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Summary of the Habilitation Thesis

I) Scientific degrees and diplomas

PhD in Mathematics (with Honors). Institute of Mathematics PAN, Warsaw, January 2009. Dissertation title: *Spectral analyzis of operators generating irreversible dynamical systems*. Supervisor: Prof. A. V. Lebedev.

MSc in Mathematics (with Honors). Faculty of Mathematics and Physics of the University of Bialystok, June 2003. Thesis title: *Spectral and algebraic properties of non-local operators*. Supervisor: Prof. A. V. Lebedev.

Bachelor's degree in Financial Mathematics. Faculty of Mathematics and Physics of the University of Bialystok, July 2002. Thesis title: *On the existence of stochastic processes - Kolmogorov theorem*. Supervisor: Dr. J. Kotowicz,

II) Employment in academic institutions

10/2014 – 9/2016: Postdoctoral fellow, Marie Curie Actions (FP7) UE, Department of Mathematics and Computer Science University of Southern Denmark, Odense, Denmark

10/2013 – 9/2014: Assistant Professor at the Laboratory of Noncommutative Geometry, Institute of Mathematics PAN, Warsaw

10/2009 – present: Assistant Professor, Institute of Mathematics, University of Białystok

10/2003 – 9/2009: Assistant, Institute of Mathematics, University of Białystok

10/2002 – 7/2003: Assistant Trainee, Institute of Mathematics, University of Białystok

III) Scientific Achievements: (in the Sense of Article 16 Paragraph 2 of the Act of 14 March 2003 on Academic degrees and Titles and on Degrees and Titles in the Field of Arts):

Title of the Habilitation Thesis:

Structure of C^*-algebras defined by relations of dynamic type

List of publications that constitute the Habilitation Thesis:

1. [KL13] B. K. Kwaśniewski, A. V. Lebedev *Crossed products by endomorphisms and reduction of relations in relative Cuntz-Pimsner algebras* J. Funct. Anal., 264 (2013), 1806–1847.

2. [Kwa13] B. K. Kwaśniewski, *C^*-algebras generalizing both relative Cuntz-Pimsner and Doplicher-Roberts algebras*, Trans. Amer. Math. Soc. 365 (2013), no. 4, 1809–1873.

3. [Kwa14] B. K. Kwaśniewski, *Topological freeness for Hilbert bimodules* Israel J. Math. 199 (2014), no. 2, 641–650.

4. [Kwa14'] B. K. Kwaśniewski, *Crossed products for interactions and graph algebras* Integr. Equ. Oper. Theory 80 (2014), no. 3, 415–451.

5. [Kwa15] B. K. Kwaśniewski, *Ideal structure of crossed products by endomorphisms via reversible extensions of C^*-dynamical systems* Int. J. Math. 26 (2015), no. 3, 1550022 [45 pages].

6. [Kwa16] B. K. Kwaśniewski, *Crossed products by endomorphisms of $C_0(X)$-algebras* J. Funct. Anal., 270 (2016), 2268–2335.

7. [KS16] B. K. Kwaśniewski, W. Szymański, *Topological aperiodicity for product systems over semigroups of Ore type*, J. Funct. Anal. 270 (2016), no. 9, 3453–3504.

8. [KS17] B. K. Kwaśniewski, W. Szymański, *Pure infiniteness and ideal structure of C^*-algebras associated to Fell bundles*, J. Math. Anal. Appl. 445 (2017), no. 1, 898–943.

9. [Kwa17] B. K. Kwaśniewski, *Exel's crossed products and crossed products by completely positive maps*, accepted in Houston Journal of Mathematics (accepted on 15–08–2014, the corresponding letters enclosed). Available at arXiv:1404.4929.

In the case of article [KL13] the contribution of co-authors should be regarded as follows: B. K. Kwaśniewski 70%, A. V. Lebedev 30%. In the case of papers [KS16], [KS17] the contribution of co-authors should be regarded respectively as follows: B. K. Kwaśniewski 65%, 70%, W. Szymański 35%, 30%. The relevant statements are attached to the application.

1 Introduction

Crossed products of W^*-algebras by discrete group actions were introduced by John von Neumann in [Neu40]. They served as a tool to construct the first examples of factors of type III, also called purely infinite factors. C^*-algebraic versions of crossed products appeared, among others, in the work of Segal, but the first formal definition was probably given in [Tur58]. By the end of the 1970's the theory of crossed products of C^*-algebras by group actions became a fully fledged branch of modern mathematics, see e.g. [Ped79], [Gre78]. This theory is still being developed and new significant applications are being found. These applications concern many different fields such as: noncommutative harmonic analysis, [Wil07], noncommutative geometry [NSS08] or the theory of functional differential equations [AL94], [AL98]. However, in the physical world and therefore also in mathematical physics and pure mathematics many structures are modeled by irreversible dynamics implemented by objects much more general and complex than automorphisms. The list of publications that constitute the Habilitation Thesis form an important contribution to the theory of generalized crossed products of C^*-algebras. This specifically concerns actions by endomorphisms and completely positive maps, semigroups actions of C^*-correspondences (product systems) and group actions by Hilbert bimodules (Fell bundles).

Crossed products of C^*-algebras by single endomorphisms firstly appeared, in an informal way, in the seminal paper [Cu77]. In this paper Cuntz introduced the notion of a *simple purely infinite C^*-algebra* (C^*-algebraic analogue of W^*-factor of type III) as well as the celebrated *Cuntz algebras* $\mathcal{O}_n$. The original constructions were spatial and involved injective endomorphisms acting on unital C^*-algebras [Cu77], [Pas80], [Cun82], [Rør95], [Mur96]. Their main aim was to produce new interesting examples of simple C^*-algebras. In particular, the first complete set of models for *Kirchberg algebras* (separable, nuclear, simple purely infinite C^*-algebras), satisfying the UCT was obtained as crossed products by endomorphisms [Rør95], [ER95]. We recall that such algebras are completely classified by their K-theory [Kir00], [Phi00].

A universal definition of crossed products by 'an arbitrary' endomorphism was proposed by Stacey [Sta93]. However, as noticed in [Adj95] Stacey's formalism requires an additional assumption that the endomorphism is extendible to the multiplier algebra. Moreover, if the endomorphism is not injective, Stacey's crossed product degenerates and in some cases it might even be the zero algebra. The authors of [LR04] proposed another definition, which does not have the latter flaw, but in turn has a disadvantage that it is not a generalization of classical crossed products by automorphisms. In the meantime, Exel [Exe03] suggested yet another definition of crossed products by endomorphisms, which requires an additional component – a transfer operator. This definition was then modified, for instance, in [ER07]. It is known that this multitude of different variants and constructions can be "unified" in the realm of relative Cuntz-Pimsner algebras $\mathcal{O}(X, J)$ [Pim97], [MS98]. Nevertheless, different constructions are related with different C^*-correspondences X and different ideals J. Therefore, such a general approach does not give a complete, clear picture - it does not explain the relationships between various structures on the level of generators and relations.

An important part of the Habilitation Thesis is an elaboration of canonical constructions and development of a general theory of crossed products of C^*-algebras by singly generated dynamics. This is achieved by: introducing (relative) crossed products by completely positive maps [Kwa17]; elimination of unnecessary assumptions and superfluous relations appearing in the literature [KL13], [Kwa16], [Kwa17]; obtaining strong structural results concerning the ideal structure, pure infiniteness and K-theory for crossed products by endomorphisms [Kwa15], [Kwa16]; and detailed analysis of important examples [Kwa14'], [Kwa17]. Furthermore, [Kwa13] introduces formalism that unifies relative Cuntz-Pimsner algebras and Doplicher-Roberts algebras. This gives strong tools to study general structure of C^*-algebras given by broadly and abstractly understood singly generated dynamics.

Another achievement is a contribution to the theory of semigroup crossed products and their generalizations. Semigroup versions of crossed products by endomorphisms were investigated in connection with Toeplitz algebras of semigroups of isometries [ALNR94], [LR96], Bost-Connes Hecke algebras arising from number fields [LR99], [ALR97], phase transitions [Lac98], short exact sequences and tensor products [Lar00], or higher rank graphs [Bro12]. Despite of these achievements, up to now this area of research is far from being fully fledged. The habilitant contributed in [KS16] with detailed analysis and description of structure of the semigroup version of Pimnser's algebras $\mathcal{O}_X$ [Fow02] associated to regular product systems X over Ore semigroups P. This analysis led to powerful tools such as: uniqueness theorem for $\mathcal{O}_X$, simplicity criterion for $\mathcal{O}_X$ and an explcite description of dilations of the product system X to a Fell bundle $\mathcal{B}_X$ over the enveloping group $G(P)$. The latter result allows one to view $\mathcal{O}_X$ as the cross-sectional C^*-algebra $C^*(\mathcal{B}_X)$ of the Fell bundle $\mathcal{B}_X$ and define reduced version $\mathcal{O}_X^r$ of $\mathcal{O}_X$. This shows that under the above assumptions the algebra $\mathcal{O}_X$ is the proper object of study; in particular, it models the corresponding smiegroup crossed products by endomorphisms in a correct way.

The above results are complemented by the analysis of the reduced cross sectional C^*-algebra $C^*_r(\mathcal{B})$ associated to a Fell bundle $\mathcal{B}$ over a discrete group G, see [KS17]. The main results include: description of the ideal structure and the primitive ideal space, and effective criteria for pure infiniteness of $C^*_r(\mathcal{B})$. These tools are directly applicable, for instance, to (twisted) crossed products by partial actions, whose theory is well developed, cf. [Exel], and indirectly, through the previously established dilation of product systems also to algebras of type $\mathcal{O}_X$. The resulting criteria of pure infiniteness generalize, strengthen and unify the analogous results obtained in the context of the classical crossed products by the following authors: Laca, Spielberg [LS96]; Jolissaint, Robertson [JR00]; Rørdam, Sierakowski [RS12]; Giordano, Sierakowski [GS14] and Ortega, Pardo [OP14]. In particular, they clarify the relationship between paradoxical actions and pure infiniteness. The results of [KS17] are optimal when applied to graph algebras. They are also applied to a certain class of semigroup crossed products of Exel type [Lar10], [KL09].

