normaliser)}$$
  2. Consider the (partial) action of S on the spectrum $\mathbf{\Phi}^B$ given by
  $$s\phi(a) = \phi(s^*as)/\phi(s^*s) \qquad (\text{when } \phi(s^*s) \neq 0).$$
  3. Recover G as the groupoid of germs of this action.

*-Domination

- What if we reverse the roles of $B = C_0(G^0)$ and
$$S = \{a \in C_0(G, \mathbb{D}) : \mathrm{supp}(a) \text{ is a bisection}\}?$$
- Given $S \subseteq A$, we can certainly recover $B = C^*(|S|^2)$.
- Or let us just take the *-semigroups S and $E = B^1$ as given.
- Claim: This is all we need to recover any LCH étale G.
- Idea: Again consider ultrafilters w.r.t. the relation given by
$$a \precsim b \qquad \Leftrightarrow \qquad \mathrm{supp}(a) \subseteq b^{-1}[\mathbb{T}].$$
- First we need an algebraic characterisation of $\precsim$.

Proposition

$$a \precsim b \qquad \Leftrightarrow \qquad a = ab^*b \quad \text{and} \quad ab^* \in E.$$

Proof.

- Roughly, $ab^* \in E$ corresponds to $\mathrm{supp}(a) \subseteq \mathrm{supp}(b)$.
- Then $a = ab^*b$ implies $b[\mathrm{supp}(a)] \subseteq \mathbb{T}$. $\square$

Reconstruction

- Recall: G is a LCH étale groupoid,

$$S = \{a \in C_0(G, \mathbb{D}) : \mathrm{supp}(a) \text{ is a bisection}\}$$

and $\mathbf{U}^S$ denotes the $\precsim$-ultrafilters in S topologised by

$$O_a = \{U \in \mathbf{U}^S : a \in U\}.$$

Theorem (B. 2019)
We have a homeomorphism from G onto $\mathbf{U}^S$ given by

$$g \mapsto U_g = \{a \in S : g \in \mathrm{int}(a^{-1}[\mathbb{T}])\}.$$

Moreover, for all $g, h \in G$ with $g^*g = hh^*$,

$$U_{g^*} = (U_g)^* \qquad \text{and} \qquad U_{gh} = (U_g U_h)^{\precsim}.$$

- So ultrafilters recover the topology+groupoid structure of G.

Reconstruction

We have a homeomorphism from G onto $\mathbf{U}^S$ given by

$$g \mapsto U_g = \{a \in S : g \in \mathrm{int}(a^{-1}[\mathbb{T}])\}.$$

Proof.

- Outline: By Urysohn, each U_g is an ultrafilter.
- Conversely, if U is an ultrafilter, $\bigcap_{a \in U} a^{-1}[\mathbb{T}] \neq \emptyset$.
- Thus $U \subseteq U_g$, for some $g \in G$, so $U = U_g$ by maximality.
- So $\mathbf{U}^S = \{U_g : g \in G\}$. By Urysohn, $g \mapsto U_g$ is homeo. $\square$

- Certainly $U_{g^*} = (U_g)^*$ and $U_g U_h \subseteq U_{gh}$.
- Conversely, given $a \in U_{gh}$, take $b \in U_{gh}$ with $b \precsim a$.
- Taking $c \in U_g$, note $c^* b \in U_{g^*} U_{gh} \subseteq U_h$ so

$$a \succsim cc^* b \in U_g U_h.$$

- Thus $U_{gh} = (U_g U_h)^{\precsim}$.