We start the detailed description of the aforementioned results from single endomorphisms, so that going up along successive levels of abstraction we reach more general structures. In section 2 we discuss crossed products by endomorphisms. Section 3 presents relative crossed products by completely positive maps. Section 4 is devoted to a general theory that unifies relative Cuntz-Pimsner algebras and Doplicher-Roberts algebras. In section 5 we extend the construction of Doplicher and Roberts so that we get an effective description the semigroup version of Pimsners algebra $\mathcal{O}_X$. We also discuss here the notion of topological freeness for Hilbert bimodules, product systems and Fell bundles. Section 6 presents the results concerning the structure of ideals and pure infiniteness criteria for C^*-algebras associated with Fell bundles over discrete groups.

Contents

1.1 Notation and basic notions

Ideals in C^*-algebras are always assumed to be closed two-sided. If I is an ideal in a C^*-algebra A we write $I \lhd A$. We denote by $I^\perp = \{a \in A : aI = \{0\}\}$ the *annihilator* of an ideal I in a C^*-algebra A (it is the largest ideal in A with the property that $I^\perp \cap I = \{0\}$). By homomorphisms between C^*-algebras we mean involution preserving maps. For actions $\gamma\colon E \times F \to G$ such as multiplications, inner products, etc., we use the notation:

$$\gamma(E, F) = \overline{\operatorname{span}}\{\gamma(a, b) : a \in E, b \in F\}.$$

If A acts non-degenerately on a Hilbert space H, that is when $A \subseteq \mathcal{B}(H)$ and $AH = H$, then the algebras of left, right and two-sided multipliers can be defined respectively as follows: $M_\ell(A) := \{x \in \mathcal{B}(H) : xA \subseteq A\}$, $M_r(A) := M_\ell(A)^*$ i $M(A) := M_\ell(A) \cap M_r(A)$. These algebras can be defined abstractly as algebras of certain operators on A, or subalgebras of the enveloping von Neumann algebra A^{**}, cf. [Ped79]. The algebra $M(A)$ is a C^*-algebra (with unit 1) which could be defined as a completion of A in the *strict topology*, i.e. the topology determined by seminorms $m \mapsto \|ma\|$, $m \mapsto \|am\|$ for all $a \in A$.

A (right) *C^*-correspondence* over a C^*-algebra A is a right Hilbert A-module X, see [Lan94], equipped with left action of A on X given by a homomorphism $\phi : A \to \mathcal{L}(X)$, where $\mathcal{L}(X)$ is a C^*-algebra of adjointable operators. We write $a \cdot x = \phi(a)x$, $a \in A$, $x \in X$. Similarly, we can define left C^*-correspondences as left Hilbert modules. To each (right) C^*-correspondence X one can attach 'the inverse' left C^*-correspondence $\widetilde{X}$ (X and $\widetilde{X}$ are anti-isomorhic as linear spaces). By a *Hilbert A-bimodule* we mean a space X which is both a right and left C^*-correspondence and the right $\langle\cdot,\cdot\rangle_A$ and left ${}_A\langle\cdot,\cdot\rangle$ A-valued sesquilinear forms satisfy $x \cdot \langle y, z\rangle_A = {}_A\langle x, y\rangle \cdot z$, for every $x, y, z \in X$. In particular, a Hilbert A-bimodule is an equivalence bimodule (in the Morita-Rieffel sense) between ideals $\langle X, X\rangle_A$ and ${}_A\langle X, X\rangle$ in A, see, for instance, [RW98]. Hence Hilbert bimodules may be viewed as partial equivalences.

Representation of a C^*-correspondence X is a pair (π, π_X) where $\pi : A \to \mathcal{B}(H)$ is a representation of a C^*-algebra A on a Hilbert space H and $\pi_X : X \to \mathcal{B}(H)$ is a linear map (which is necessarily contractive) such that

$$\pi_X(a \cdot x) = \pi(a)\pi_X(x), \quad \pi_X(x \cdot b) = \pi_X(x)\pi(b), \quad \pi_X(x)^*\pi_X(y) = \pi(\langle x, y\rangle_A),$$

for all $a, b \in A$, $x \in X$. We say that the C^*-algebra $C^*(\pi(A) \cup \pi_X(X))$, generated by $\pi(A)$ and $\pi_X(X)$, is the C^*-algebra generated by (π, π_X). The set $\mathcal{K}(X)$ of *generalized compact operators* on X is by definition the closed linear span of operators $\Theta_{x,y}$ where $\Theta_{x,y}(z) = x \cdot \langle y, z\rangle_A$ for all $x, y, z \in X$. In particular $\mathcal{K}(X)$ is an ideal in the C^*-algebra $\mathcal{L}(X)$. Every representation (π, π_X) of a C^*-correspondence X induces homomorphism $(\pi, \pi_X)^{(1)} : \mathcal{K}(X) \to \mathcal{B}(H)$ which satisfies

$$(\pi, \pi_X)^{(1)}(\Theta_{x,y}) = \pi_X(x)\pi_X(y)^*,$$

for all $x, y \in X$. Let $J(X) := \phi^{-1}(\mathcal{K}(X))$ be the ideal in A consisting of those elements that act by generalized compacts. For every representation (π, π_X) of a C^*-correspondence X the restrictions $(\pi, \pi_X)^{(1)} \circ \phi|_{J(X)}$ and $\pi|_{J(X)}$ give two representations of the ideal $J(X)$. For every ideal J in $J(X)$ we say that a representation (π, π_X) of X is *J-covariant* if $(\pi, \pi_X)^{(1)} \circ \phi|_J = \pi|_J$. The *relative Cuntz-Pimsner algebra* $\mathcal{O}(J, X)$ [MS98, Definicja 2.18] can be defined as a C^*-algebra generated by a universal J-covariant representation (i_A, i_X) of X, cf. [FMR03]. Then $\mathcal{O}(J(X), X)$ is the C^*-algebra studied by Pimsner [Pim97] and $\mathcal{O}_X := \mathcal{O}(J_X, X)$ where $J_X := (\ker \phi)^\perp \cap J(X)$, is the (unrelative) *Cuntz-Pimsner algebra* popularized by Katsura, see [Kat03, Definition 2.6]. The algebra

$\mathcal{O}(J, X)$ is equipped with the action γ of the circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$, where $\gamma_z(a) = a$ and $\gamma_z(x) = zx$, for all $a \in A$, $x \in X$.

Product systems can be viewed as semigroup actions of C^*-correspondences on C^*-algebras, where composition of morphisms is replaced by inner tensor product of C^*-correspondences. For instance, if X is a C^*-correspondence over A then the family of inner tensor product powers $\{X^{\otimes n}\}_{n\in\mathbb{N}}$, where $X^{\otimes 0} := A$ is the trivial bimodule, is a semigroup with the semigroup operation given by inner tensor product. In general [Fow02], if P is a semigroup with the unit e and $\{X_p\}_{p\in P}$ is a family of C^*-correspondences over A, then we say that $X = \bigsqcup_{p\in P} X_p$ is a *product system* if $X_e = A$ is the trivial bimodule and X is a semigroup such that the semigroup operation determines isomorphisms $X_p \otimes_A X_q \cong X_{pq}$ for all $p, q \in P\backslash\{e\}$ and it coincides with the right and left action of $X_e = A$ on X_p,for $p \in P$. A *representation of a product system* X in a C^*-algeba B is a semigroup homomorphism $\psi : X \to B$ such that

$$(\psi|_A, \psi|_{X_p}) \text{ is a representation of } X_p, \text{ for all } p \in P.$$

Fowler [Fow02] defined *Pismner C^*-algebra $\mathcal{O}_X$ associated to the product system* X as a C^*-algebra generated by the image of a universal representation ι of the product system X such that $(\iota|_A, \iota|_{X_p})$ is a $J(X)$-covariant representation of the C^*-correspondence X_p for every $p \in P$.

Fell bundles can be viewed as group actions by partial equivalences - Hilbert bimodules. For instance, if $P = G$ is a group, then product systems over P coincide with saturated Fell bundles over G - group action by Morita-Rieffel equivalence bimodules. In general a *Fell bundle over a (discrete) group* G can be defined as a family $\mathcal{B} = \{B_g\}_{g\in G}$ of closed linear subspaces of a C^*-algebra B such that $B_g^* = B_{g^{-1}}$ and $B_g B_h \subseteq B_{gh}$ for all $g, h \in G$. An axiomatic (equivalent) definition, see for instance [Exel, Definicja 16.1], says that $\mathcal{B} = \{B_g\}_{g\in G}$ is a family of Banach spaces equiped with bilinear multiplication maps $B_g \times B_h \to B_{gh}$ and conjugate-linear involutions $B_g \to B_{g^{-1}}$ for $g, h \in G$, with a list of properties; in particular, $A := B_e$ is a C^*-algebra and each B_g is a Hilbert A-bimodule such that the multiplication maps yield isomorphisms $B_g \otimes_A B_h \cong B_g B_h \subseteq B_{gh}$, $g, h \in G$. We say that a Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ is *saturated* if $B_g B_h = B_{gh}$ for every $g, h \in G$. A Fell bundle $\mathcal{B} = \{B_n\}_{n\in\mathbb{Z}}$ is *semi-saturated* when $B_n B_1 = B_{n+1}$ for every $n > 0$.

If $\mathcal{B} = \{B_g\}_{g\in G}$ is a Fell bundle then the direct sum $\bigoplus_{g\in G} B_g$ is naturally equipped with the $*$-algebra structure. This $*$-algebra admits the largest C^*-norm $\|\cdot\|_{max}$ and the least C^*-norm $\|\cdot\|_r$ which satisfies the inequality

$$\|a_e\| \leqslant \|\sum_{t\in G} a_t\|_r, \qquad \text{for every } \sum_{t\in G} a_t \in \bigoplus_{t\in G} B_t,\ a_t \in B_t,\ t \in G, \tag{1}$$

Completion of the $*$-algebra $\bigoplus_{g\in G} B_g$ in the norm $\|\cdot\|_{max}$ is denoted by $C^*(\mathcal{B})$ and called the full *cross-section C^*-algebra* of $\mathcal{B}$. Completion in the norm $\|\cdot\|_r$ is denoted by $C_r^*(\mathcal{B})$ and we call it the *reduced cross-section C^*-algebra* of $\mathcal{B}$. One can equivalently define $C_r^*(\mathcal{B})$ as a C^*-completion of $\bigoplus_{g\in G} B_g$ such that the map $\bigoplus_{t\in G} B_t \ni \sum_{t\in G} a_t \to a_e \in B_e$ extends to a faithful conditional expectation $E : C_r^*(\mathcal{B}) \to B_e$. We recall that a *conditional expectation* from a C^*-algebra B onto its subalgebra A is a projection onto A of norm one. We say that a map $E : B \to A$ is *faithul* if $E(b^*b) = 0$ implies $b = 0$ for every $b \in B$.

If X is a Hilbert bimodule over A, then putting $X_n := X^{\otimes n}$ and $X_{-n} := \widetilde{X}^{\otimes n}$ for all $n \in \mathbb{N}$, the family $\{X_n\}_{n\in\mathbb{Z}}$ is naturally equipped with the structure of a Fell bundle. Then both of the C^*-algebras $\mathcal{O}_X$ and $C^*(\{X_n\}_{n\in\mathbb{Z}})$ are naturally isomorhic to the product system $A \rtimes_X \mathbb{Z}$ defined in [AEE98]. The Fell bundle $\{X_n\}_{n\in\mathbb{Z}}$ is semi-saturated and every semi-saturated Fell bundle over $\mathbb{Z}$ is of this form.

Cuntz [Cu77] defined *simple purely infinite C^*-algebras* as simple C^*-algebras whose every non-zero hereditary subalgebra contains an inifinite projection. The notion of pure infiniteness for arbitrary C^*-algebras was defined by Kirchberg and Rørdam in [KR00]. Let us briefly recall it. For every two elements $a, b \in A$ of a C^*-algebra A and $\varepsilon > 0$ we write $a \approx_\varepsilon b$ when $\|a - b\| < \varepsilon$. For $a, b \in B^+$, we write $a \precsim b$ and say that *a supports b* in Cuntz sense if for every $\varepsilon > 0$ there is $x \in A$ such that $xbx^* \approx_\varepsilon a$ (that is, we may compress b to a up to an arbitrarily small epsilon).

Definition 1.1. Let A be a C^*-algebra. An element $a \in A^+$ *is infinite* if there is $b \in A^+\backslash\{0\}$ such that $a \oplus b \precsim a \oplus 0$ in the algebra $M_2(A)$. An element $a \in A^+\backslash\{0\}$ is *properly infinite* if $a \oplus a \precsim a \oplus 0$. We say that the C^*-algebra A is *purely infinite* if every element $a \in A^+\backslash\{0\}$ is properly infinite.

Remark 1.2. Proper infiniteness of an element may be viewed as a *'residual infiniteness'*. Indeed, by [KR00, Stwierdzenie 3.14], an element $a \in A^+\backslash\{0\}$ is properly infinite in A if and only if the element $a + I$ is infinite in A/I for every ideal I in A, with $a \notin I$.

2 Crossed products by endomorphisms

In this section we present a general construction of relative crossed products $C^*(A, \alpha; J)$ by an endomorphism $\alpha : A \to A$ where ideals $J \lhd A$ play a role of "parameters" that describe all possible crossed products by α. We discuss here: relationships between these objects and previously considered constructions (cf. Tables 1, 2); the ideal structure of $C^*(A, \alpha; J)$ (Figure 1, Theorems 2.11, 2.13, 2.14, 2.19); pure infiniteness criteria and faithfulness criteria for representations of $C^*(A, \alpha; J)$ (Theorems 2.19, 2.21); as well as K-theory of ideals and quotients of $C^*(A, \alpha; J)$ (Theorems 2.16, 2.24, 2.25).

The aforementioned results are the culmination of the research described in [KL13], [Kwa15], [Kwa16]. In [KL13] we studied the case when A is unital and in [Kwa15] the case when the endomorphism α is *extendible*, i.e. it extends to a strictly continuous endomorphism $\alpha : M(A) \to M(A)$ of the multiplier algebra $M(A)$. In [Kwa16] general endomorphisms are considered, however the second part of [Kwa16] is devoted to the case when A is a $C_0(X)$-algebra.

2.1 Basic definitions and facts

Let $\alpha : A \to A$ be an endomorphism of a C^*-algebra A.

Definition 2.1 (Definition 2.4 in [Kwa16])**.** A *representation* of an endomorphism (A, α) in a Hilbert space H is a pair (π, U) where $\pi : A \to \mathcal{B}(H)$ is non-degenerate representation of A and $U \in \mathcal{B}(H)$ is an operator such that

$$U\pi(a)U^* = \pi(\alpha(a)), \qquad \text{for all } a \in A. \tag{2}$$

We call $C^*(\pi, U) := C^*(\pi(A) \cup U\pi(A))$ the *C^*-algebra generated by* (π, U).

Remark 2.2. For every representation (π, U) we have

$$C^*(\pi, U) = \overline{\text{span}}\{U^{*n}\pi(a)U^m : a \in \alpha^n(A)A\alpha^m(A)\}$$

and $U \in M_\ell(C^*(\pi, U))$ is a left multiplier of the algebra $C^*(\pi, U)$. If α is extendible, then $U \in M(C^*(\pi, U))$. If A is unital then $U \in C^*(\pi, U)$.

The above definition (and multiplicativity of α) automatically implies the following relationships. This fact was often not noticed in the literature, cf. [KL13, Remark 1.3]. Firstly, U is necessarily a partial isometry. Secondly, we have the following commutation relation

$$U\pi(a) = \pi(\alpha(a))U \qquad \text{for every } a \in A. \tag{3}$$

Thirdly, the projection U^*U belongs to the commutant $\pi(A)'$ of the algebra $\pi(A)$. Accordingly, the set $\{a \in A : U^*U\pi(a) = \pi(a)\}$ is in fact an ideal in the C^*-algebra A. This ideal carries an important information on the representation (π, U).

Definition 2.3. Let J be an ideal in A and let (π, U) be a representation of the endomorphism (A, α). We say that (π, U) is *J-covariant representation* if $J \subseteq \{a \in A : U^*U\pi(a) = \pi(a)\}$. We say that (π, U) is a *covariant representation* if it is J-covariant for $J = (\ker \alpha)^\perp$.

Examples 2.4. It is not difficult to see that if α is a monomorphism, then the representation (π, U) is covariant if and only if U is an isometry. If α is an automorphism, then the representation (π, U) is covariant if and only if U is a unitary operator, and then $U^*\pi(a)U = \pi(\alpha^{-1}(a))$ for $a \in A$.

We define the crossed product associated to (A, α) and $J \lhd A$ as a C^*-algebra generated by a universal J-covariant representation. Criteria for existence of universal representations, or universal C^*-algebras defined in terms of generators and relations, are well known, cf. the discussion in [Kwa17, Subsection 2.2].

The analysis in [KL13], [Kwa15], [Kwa16] is focused mainly on the case when $J \subseteq (\ker \alpha)^\perp$. The reason for this is that only in this case the algebra A embeds into the crossed product. Moreover, one can easily reduce the general situation to this case by passing to a quotient endomorphism, cf. Remarks 2.6, 2.7 below. Here we formulate the definition of $(C^*(A, \alpha; J)$ without any restrictions on the ideal J.

Definition 2.5 (cf. Definition 2.7 in [Kwa16])**.** *The relative crossed product* of an endomorphism (A, α) relative to an ideal $J \lhd A$ is a triple $(C^*(A, \alpha; J), \iota_A, u)$ where $C^*(A, \alpha; J)$ is a C^*-algebra, $\iota_A : A \to C^*(A, \alpha; J)$ is a non-degenerate homomorphism and $u \in M_\ell(C^*(A, \alpha; J))$ is a left multiplier of the algebra $C^*(A, \alpha; J)$ such that

i) $\iota_A(\alpha(a)) = u\iota_A(a)u^*$ for every $a \in A$ and $J \subseteq \{a \in A : u^*u\iota_A(a) = \iota_A(a)\}$;

ii) $C^*(A, \alpha; J)$ is generated by $\iota_A(A) \cup u\iota_A(A)$;

iii) for every J-covariant representation (π, U) of (A, α) there is a representation $\pi \rtimes U$ of the C^*-algebra $C^*(A, \alpha; J)$ such that $(\pi \rtimes U)(\iota_A(a)) = \pi(a)$, for all $a \in A$, and $(\pi \rtimes U)(u) = U$.

Conditions (i), (ii), (iii) determine $C^*(A,\alpha;J)$ up to a natural isomorphism. The relative crossed product is usually identified with this C^*-algebra. In the case when $J=(\ker\alpha)^\perp$ we write $C^*(A,\alpha):=C^*(A,\alpha;(\ker\alpha)^\perp)$ and we call it the (unrelative) *crossed product of A by α*.

Remark 2.6. The algebra $C^*(A,\alpha;J)$ was constructed in [KL13] in an explicit way: by elaborating a special "matrix calculus" [KL13, 2.1] and establishing concrete formulas of norms of elements in spectral subspaces [KL13, 2.2]. As shown in [Kwa13] this construction is very general: it was extended so that it covers both the relative Cuntz-Pimnser algebras and Doplicher-Roberts algebras. We will discuss it in more detail in subsection 4.2. It follows from these constructions that the universal homomorphism $\iota_A: A\to C^*(A,\alpha;J)$ is injective if and only if $J\subseteq(\ker\alpha)^\perp$, see also [Kwa16, Lemma 2.5, Proposition 2.6].