Further Comments

- $S \subseteq \mathcal{N}^*(E) \subseteq C_r^*(G)$ but $S = \mathcal{N}^*(E)$ only if G is effective.
- E.g. if G is a group, $\mathcal{N}^*(E) = \mathcal{N}^*(\mathbb{D}1_A) \supseteq$ all unitaries.
- Reconstruction valid if just G^0 is Hausdorff (even if G is not).
- Even works for a Fell bundle F (if the fibres have unitaries):
  1. $\pi : F \to G$ is a Banach bundle.
  2. F is a *-category and π is a *-isocofibration.
  3. Product is bilinear, * is conjugate linear.
  4. $\|ef\| \leq \|e\|\|f\|$, $\|ff^*\| = \|f\|^2$ and $ff^* \geq 0$.
- Then take $S =$ *-semigroup of continuous sections a of F with $\mathrm{supp}(a)$ a bisection and $a(g) = \lambda u$, for $\lambda \in \mathbb{D}$ and u unitary.
- Can recover the fibres too if we do now consider $A = C_r^*(F) = C^*(S)$ together with $\Phi(a) = a|_{G^0}$ as

$$a(g) = b(g) \qquad \Leftrightarrow \qquad \inf_{c \in U_g} \|\Phi(ac^*) - \Phi(bc^*)\| = 0.$$

Going Abstract

- Like with the Gelfand representation and the Kumjian-Renault Weyl groupoid, we can start with an abstract *-semigroup S.

Definition

Call a *-subsemigroup $E \subseteq S$ Cartan if

1. E is *-normal in S, i.e. for all $s \in S$, $sEs^* \subseteq E$.
2. The *-squares $|S|^2 = \{ss^* : s \in S\}$ are central in E, i.e.

$$|S|^2 \subseteq E \cap E'.$$

- This all we need for our general Weyl groupoid construction, i.e. a *-semigroup S together with a Cartan subsemigroup E.
- Note E itself does not have to be commutative.
- This is important well dealing with 'non-commutative Cartan subalgebras' e.g. coming from Fell bundles.

Examples

- If S is an inverse semigroup, the idempotents are Cartan.
  - If S satisfies Lawson's trapping condition then our Weyl groupoid will be the same as Exel's tight groupoid.
- If S is a group, any normal subgroup is Cartan.
- Every (commutative) Cartan subalgebra E of a C*-algebra A is a Cartan subsemigroup of its *-normaliser *-semigroup

$$S = \mathcal{N}^*(E) = \{s \in A : s^*Es \cup sEs^* \subseteq S\}.$$

  - More generally, if A is a topological *-semigroup then any closed commutative *-subsemigroup E containing an approximate unit for A is Cartan in $S = \mathcal{N}^*(E)$.
  - Alternatively, any maximal commutative *-subalgebra E of any *-algebra A is Cartan in the intertwiners of E_{sa} –

$$S = \mathcal{I}(E_{\mathrm{sa}}) = \{s \in A : sE_{\mathrm{sa}} = E_{\mathrm{sa}}s\}.$$

- When E is a Cartan subalgebra of a C*-algebra A,

$$\mathcal{N}^*(E) = \overline{\mathcal{I}(E_{\mathrm{sa}})}$$

  (Donsig and Pitts (2008) showed $\mathcal{N}^*(E) = \overline{\mathcal{N}(E)}$).

Abstract *-Domination

- Given a *-semigroup S and Cartan $E \subseteq S$, define

$$a \prec b \quad \Leftrightarrow \quad a = ab. \qquad \text{(Domination)}$$
$$a \sim b \quad \Leftrightarrow \quad ab^* \in E. \qquad \text{(Compatibility)}$$
$$a \precsim b \quad \Leftrightarrow \quad b \sim a \prec b^*b. \qquad \text{(*-Domination)}$$

- Note: if S is an inverse semigroup, $\precsim$ is the canonical order.
- As E is Cartan, $\precsim$ has the required properties, e.g.

$$a \precsim b \precsim c \quad \Rightarrow \quad a \precsim c. \qquad \text{(Transitivity)}$$
$$a \precsim b \quad \Rightarrow \quad a^* \precsim b^*. \qquad \text{(*-Invariance)}$$
$$a \precsim b \quad \text{and} \quad c \precsim d \quad \Rightarrow \quad ac \precsim bd. \qquad \text{(Multiplicative)}$$

- E.g. for (Multiplicative), say $a \precsim b$ and $c \precsim d$.
- As *-squares are central in E,

$$ac(bd)^*bd = acd^*b^*bd = ab^*bcd^*d = ac.$$

- As E is *-normal in S,

$$ac(bd)^* = acd^*b^* = ab^*bcd^*b^* \in EbEb^* \subseteq E.$$

Cosets and Ultrafilters

Definition

$C \subseteq S$ is a coset if $C = C^{\precsim} = CC^*C$.