Remark 2.7. As noticed in [KL13, Corollary 4.14] and shown in general in [Kwa16, Appendix], see also [Kwa17, Subsection 3.4], we have natural isomorphisms

$$C^*(A,\alpha;J)\cong\mathcal{O}(J,E_\alpha),\qquad C^*(A,\alpha)\cong\mathcal{O}_{E_\alpha}, \tag{4}$$

where E_α is a C^*-correspondence defined by the formulas:

$$E_\alpha:=\alpha(A)A,\ \ \langle x,y\rangle_A:=x^*y,\ \ a\cdot x\cdot b:=\alpha(a)xb,\quad x,y\in\alpha(A)A,\ a,b\in A.$$

For this C^*-correspondence we have $J(E_\alpha)=A$. For a general C^*-correspondence X over A, it is known [Kat04] that A embeds into the relative Cuntz-Pimsner algebra $\mathcal{O}(J,X)$ if and only if $J\subseteq J_X=J(X)\cap\ker\phi^\perp$. In [KL13, 5.1], by determining the kernel of the homomorphism ι_A, it is described how to reduce the C^*-correspondence X to a quotient C^*-correspondence X/XJ_∞ such that

$$\mathcal{O}(J,X)\cong\mathcal{O}(q_{J_\infty}(J),X/X_{J_\infty})\quad\text{ and }\quad q_{J_\infty}(J)\subseteq J_{X/XJ_\infty}.$$

This method, called *reduction of relations*, was generalized in [Kwa13, Theorem 6.23] to the case of C^*-algebras associated to ideals in right tensor C^*-precategories, cf. Remark 4.18 below. It allows one to reduce the considerations to the case when initial algebra A embeds into the universal object. When $X=E_\alpha$ we have $J_\infty=\{a\in A:\alpha^n(a)\in J$ for every $n\in\mathbb{N}$ and $\lim_{n\to\infty}\alpha^n(a)=0\}$. Hence, by passing if necessary to the quotient endomorphism of A/J_∞, we may always assume that $J\subseteq\ker\alpha^\perp$.

In [KL13] and [Kwa17], a number of relationships between various crossed products by endomorphisms was established. Some special cases of (unrelative) crossed products considered in the unital case are summarized in Table 1, cf. [KL13, Subsection 3.4].
In [ABL11] (see case 4 in Table 1) the authors studied a crossed product by an endomorphism that admits the so called complete transfer operator. As noted by the habilitant, an endomorphism admits such an operator if and only if it is partially reversible in the following sense.

Definition 2.8 (Definition 2.1 in [Kwa16])**.** We say that an endomorphism (A,α) is *partially reversible* if the kernel $\ker\alpha$ is complemented in A (i.e. $A=\ker\alpha\oplus(\ker\alpha)^\perp$) and the range $\alpha(A)$ is hereditary subalgebra of A (i.e. $\alpha(A)A\alpha(A)=\alpha(A)$).

N.	$\alpha : A \to A$	$C^*(A,\alpha)$	relations
1.	automorphism	classical crossed product [Tur58]	$UAU^* = A$ $UU^* = U^*U = 1$
2.	$\ker\alpha = \{0\}$, $\alpha(A)$ hereditary in A	Cuntz [Cu77], Paschke [Pas80]	$UAU^* \subseteq A$, $U^*AU \subseteq A$ $U^*U = 1$
3.	$\ker\alpha = \{0\}$	Cuntz [Cun82], Murphy [Mur96]	$UAU^* \subseteq A$, $U^*U = 1$
4.	$\ker\alpha \lhd A$ complemented, $\alpha(A)$ hereditary in A	Antonevich, Bakhtin, Lebedev [ABL11]	$UAU^* \subseteq A$, $U^*AU \subseteq A$ $U^*U \in A'$

Table 1: Special cases of crossed products (A is unital)

Remark 2.9. By [Kwa16, Proposition A.11] an endomorphism is partially reversible if and only if the C^*-correspondence E_α is a Hilbert bimodule.

Stacey proposed in [Sta93] a universal definition of crossed product for an 'arbitrary' endomorphism α on an arbitrary C^*-algebra A. However, he tacitly assumed that α extends to an endomorphism of $M(A)$ which led to the introduction of the concept of extendible endomorphisms [Adj95]. From that moment the assumption of extendability has become an inherent part of any discussion concerning crossed products by endomorphisms. Nevertheless, as shown by the habilitant, the assumption of extendability is not only not necessary, but it is in fact an obstruction in the development of the theory. For instance it prevents a harmonious description of ideals in crossed products (restriction of an extendible endomorphism to an invariant ideal need not be extendible).

Another shortcoming of Stacey's definition is the fact that the algebra A embeds in his crossed product if and only if α is injective. In particular, if α is *pointwise quasinilpotent*, i.e. when $\lim_{n\to\infty} \alpha^n(a) = 0$ for every $a \in A$, Stacey's crossed product degenerates to $\{0\}$. This remark has become a motivation for another popular definition proposed by Lindiarni and Raeburn [LR04]. The latter, however, is not a generalization of classical crossed products - it leads to algebras of Toeplitz type. Both constructions are covered by our algebras $C^*(A,\alpha;J)$ as presented in Table 2, see [Kwa17, Proposition 3.26].

N.	$J \lhd A$	$C^*(A,\alpha;J)$	relations
1.	$J = A$	Stacey's crossed product [Sta93]	$UAU^* \subseteq A$ $U^*U = 1$
2.	$J = \{0\}$	partial-isometric crossed products [LR04]	$UAU^* \subseteq A$, $(U^*U)^2 = U^*U \in A'$

Table 2: Cases of relative crossed products (α is extendible)

Considering relative crossed products one allows to comprise different constructions. But it is also necessary in the analysis of ideal structure of unrelative crossed products, cf. [Kwa15, Remark 3.2].

2.2 Structure of crossed products by endomorphisms

Let α be an endomorphism of a C^*-algebra A and let $J \lhd (\ker \alpha)^{\perp}$. Then A embeds into $C^*(A, \alpha; J)$ and we may treat it as a subalgbera of $C^*(A, \alpha; J)$. The crossed product is naturally equipped with the circle action which is identity on A and maps u to λu, for $\lambda \in \mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$; this makes isomorphisms (4) equivariant. C^*-subalgebra $B \subseteq C^*(A, \alpha; J)$ consisting of fixes points for this action contains A and the universal partial isometry u generates on B an endomorphism $\beta : B \to B$ such that the system (B, β) is partially reversible and

$$C^*(A, \alpha; J) \cong C^*(B, \beta).$$

In [Kwa15, 3.1] and [Kwa16, 2.6] an explicit construction of the system (B, β) in terms of the triple (A, α, J) is presented. It uses operations of taking quotients, passing to hereditary subalgebras, direct sums and direct limits. The system (B, β) is called the *(universal) reversible J-extension* of the endomorphism α.

In [Kwa15, 3.1] this construction was studied under the assumption that α is extendible. Then the endomorphism β is also extendible (admits a complete transfer operator) and the system (B, β) is a universal object in the category of C^*-algebras with morphisms that are extendible homomorphisms, see [Kwa15, Theorem 3.1]. By a direct analysis of (B, β) a number of relationships between various ideals are established, cf. Theorem 2.13 below. These relationships are summarized on Figure 1: all mentioned sets form lattices; the arrow $A \Longrightarrow B$ means that there is an order retraction[1] from the lattice A onto the lattice B (which in general is not an isomorphism), and $A \Longleftrightarrow B$ means that A and B are order isomorphic.

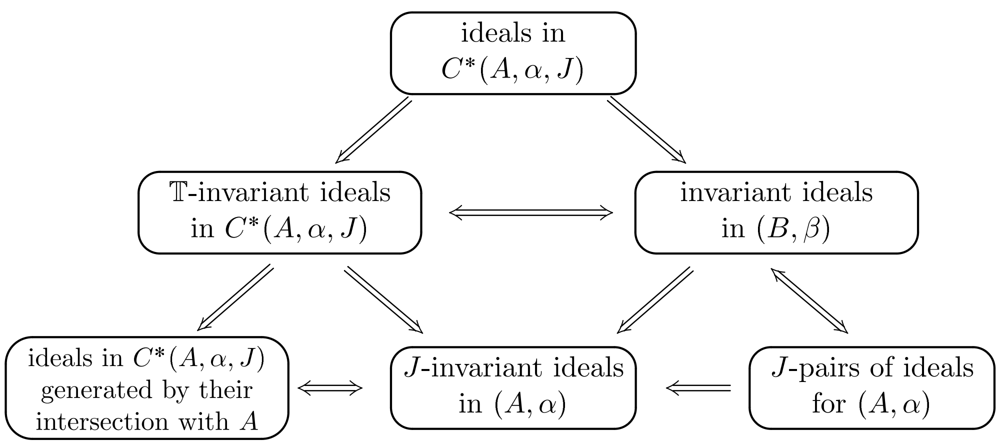

Figure 1: Relationships between the lattices of ideals

Definition 2.10 (Definitions 3.3 and 3.4 in [Kwa15]). Let I, I', J be ideals in A. We say that I is *J-invariant* in (A, α) if $\alpha(I) \subseteq I$ and $J \cap \alpha^{-1}(I) \subseteq I$. We say that (I, I') is a *J-pair* in (A, α) if $\alpha(I) \subseteq I$, $J \subseteq I'$ and $I' \cap \alpha^{-1}(I) = I$. When $J = (\ker \alpha)^{\perp}$ we drop the prefix 'J-'.

[1] an order preserving surjection $r : A \to B$ which has an order preserving right inverse

It follows from diagram on Figure 1 that one of the key issues is to establish conditions implying that all the ideals in $C^*(A,\alpha;J)$ are $\mathbb{T}$-invariant. Such conditions at the level of the partially reversible system (B,β) can be derived from [Kwa14], because E_β is a Hilbert bimodule, see Theorems 5.4 and 3.15 below. This type of analysis led to characterization of simplicity of $C^*(A,\alpha)$ in the case $A = C_0(X)$ is commutative, in terms of the dual topological dynamical system (X,φ), [Kwa15, Theorem 4.4]. In general we have the following simplicity criterion[2]. We say that α is *minimal* if there are no non-trivial invariant ideals in (A,α); we say that α is *outer* if there is no $v \in M(A)$ such that $\alpha(a) = vav^*$, $a \in A$.

Theorem 2.11 (Theorem 4.2 in [Kwa15]). *If $C^*(A,\alpha,J)$ is simple then $J = (\ker\alpha)^\perp$, α is minimal and either α is pointwise quasinilpotent or α is a monomorphism and no power α^n, $n > 0$, is inner.*

Conversely, if α is minimal, then each of the following conditions implies that $C^*(A,\alpha)$ is simple:

i) α is pointwise quasinilpotent,

ii) α is injective, A is unital, and every power α^n, $n > 0$, is outer,

iii) α is injective with hereditary range, A is separable, and every power α^n, $n > 0$, is outer.