- When S is a group with normal subgroup E, these are precisely the cosets of subgroups containing E.
- We consider the topology on cosets $\mathbf{C}^S$ generated by

$$O_a = \{C \in \mathbf{C}^S : a \in C\}.$$

Theorem (B. 2019)

The cosets form an étale groupoid with inverse and product:

$$C \mapsto C^* \qquad \text{and} \qquad (B, C) \mapsto (BC)^{\precsim} \quad (\text{when } (B^*B)^{\precsim} = (CC^*)^{\precsim})$$

- Proof uses the properties of $\precsim$ just mentioned.
- This extends inverse semigroup results of Lawson-Margolis-Steinberg (2013).
- The $\precsim$-ultrafilters form an 'ideal' subgroupoid of the cosets.
- Thus the ultrafilter groupoid $\mathbf{U}^S$ is also étale.
- If $S \ni 0$ then the unit ultrafilters are also Hausdorff.

Local Compactness

Theorem (B. 2019)

If S is a *-subsemigroup of the unit ball of a C*-algebra and $E \subseteq S$ is the unit ball of the C*-subalgebra it generates,

$$a \precsim b \qquad \Leftrightarrow \qquad O_a \Subset O_b,$$

i.e. $\exists$ compact C with $O_a \subseteq C \subseteq O_b$ (Hausdorff case: $\overline{O_a} \subseteq O_b$).

- Any $U \in \mathbf{U}^S$ has neighbourhood base $(O_a)_{a \in U}$.
- Thus U has a compact neighbourhood base.

Corollary

The ultrafilter groupoid $\mathbf{U}^S$ is locally compact.

- Even applies to a general class of *-rings A with a commutative lattice ordered *-subring $L \supseteq |S|^2$.
- E.g. applies to Steinberg algebras, real C*-algebras etc.

The Weyl Bundle

- Assume: A is a C*-algebra with conditional expectation Φ and *-subsemigroup $S \subseteq A^1$, $|S|^2 \subseteq \Phi[S]$ and, for all $s \in S$,
$$\Phi(sas^*) = s\Phi(a)s^*.$$
$$ss^*\Phi(a) = \Phi(a)ss^*.$$
- Then $E = \Phi[S]$ is *-normal and $|S|^2 \subseteq E \cap E'$.
- Each $U \in \mathbf{U}^S$ defines a closed subspace $A_U \subseteq A$ by
$$A_U = \{a \in A : \inf_{u \in U} \|\Phi(au^*)\| = 0\}.$$
- Topologise $\mathbf{A}^S = \{(U, a + A_U) : U \in \mathbf{U}^S, a \in A\}$ by $(a_s^\delta)_{s \in S}^{\delta > 0}$:
$$a_s^\delta = \{(U, b + A_U) : s \in U \text{ and } \exists t \in U \ \|\Phi(at^*) - \Phi(bt^*)\| < \delta\}.$$
- Each $a \in A$ defines a continuous section $\widehat{a}(U) = (U, a + A_U)$.

Theorem (B. 2019)

The Weyl bundle $\mathbf{A}^S$ is a Fell bundle with involution and product
$$(U, a + A_U)^* = (U^*, a^* + A_{U^*}).$$
$$(U, a + A_U)(V, b + A_V) = ((UV)^{\precsim}, ab + A_{(UV)^{\precsim}}).$$

Hilbert C*-Modules and Hilbert-Fell Bundles

- Say X is a Hilbert A-module with 'conditional expectation' Ψ, i.e. Ψ is an idempotent linear map on X compatible with Φ:

$$\Psi(\Psi(x)a) = \Psi(x)\Phi(a) = \Psi(x\Phi(a)).$$
$$\Phi(\langle\Psi(x)|y\rangle) = \langle\Psi(x)|\Psi(y)\rangle = \Phi(\langle x|\Psi(y)\rangle).$$

- Again each $U \in \mathbf{U}^S$ defines a closed subspace $X_U \subseteq X$ by

$$X_U = \{x \in X : \inf_{u \in U} \|\Psi(xu^*)\| = 0\}.$$

- Topologise $\mathbf{X}^S = \{(U, a + A_U) : U \in \mathbf{U}^s, a \in A\}$ by $(x_s^\delta)_{s \in S}^{\delta > 0}$.

$$x_s^\delta = \{(U, y + X_U) : s \in U \text{ and } \exists t \in U \ \|\Psi(xt^*) - \Psi(yt^*)\| < \delta\}.$$

- Each $x \in X$ defines a continuous section $\widehat{x}(U) = (U, x + X_U)$.

Theorem (B. 2019)

The Weyl bundle $\mathbf{X}^S$ is a 'Hilbert $\mathbf{A}^S$-bundle' where

$$(U, x + X_U)(V, a + A_V) = ((UV)^{\precsim}, xa + X_{(UV)^{\precsim}}) : \ (U^*U)^{\precsim} = (VV^*)^{\precsim}.$$
$$\langle(U, x + X_U)|(V, y + X_V)\rangle = ((U^*V)^{\precsim}, \langle x|y\rangle + A_{(U^*V)^{\precsim}}) : \ (UU^*)^{\precsim} = (VV^*)^{\precsim}$$

A Very Twisted Fell Bundle

Question (Sims 2018)

Does every Weyl twist admit a continuous global section?

- $\equiv$ Do all Fell line bundles have cts nowhere vanishing sections?
- A counterexample comes from Pedersen-Petersen (1971).
- There they defined a C*-algebra A as cts sections of

$$B = \{(\begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix}, V) : a, d \in \mathbb{C} \text{ and } \mathbf{b}, \overline{\mathbf{c}} \in V \in \mathbb{C}P^1\},$$

$$\text{where} \qquad \begin{bmatrix} a & \mathbf{b} \\ \mathbf{c} & d \end{bmatrix} \begin{bmatrix} a' & \mathbf{b}' \\ \mathbf{c}' & d' \end{bmatrix} = \begin{bmatrix} aa' + \mathbf{b} \cdot \mathbf{c}' & a\mathbf{b}' + d'\mathbf{b} \\ a'\mathbf{c} + d\mathbf{c}' & \mathbf{c} \cdot \mathbf{b}' + dd' \end{bmatrix}.$$

- $A = C_r^*(F)$ where F is a Fell line bundle over (principal)

$$G = \mathbb{C}P^1 \times \{0, 1\}^2.$$

- A is 2-homogeneous but $A \not\approx PC(X, M_n)P$ (B.-Farah 2012).

A Very Twisted Fell Bundle

- Say we have cts $\mathbf{b}$ on $\mathbb{C}P^1$ with $\mathbf{b}(V) \in V$ and $\|\mathbf{b}(V)\| = 1$.
- Identify the 3-sphere S^3 with $\{\mathbf{e} \in \mathbb{C}^2 : \|\mathbf{e}\| = 1\}$.
- Define $f : S^3 \to \mathbb{T}(\approx S^1)$ by $f(\mathbf{e})\mathbf{e} = \mathbf{b}(\mathbb{C}\mathbf{e})$.
- For all $\mathbf{e} \in S^3$, note that

  $$f(-\mathbf{e}) = -f(\mathbf{e})$$

  i.e. we have a continuous map from a higher to lower dimensional sphere preserving antipodal points.
- This contradicts the Borsuk-Ulam Theorem.
- Thus continuous sections always vanish somewhere.
- So Kumjian's twists are more general than Renault's cocycles.