Remark 2.12. In light of the recent result [KM, Theorem 9.14], the assumption of separability in item (iii) is superfluous.

Restriction of an extendible endomorphism to an invariant ideal need not be extendible. This is one of the main reasons why in [Kwa16] we studied arbitrary, *not necessarily extendible endomorphisms*. This made it possible to describe not only the quotients of crossed products but also the ideals themselves. Here we identify A with the C^*-subalgebra $\iota_A(A) \subseteq C^*(A,\alpha;J)$, which is allowed by our assumption that $J \lhd (\ker\alpha)^\perp$.

Theorem 2.13 (Theorem 2.19 in [Kwa16]). *We have a bijective correspondence between J-pairs of ideals (I,I') in (A,α) and $\mathbb{T}$-invariant ideals $\mathcal{I}$ in $C^*(A,\alpha;J)$ where $I = A\cap\mathcal{I}$, $I' = \{a \in A : (1-u^*u)a \in \mathcal{I}\}$. For objects satisfying these relations we have*

$$C^*(A,\alpha;J)/\mathcal{I} \cong C^*(A/I,\alpha_I;q_I(I'))$$

and if $I' = I + J$ (equivalently $\mathcal{I}$ is generated by A), then $\mathcal{I}$ is Morita-Rieffel equivalent to the crossed product $C^*(I,\alpha|_I;I\cap J)$.

When $\ker\alpha$ is a complemented ideal in A and $J = (\ker\alpha)^\perp$, then for every J-pair of ideals in (A,α) we have $I' = I + J$. Hence the above theorem simplifies to the following:

Theorem 2.14 (Corollary 2.21 in [Kwa16]). *If $\ker\alpha$ is a complemented ideal in A, then relations $I = A\cap\mathcal{I}$, $\mathcal{I}$ is generated by I give bijective correspondence between invariant ideals I in (A,α) and $\mathbb{T}$-invariant ideals $\mathcal{I}$ in $C^*(A,\alpha)$, under which $C^*(A,\alpha)/\mathcal{I} \cong C^*(A/I,\alpha_I)$ and $\mathcal{I}$ is Morita-Rieffel equivalent to $C^*(I,\alpha|_I)$.*

[2] in [Kwa15] it is formulated under the assumption that α is extendible, but it remains true in general

Remark 2.15. Let α and $J \lhd (\ker\alpha)^{\perp}$ be arbitrary. There is a canonical construction of an endomorphism $\alpha^J : A^J \to A^J$ which extends α ($A \subseteq A^J$) and such that $\ker\alpha^J$ is complemented in $A^J$and $C^*(A,\alpha;J) \cong C^*(A^J,\alpha^J)$, see [Kwa16, 2.4]. Accordingly, by passing from (A,α) to (A^J,α^J) we may reduce the situation of Theorem 2.13 to the situation of Corollary 2.14, see [Kwa16, Proposition 2.25].

The analysis in [Kwa15] is self-contained. It is completely independent of the theory of C^*-correspondences and Cuntz-Pimsner algebras. In contrast, in [Kwa16] the general analysis of the structure of $C^*(A,\alpha;J)$ is based on isomorphisms (4) and general facts from the theory of Cuntz-Pimsner algebras [Kat04], [Kat07], also developed by the habilitant in [Kwa13], [Kwa16, Appendix]. One of the results is the following generalization of the celebrated Pimsner-Voiculescu exact sequence.

Theorem 2.16 (Proposition 2.26 in [Kwa16])**.** *For any endomorphism α and any ideal J in $(\ker\alpha)^{\perp}$ we have the following exact sequence:*

$$\begin{array}{ccccc}
K_0(J) & \xrightarrow[K_0(\iota)-K_0(\alpha|_J)]{} & K_0(A) & \xrightarrow[K_0(\iota)]{} & K_0(C^*(A,\alpha;J))\ , \\
\uparrow & & & & \downarrow \\
K_1(C^*(A,\alpha;J)) & \xleftarrow{K_1(\iota)} & K_1(A) & \xleftarrow{K_1(\iota)-K_1(\alpha|_J)} & K_1(J)
\end{array}$$

where ι denotes inclusion.

In view of Corollary 2.14 and Remark 2.15, the above theorem leads to the description of exact sequences for all $\mathbb{T}$-invariant ideals and the corresponding quotinets, see [Kwa16, Theorem 2.27]. This is an important result because information concerning these sequences has a fundamental meaning in classification of non-simple C^*-algebras. In this context, it is also important that we have the following results established in [Kwa16, Proposition 2.10] for an arbitrary endomorphisms α:

(i) A is exact $\Longleftrightarrow C^*(A,\alpha;J)$ is exact.

(ii) A is nuclear $\Longrightarrow C^*(A,\alpha;J)$ is nuclear.

(iii) If A is separable and nuclear and both A and J satisfy the UCT then $C^*(A,\alpha;J)$ satisfies the UCT.

If (A,α) is partially reversible, general pure infiniteness criteria for $C^*(A,\alpha)$ are established in [Kwa16, Proposition 2.46] and natural conditions ensuring that all ideals in $C^*(A,\alpha)$ are $\mathbb{T}$-invariant are given in [Kwa16, Proposition 2.35]. These pure infiniteness criteria were generalized in [KS17] and they will be discussed in subsection 6 (cf. Theorem 6.6 below).

To obtain more accurate and concrete results in the second part of [Kwa16] endomorphisms of $C_0(X)$-algebras were studied.

2.3 Crossed products of endomorphisms of $C_0(X)$-algebras

Let X be a locally compact Hausdorff space and let A be a $C_0(X)$*-algebra*. That is A is a C^*-algebra equipped with a non-degenerate homomorphism from $C_0(X)$ into the center $Z(M(A))$ of the multiplier algebra $M(A)$ of A (A has the structure of a $C_0(X)$-module). Equivalently, the algebra A is the C^*-algebra of sections of a certain upper semicontinuous bundle of C^*-algebras $\mathcal{A} = \bigsqcup_{x\in X} A(x)$. In [Kwa16, Definition 3.1] we introduced a notion of morphism of $C_0(X)$-algebras which by [Kwa16, Proposition 3.5] can be defined equivalently as follows.

Definition 2.17 ([Kwa16])**.** Let A be a $C_0(X)$-algebra and B a $C_0(Y)$-algebra. We say that a homorphism $\alpha : B \to A$ is *induced by a morphism* if there exits a homomorphism $\Phi : C_0(Y) \to C_0(X)$ such that $\alpha(f \cdot b) = \Phi(f) \cdot \alpha(b), f \in C_0(Y),\ b \in B$.

Remark 2.18. The above definition plays a fundamental role in [McC] where the author studies universal objects in the category of $C_0(X)$-algebras with morphisms of $C_0(X)$-algebras understood in this way.

Homomorphism $\Phi : C_0(Y) \to C_0(X)$ is given by a proper and continuous map $\varphi : \Delta \to Y$ defined on an open set $\Delta \subseteq X$. We may treat A and B as algebras of sections of bundles $\mathcal{A} = \bigsqcup_{x\in X} A(x)$ and $\mathcal{B} = \bigsqcup_{y\in Y} B(y)$. The relation $\alpha(f \cdot b) = \Phi(f) \cdot \alpha(b)$ implies existence of homomorphisms $\alpha_x : B(\varphi(x)) \to A(x)$, $x \in \Delta$ such that

$$\alpha(b)(x) = \begin{cases} \alpha_x(b(\varphi(x)), & x \in \Delta, \\ 0 & x \notin \Delta, \end{cases} \qquad b \in B,\ x \in X. \tag{5}$$

We interpret the pair $(\varphi, \{\alpha_x\}_{x\in\Delta})$ as a morphism from $\mathcal{B}$ to $\mathcal{A}$, cf. [Kwa16, Proposition 3.2]. If A is a $C_0(X)$-algebra and $\alpha : A \to A$ is induced by a morphism, we call an endomorphism (A, α) a $C_0(X)$*-dynamical system*, [Kwa16, Definition 3.2].

Let (A, α) be a $C_0(X)$-dynamical system induced by $(\varphi, \{\alpha_x\}_{x\in\Delta})$ and let $J \lhd (\ker\alpha)^\perp$. It is shown in [Kwa16, Theorem 4.9] that reversible J-extension (B, β) is in a natural way a $C_0(\widetilde{X})$-dynamical system induced by a morphism $(\widetilde{\varphi}, \{\beta_{\widetilde{x}}\}_{\widetilde{x}\in\widetilde{\Delta}})$ where $(\widetilde{X}, \widetilde{\varphi})$ is a topological universal reversible Y-extension of the topological system (X, φ) ($Y \subseteq X$ is given by J). This description and general theorems for partially reversible systems led to the following results.

We say that a C^*-algebra A has the *ideal property*, if every ideal in A is generated (as an ideal) by the projections it contains. A commutative C^*-algebra $C_0(X)$ has the ideal property if and only if the space X is totally disconnected.

Theorem 2.19 (Theorem 4.12 in [Kwa16])**.** *Let φ be a free map (i.e. φ have no periodic points). All ideals in $C^*(A, \alpha; J)$ are $\mathbb{T}$-invariant and they are in bijective correspondence with J-pairs of ideals in (A, α). Moreover,*

(i) If A has the ideal property and is purely infinite then the same holds for $C^*(A, \alpha; J)$.

(ii) If there are only finitely many J-pairs of ideals in (A, α) and A is purely infinite, then $C^*(A, \alpha; J)$ is purely infinite.

In order to simplify the presentation, from now on we will assume that $\mathrm{Prim}(A)$ is a Hausdorff space and that $X = \mathrm{Prim}(A)$, cf. [Kwa16, Example 3.8].

Definition 2.20. We say that a periodic orbit $\mathcal{O} = \{x, \varphi(x), ..., \varphi^{n-1}(x)\}$ of a periodic point $x = \varphi^n(x)$ *has an entry* $y \in \Delta$ if $y \notin \mathcal{O}$ and $\varphi(y) \in \mathcal{O}$. We say that φ is *topologically free outside* $Y \subseteq X$ if the set of periodic points whose orbits do not have entries and do not intersect Y has empty interior [Kwa15, Definitions 4.7, 4.8], [Kwa16, Definition 2.37].

A particular case of [Kwa16, Theorem 4.11] (cf. [Kwa15, Proposition 4.8]) gives:

Theorem 2.21 (Uniqueness theorem)**.** *Let us assume that φ is topologically free outside $Y = \mathrm{Prim}(A/J)$. A representation (π, U) of an endomorphism (A, α) integrates to a faithful representation $\pi \rtimes U : C^*(A, \alpha; J) \to C^*(\pi, U)$ if and only if π is faithful and $J = \{a \in A : U^*U\pi(a) = \pi(a)\}$.*

Remark 2.22. Let us note that if $J = \{0\}$, then $Y = \mathrm{Prim}(A/J) = \mathrm{Prim}(A) = X$ and any map φ is topologically free outside X. In particular, in the case when $J = \{0\}$ and $A = \mathbb{C}$, Theorem 2.21 gives classical Coburn's theorem (on uniqueness of the C^*-algebra generated by a non-invertible isometry).

The assumption that $X = \mathrm{Prim}(A)$ implies that ideals in A, and therefore also J-invariant ideals and J-pairs of ideals, can be described by open (equivalently closed) sets in X. In [Kwa15, Definition 4.9], see also [Kwa16, Definition 5.6], the notions of Y-invariant sets and Y-pairs of closed subsets of X were introduced. This led to the following results:

- Description of the ideal structure of $C^*(A, \alpha; J)$ in terms of Y-pairs of sets for the topological dynamical system (X, φ), [Kwa16, Proposition 5.6];
- Pure infiniteness criterion for $C^*(A, \alpha; J)$, [Kwa16, Proposition 5.8];
- Characterisation of simplicity of $C^*(A, \alpha)$, [Kwa16, Proposition 5.9];
- Necessary and sufficient conditions for $C^*(A, \alpha)$ to be a Kirchberg algebra (under the assumption that A is a Kirchberg algebra), [Kwa16, Corollary 5.10].

If the bundle $\mathcal{A} = \bigsqcup_{x \in X} A(x)$ is trivial, A is necessarily of the form $A = C_0(X, D)$ where D is a simple C^*-algebra. In this case, under additional assumptions that X is a totally disconnected space, the group $G = K_0(D)$ has no torsion and $K_1(D) = 0$, explicit formulas for K-theory of all ideals and quotients of $C^*(A, \alpha; J)$ were given in [Kwa16, Subsection 5.3]. More specifically, let us retain the above assumptions and consider the following groups:

Definition 2.23 (Definition 5.12 in [Kwa16])**.** Let δ_α be a group homomorphism $\delta_\alpha : C_0(X, G) \to C_0(X, G)$ given by the formula

$$\delta_\alpha(f)(x) = \begin{cases} f(x) - K_0(\alpha_x)(f(\varphi(x))), & x \in \Delta \\ 0 & x \notin \Delta. \end{cases}$$

Let $\delta_\alpha^Y : C_0(X\backslash Y, G) \to C_0(X, G)$ be the restriction of δ_α. We put

$$K_0(X, \varphi, \{\alpha_x\}_{x\in\Delta}; Y) := \operatorname{coker}(\delta_\alpha^Y), \qquad K_1(X, \varphi, \{\alpha_x\}_{x\in\Delta}; Y) := \ker(\delta_\alpha^Y).$$

Theorem 2.24 (Proposition 5.13 in [Kwa16])**.** *Suppose that $A = C_0(X, D)$ where X is totally disconnected and D is a simple C^*-algebra such that $K_0(D)$ has no torsion and $K_1(D) = 0$. Then we have natural isomorphisms:*

$$K_i(C^*(A, \alpha; J)) \cong K_i(X, \varphi, \{\alpha_x\}_{x\in\Delta}; Y), \qquad i = 1, 2. \tag{6}$$

Using the above theorem and description of the ideal structure by Y-pairs of sets, one can achieve *formulas for K-theory of all ideals and quotients of $C^*(A, \alpha; J)$*, see [Kwa16, Theorem 5.14]. In the case when $\ker\alpha$ is a complemented ideal (equivalently $\varphi(\Delta)$ is an open set in X) this result simplifies to the following one:

Theorem 2.25 (Corollary 5.15 in [Kwa16])**.** *Let us retain the assumptions of Theorem 2.24 and additionally assume that φ is free and $\varphi(\Delta)$ is open in X. For every ideal $\mathcal{I}$ in $C^*(A, \alpha)$ we have isomorphisms $K_i(C^*(A, \alpha)/\mathcal{I}) \cong K_i(V, \varphi|_{\Delta\cap V}, \{\alpha_x\}_{x\in\Delta\cap V})$ and $K_i(\mathcal{I}) \cong K_i(X\backslash V, \varphi|_{\Delta\backslash\varphi^{-1}(V)}, \{\alpha_x\}_{x\in\Delta\backslash\varphi^{-1}(V)})$ where $C_0(X\backslash V, D) = C_0(X, D)\cap\mathcal{I}$, $i = 1, 2$.*

Remark 2.26. The above results give a complete set of tools for the construction and classification of a large class of purely infinite non-simple C^*-algebras. Ideal structure is completely controlled by the topological dynamical system (X, φ), while the field of endomorphism $\{\alpha_x\}_{x\in\Delta}$ can be used to influence the K-groups, cf. [Kwa16, Examples 5.17, 5.18]. Let us recall that that there is a general machinery, developed by Kirchberg [Kir00], that classifies (separable and nuclear) purely infinite C^*-algebras. However, invariants used by Kirchberg - the ideal related KK-theory, are complicated and there is still a lot of effort put into classification of certain classes of such algebras by means of some less complex invariants, cf. for instance [MN12], [Bon02], [Rør97]. Fundamental information, which is necessary and often also sufficient for classification of purely infinite C^*-algebras is K-theory of all of their ideals and quotients.

3 Crossed products by completely positive maps

In statistical mechanics and noncommutative harmonic analysis evolution of a system is often implemented by completely positive maps which are not multiplicative (for instance by transfer operators or interactions). Exel's crossed product [Exe03] is an important construction whose essential ingredient, apart from an endomorphism, is a transfer operator. This crossed product provides a direct link between Cuntz-Krieger algebras and Markov chains.

In this section, which is based on the results of [Kwa17], [Kwa14'], we present a new construction of crossed products $C^*(A, \varrho; J)$ by an arbitrary completely positive map $\varrho : A \to A$. This construction (Definition 3.2, Theorem 3.3) unifies crossed products by endomorphisms (Theorem 3.5) and Exel's crossed products, for which it also gives a new description of their internal structure (Theorem 3.9). In particular, Exel-Royer crossed products are completely independent on the choice of an endomorphism (Theorem 3.11).

The construction of $C^*(A, \varrho; J)$ allows uniform analysis and give new insight into the structure of objects such as: C^*-algebras associated to topological relations and Markov operators (paragraph 3.1.1), graph algebras (Theorem 3.12) and crossed products by interactions (subsection 3.3). For corner interactions we obtain strong tools to study the ideal structure and K-theory of their crossed products (Theorem 3.15). When applied to a newly introduced canonical interaction associated to a graph this leads to description of ideal structure and K-theory computation for graph algebras (paragraph 3.3.1).

3.1 Basic definitions and facts

In this section $\varrho : A \to A$ denotes a completely positive map on a C^*-algebra A. The definition of crossed products by ϱ was inspired by Exel's construction [Exe03]. We emphasize that when ϱ is an endomorphism, the following definition agrees with Definition 2.1 by putting $U = S^*$.

Definition 3.1 (Definition 3.1 in [Kwa17])**.** A *representation of the completely positive map* (A, ϱ) is a pair (π, S) where $\pi : A \to \mathcal{B}(H)$ is a non-degenerate representation and $S \in \mathcal{B}(H)$ is such that

$$S^*\pi(a)S = \pi(\varrho(a)) \qquad \text{for all } a \in A. \tag{7}$$

We denote by $C^*(\pi, S)$ the C^*-algebra generated by $\pi(A) \cup \pi(A)S$. *Toeplitz algebra for* (A, ϱ) is the C^*-algebra $\mathcal{T}(A, \varrho) := C^*(i_A(A), t)$ generated by a universal representation (i_A, t) of (A, ϱ).

A *redundancy* of a representation (π, S) of (A, ϱ) is a pair $(\pi(a), k)$ where $a \in A$, $k \in \overline{\pi(A)S\pi(A)S^*\pi(A)}$ and $\pi(a)\pi(b)S = k\pi(b)S$ for every $b \in A$, [Kwa17, Definition 3.3]. We note, see [Kwa17, Proposition 2.2], that the set

$$N_\varrho := \{a \in A : \varrho((ab)^*ab)) = 0 \text{ for every } b \in A\} \tag{8}$$

is the largest ideal in A contained in the kernel of the map $\varrho : A \to A$.

Definition 3.2 (Definition 3.5 in [Kwa17])**.** For any ideal J in A we define the *relative crossed product* $C^*(A, \varrho; J)$ as the quotient of Toeplitz algebra $\mathcal{T}(A, \varrho)$ by an ideal generated by the set

$$\{i_A(a) - k : a \in J \text{ and } (i_A(a), k) \text{ is a redundancy for } (i_A, t)\}.$$

We denote by (j_A, s) the representation of (A, ϱ) that generates $C^*(A, \varrho; J)$. In the case when $J = N_\varrho^\perp$ we write $C^*(A, \varrho) := C^*(A, \varrho; N_\varrho^\perp)$ and we call this algebra (the unrelative) *crossed product of* A *by* ϱ.

There is a natural C^*-correspondence X_ϱ associated to ϱ. It is called *GNS-correspondence* [Pas73, section 5] or *KSGNS-correspondence* [Lan94] (for Kasparov, Stinespring, Gelfand, Naimark, Segal). We let X_ϱ to be a Hausdorff completion of the algebraic tensor product $A \odot A$ with respect to the seminorm associated to the A-valued sesquilinear form given by $\langle a \odot b, c \odot d\rangle_\varrho := b^*\varrho(a^*c)d$, $a, b, c, d \in A$. We denote by $a \otimes b$ the image of the simple tensor $a \odot b$ in X_ϱ. Then the left and right actions of A on X_ϱ are determined by: $a \cdot (b \otimes c) = (ab) \otimes c$ and $(b \otimes c) \cdot a = b \otimes (ca)$ where $a, b, c \in A$. The fundamental structural fact is:

Theorem 3.3 (Theorem 3.13 in [Kwa17]). *Let X_ϱ be a GNS-correspondence associated to ϱ and let J be an ideal in A. We have natural isomorphisms*

$$C^*(A,\varrho;J) = C^*(A,\varrho;J\cap J(X_\varrho)) \cong \mathcal{O}(J\cap J(X_\varrho),X_\varrho), \qquad C^*(A,\varrho)\cong \mathcal{O}_{X_\varrho}.$$

In particular, the homomorphism $j_A : A \to C^*(A,\varrho;J)$ is injective if and only if $J \cap J(X_\varrho) \subseteq N_\varrho^\perp$.

Remark 3.4. The above theorem follows from [Kwa17, Proposition 3.10]. The latter result implies that for every representation (π, S) of (A,ϱ) the operator $S \in M_r(C^*(\pi,S))$ is a right multiplier of $C^*(\pi,S)$ and when ϱ is a *strict map*, i.e. when $\{\varrho(\mu_\lambda)\}_{\lambda\in\Lambda}$ converges in strict topology in $M(A)$ for an approximate unit $\{\mu_\lambda\}_{\lambda\in\Lambda}$ in A, then $S \in M(C^*(\pi,S))$ (an endomorphism is a strict map if and only if it is extendible).

Using Theorem 3.3 it is shown in [Kwa17, Proposition 3.17] that the crossed product $C^*(A,\varrho;J)$ is a universal object for appropriately defined *J-covariant representations* [Kwa17, Definition 3.16]. A version of uniqueness theorem is established in [Kwa17, Proposition 3.18]. However, it seems that in general it is not possible to phrase relations that define $C^*(A,\varrho;J)$ without the use of the ideal $J(X_\varrho)$, that is without referring to the theory of C^*-correspondences.

In the case when $\varrho = \alpha$ is an endomorphism, the C^*-correspondences E_α and X_ϱ are isomorphic [Kwa17, Lemma 3.25] and therefore $J(X_\varrho) = J(E_\alpha) = A$. In particular, we have:

Theorem 3.5 (Proposition 3.26 in [Kwa17]). *Definitions 2.5 and 3.2 are consistent: If $\varrho = \alpha$ is an endomorphism of A, then for every ideal J in A, the assignments $\iota_A(a) \mapsto j_A(a)$, $a \in A$, $u \mapsto s^*$ give an isomorphism $C^*(A,\alpha;J) \cong C^*(A,\varrho;J)$.*

3.1.1 Algebras associated with topological relations and Markov operators

Assume now that $A = C_0(\Omega)$ is commutative (Ω is a locally compact Hausdorff space). This case is discussed in [Kwa17, Subsection 3.5]. Then the map $\varrho : C_0(\Omega) \to C_0(\Omega)$ can be identified with a structure $\mathcal{Q} = (E^0, E^1, r, s, \lambda)$ resembling topological graph with a system of measures λ, that is a *topological quiver* [MT05], cf. [Kwa17, Definicja 3.29]. Here $E^0 := \Omega$ and $E^1 := \overline{R}$ is the closure of the following subset $R \subseteq \Omega \times \Omega$:

$$(x,y) \in R \stackrel{def}{\iff} \left(\forall_{a\in C_0(\Omega)_+}\ \varrho(a)(x) = 0 \implies a(y) = 0\right).$$

The maps $r, s : E^1 \to E^0$ are projections onto the first and second coordinate, respectively. It is shown that in general the map $s(x,y) = x$, $x, y \in \Omega$, is open on R but not on $\overline{R}$, [Kwa17, Lemma 3.30].

If $R = \overline{R}$, then $\mathcal{Q}$ is a topological quiver and $C^*(A,\varrho) \cong C^*(\mathcal{Q})$ where $C^*(\mathcal{Q})$ is a C^*-algebra studied in [MT05]. Moreover, $\mu = (R, \lambda)$ is a topological relation in the sense of [Bre04] and $C^*(A,\varrho;A) \cong C^*(\mu)$ where $C^*(\mu)$ is a C^*-algebra considered in [Bre04], see [Kwa17, Propositions 3.33, 3.34].

If the map s is not open on $\overline{R}$ ($R \neq \overline{R}$), then the algebra $C^*(A,\varrho)$ is not modeled by topological quivers of [MT05] and analysis of this algebra requires a generalization of the theory in [MT05]. In particular, a mistake in the paper [IMV12] is detected and explained, cf. [Kwa17, Example 3.35 and Proposition 3.36] (it seems that the authors of [IMV12] tacitly assume that $R = \overline{R}$).

3.2 Exel's crossed products

In [Exe03] Exel proposed a new definition of a crossed product by an endomorphism, with an additional new ingredient - a transfer operator. Exel considered unital algebras. His formalism was extend onto the case where all maps extends onto the multiplier algebra in [BRV10], [Lar10]. Nevertheless, this construction can be formulated in general as follows, cf. [Kwa17, Subsection 2.4].

Definition 3.6. Let $\alpha : A \to A$ be an endomorphism of a C^*-algebra A and let $\mathcal{L} : A \to A$ be a positive linear map such that

$$\mathcal{L}(a\alpha(b)) = \mathcal{L}(a)b, \qquad \text{for all } a, b \in A. \tag{9}$$

Then $\mathcal{L}$ is called a *transfer operator* for α and the triple $(A, \alpha, \mathcal{L})$ is an *Exel system*.

Definition 3.7. A *representation of an Exel system* $(A, \alpha, \mathcal{L})$ is a pair (π, S) consisting of a non-degenerate representation $\pi : A \to \mathcal{B}(H)$ and an operator $S \in \mathcal{B}(H)$ such that

$$S\pi(a) = \pi(\alpha(a))S \quad \text{and} \quad S^*\pi(a)S = \pi(\mathcal{L}(a)) \text{ for all } a \in A. \tag{10}$$

A *redundancy* of a representation (π, S) of $(A, \alpha, \mathcal{L})$ is a pair $(\pi(a), k)$ where $a \in A$ and $k \in \overline{\pi(A)SS^*\pi(A)}$ are such that $\pi(a)\pi(b)S = k\pi(b)S$, for all $b \in A$. The *Toeplitz algebra* $\mathcal{T}(A, \alpha, \mathcal{L})$ of $(A, \alpha, \mathcal{L})$ is the C^*-algebra generated by $i_A(A) \cup i_A(A)t$ for a universal representation (i_A, t) of $(A, \alpha, \mathcal{L})$. *Exel's crossed product* $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ of $(A, \alpha, \mathcal{L})$ is the quotient C^*-algebra of $\mathcal{T}(A, \alpha, \mathcal{L})$ by the ideal generated by the set

$$\{i_A(a) - k : a \in \overline{A\alpha(A)A} \text{ and } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\}. \tag{11}$$

Remark 3.8. Let $(A, \alpha, \mathcal{L})$ be an Exel system. In [Kwa17] we noted the following three simple but important facts:

1) The map $\mathcal{L}$ is completely positive [Kwa17, Lemat 4.1].

2) The operator $\mathcal{L}$ always extends to a strictly continuous map $\overline{L} : M(A) \to M(A)$ [Kwa17, Proposition 4.2]. Thus the assumption of extendability of $\mathcal{L}$, as well as its verification, in [BRV10], [Lar10] and articles citing theses papers (16 according to MathSciNet) is superfluous.

3) The relation $S\pi(a) = \pi(\alpha(a))S$ in (10) is automatic – it follows from other axioms [Kwa17, Proposition 4.3]. Thus this relation, as well as its verification, in papers [Exe03], [BRV10], [Lar10] and articles citing theses papers (61 according to MathSciNet) is superfluous. In particular, the Toeplitz algebra $\mathcal{T}(A, \alpha, \mathcal{L})$ of the system $(A, \alpha, \mathcal{L})$ is in fact Toeplitz algebra $\mathcal{T}(A, \mathcal{L})$ of the map $\mathcal{L}$.

The above remarks suggests that Exel's crossed product $A \rtimes_{\alpha,\mathcal{L}} \mathbb{N}$ is not a crossed product by the endomorphism α, but by the completely positive map $\mathcal{L}$. The choice of the ideal $\overline{A\alpha(A)A}$ in (11), proposed in the pioneering paper [Exe03], was motivated by the fact that in a number of examples it leads to an appropriate construction. In these examples the ideal $\overline{A\alpha(A)A}$ is completely determined by $\mathcal{L}$. Formally this is explained in the following theorem, see [Kwa17, Theorem 4.7, Proposition 4.9]:

Theorem 3.9. *For any Exel system $(A, \alpha, \mathcal{L})$ we have $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}; \overline{A\alpha(A)A})$. If the operator $\mathcal{L}$ is faithful and α is extendible, then $A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L})$.*

As noticed in [BR06], if $\mathcal{L}$ is not faithful, then the algebra A may not embed into $A \times_{\alpha,\mathcal{L}} \mathbb{N}$. These considerations [BR06] led in [ER07] to a modified version the crossed product $A \times_{\alpha,\mathcal{L}} \mathbb{N}$. More specifically, we may associate to $(A, \alpha, \mathcal{L})$ a C^*-correspondence $M_{\mathcal{L}}$ that arises as a Hausdorff completion of A with respect to the seminorm determined by the A-valued sesqui-linear form $\langle m, n\rangle_{\mathcal{L}} := \mathcal{L}(m^*n)$, $m, n \in A$. Then the operations $m \cdot a := m\alpha(a)$ and $a \cdot m := am$, $n, m, a \in A$, factor through to operations on $M_{\mathcal{L}}$.

Definition 3.10 ([ER07])**.** *Exel-Royer crossed product $\mathcal{O}(A, \alpha, \mathcal{L})$ is the quotient of* $\mathcal{T}(A, \alpha, \mathcal{L})$ by the ideal generated by the set

$$\{i_A(a) - k : a \in J_{M_{\mathcal{L}}} \text{ i } (i_A(a), k) \text{ is a redundancy of } (i_A, t)\},$$

where $J_{M_{\mathcal{L}}}$ is (Katsura's ideal) associated to the C^*-correspondence $M_{\mathcal{L}}$.

The C^*-correspondence $M_{\mathcal{L}}$ is isomorphic to the GNS-correspondence $X_{\mathcal{L}}$ [Kwa17, Lemma 4.4] and the kernel of left action in these correspondences coincides with $N_{\mathcal{L}}$. This leads to the following result.

Theorem 3.11 (Theorem 4.7 in [Kwa17])**.** *For an arbitrary Exel system $(A, \alpha, \mathcal{L})$ Exel-Royer crossed product is the crossed product by $\mathcal{L}$ (it does not depend on α):*

$$C^*(A, \mathcal{L}) = \mathcal{O}(A, \alpha, \mathcal{L}) \cong \mathcal{O}_{M_{\mathcal{L}}} \cong \mathcal{O}_{X_{\mathcal{L}}}.$$

Moreover, $A \times_{\alpha,\mathcal{L}} \mathbb{N} \cong \mathcal{O}(X_{\mathcal{L}}, J \cap J(X_{\mathcal{L}})) \cong \mathcal{O}(M_{\mathcal{L}}, J \cap J(M_{\mathcal{L}}))$, where $J := \overline{A\alpha(A)A}$. In particular, A embeds into $A \times_{\alpha,\mathcal{L}} \mathbb{N}$ if and only if $\overline{A\alpha(A)A} \cap J(M_{\mathcal{L}}) \subseteq N_{\mathcal{L}}^{\perp}$.

Usually Exel systems $(A, \alpha, \mathcal{L})$ are considered with an additional assumption that the map $E = \alpha \circ \mathcal{L}$ is a conditional expectation from A onto $\alpha(A)$. We call such systems and the corresponding transfer operators $\mathcal{L}$ *regular*, cf. [Kwa17, Definition 4.1]. In [Kwa17, Subsection 4.2] the structure of such operators is studied. It is proved that for a fixed endomorphism α all regular Exel systems $(A, \alpha, \mathcal{L})$ are parametrized by conditional expectations from A onto $\alpha(A)$. On the other hand, for a fixed operator $\mathcal{L}$ Exel systems are parametrized by subalgebras of the *multiplicative domain* of $\mathcal{L}$:

$$MD(\mathcal{L}) := \{a \in A : \mathcal{L}(ab) = \mathcal{L}(a)\mathcal{L}(b) \text{ i } \mathcal{L}(ba) = \mathcal{L}(b)\mathcal{L}(a) \text{ dla każdego } b \in A\} \quad (12)$$

[Kwa17, Propositions 4.15, 4.16]. For a regular Exel system $(A, \alpha, \mathcal{L})$ such that (A, α) is partially reversible $\mathcal{L}$ and α determine each other uniquely and we have

$$A \times_{\alpha,\mathcal{L}} \mathbb{N} = C^*(A, \mathcal{L}) \cong C^*(A, \alpha),$$

see [Kwa17, Theorem 4.22].

3.2.1 Graph algebras as crossed products by Perron Frobenius operators

The main motivation in [Exe03] was to realize Cuntz-Krieger algebras as crossed products associated to systems $(A, \alpha, \mathcal{L})$ coming from topological Markov chains. This result was generalized by Brownlowe, Raebrun and Vittadello in [BRV10] to the case of graph algebras $C^*(E)$ where the graph $E = (E^0, E^1, r, s)$ is *locally finite* (the sets $r^{-1}(v)$ and $s^{-1}(v)$ are finite for every vertex $v \in E^0$) and has no sinks, that is $s(E^1) = E^0$. In [Kwa17, Section 5] general (discrete, countable) graphs $E = (E^0, E^1, r, s)$ were considered.

Let us recall that the *graph C^*-algebra* $C^*(E)$ is generated by a universal family of partial isometries $\{s_e : e \in E^1\}$ and pairwise orthogonal projections $\{p_v : v \in E^1\}$ such that $s_e^* s_e = p_{s(e)}$, $s_e s_e^* \leqslant p_{r(e)}$ and $p_v = \sum_{r(e)=v} s_e s_e^*$ whenever the sum is finite (that is when v is a *finite receiver*). Projections $s_\mu s_\mu^*$ corresponding to paths μ in the graph E generate a commutative subalgebra $\mathcal{D}_E$ of $C^*(E)$. When E has no infinite emitters, then the formula

$$\Phi_E(a) = \sum_{e \in E^1} s_e a s_e^*$$

defines a completely positive map $\Phi_E : C^*(E) \to C^*(E)$, which restricts to an endomorphism of $\mathcal{D}_E$, cf. [Kwa17, Proposition 5.3]. On the spectrum of $\mathcal{D}_E$, identified with a space of paths in E, Φ_E is given by a shift. Therefore Φ_E is usually called a *noncommutative Markov shift*.

For an arbitrary graph E one can find numbers $\lambda_e > 0$, $e \in E^1$, such that the formula $\mathcal{L}_\lambda(a) := \sum_{e,f \in E^1} \sqrt{\lambda_e \lambda_f} s_e^* a s_f$ defines a completely positive map $\mathcal{L}_\lambda : C^*(E) \to C^*(E)$. It restricts to the map $\mathcal{L}_\lambda : \mathcal{D}_E \to \mathcal{D}_E$ given by the formula

$$\mathcal{L}_\lambda(a) := \sum_{e \in E^1} \lambda_e s_e^* a s_e.$$

Necessary and sufficient conditions for the weights $\lambda := \{\lambda_e\}_{e \in E^1}$ to define the operator $\mathcal{L}_\lambda$ are given in [Kwa17, Proposition 5.4]. We call the maps $\mathcal{L}_\lambda$ *noncommutative Perron-Frobenius operators*. When E is locally finite and without sinks, then one can put $\lambda_e := |s^{-1}(s(e))|^{-1}$, $e \in E^1$. Then $(\mathcal{D}_E, \Phi_E, \mathcal{L}_\lambda)$ is a regular Exel system considered in [BRV10]. Identifying $\mathcal{D}_E$ with the algebra of functions on the space of infinite paths the operator $\mathcal{L}_\lambda$ becomes the classical Perron-Frobenious operator for the Markov shift.

The following theorem says that every graph algebra is the crossed product by a Perron-Frobenious operator. It is not the crossed product by the endomorphism - the Markov shift. Moreover, in general there is no endomorphism, for which the Perron-Frobenious operator is a transfer operator.

Theorem 3.12 (Theorem 5.6 in [Kwa17]). *Let $E = (E^0, E^1, s, r)$ be an arbitrary directed graph and let $\mathcal{L}_\lambda$ be any of the corresponding (noncommutative) Perron-Frobenius operators. Then*

$$C^*(E) \cong C^*(\mathcal{D}_E, \mathcal{L}_\lambda).$$

i) If E has no infinite emitters, then $(\mathcal{D}_E, \Phi_E, \mathcal{L}_\lambda)$ is an Exel system and $C^*(\mathcal{D}_E, \mathcal{L}_\lambda) = \mathcal{D}_E \rtimes_{\Phi_E, \mathcal{L}_\lambda} \mathbb{N}$.

ii) If E has infinite emitters but does not have infinite receivers, then there is no endomorphisms α such that $(\mathcal{D}_E, \alpha, \mathcal{L}_\lambda)$ is an Exel system.

3.3 Crossed products by interactions

In [Exe07] Exel defined a notion of *interaction* on a C^*-algebra A. It is a pair $(\mathcal{V},\mathcal{H})$ of positive maps $\mathcal{V},\mathcal{H}: A\to A$ such that

$$\mathcal{V}\mathcal{H}\mathcal{V}=\mathcal{V},\quad \mathcal{H}\mathcal{V}\mathcal{H}=\mathcal{H},\quad \mathcal{V}(A)\subseteq MD(\mathcal{H}),\quad \mathcal{H}(A)\subseteq MD(\mathcal{V}),$$

where $MD(\mathcal{H})$ and $MD(\mathcal{V})$ multiplicative domains of $\mathcal{H}$ and $\mathcal{V}$ respectively, cf. (12). Then $\mathcal{V}$ and $\mathcal{H}$ are automatically completely positive maps and $\mathcal{H}(A)$ and $\mathcal{V}(A)$ are subalgebras of A, see [Exe07]. Exel associated with $(\mathcal{V},\mathcal{H})$ a generalized C^*-correspondence $\mathfrak{X}$ and a generalized Cuntz-Pimsner algebra $C^*(A,\mathfrak{X})$. As shown in [Kwa17] for any regular Exel system $(A,\alpha,\mathcal{L})$ the pair $(\alpha,\mathcal{L})$ is an example of an interaction. The only examples of interactions considered [Exe07] were of this type. The aim of the article [Kwa14'], that we will discuss here, was twofold. Fristly, to give a detailed analysis of a canonical example of an interaction $(\mathcal{V},\mathcal{H})$ where non of the maps is multiplicative. Secondly, to study graph algebras as crossed products by interactions.

Let A be a unital C^*-algebra and let $(\mathcal{V},\mathcal{H})$ be an interaction on A (existence of unit in A is not essential for a number of the following results). We will additionally assume that the subalgebras $\mathcal{H}(A)$ and $\mathcal{V}(A)$ are hereditary in A. Such interactions are called in [Kwa14'] *corner interactions*. For every regular Exel system $(A,\alpha,\mathcal{L})$ where (A,α) is partially reversible the pair $(\alpha,\mathcal{L})$ is an example of a corner interaction.

Definition 3.13 ([Kwa14']). A *covariant representation* of the corner interaction $(\mathcal{V},\mathcal{H})$ is a pair (π,S) where $\pi: A\to\mathcal{B}(H)$ is a non-degenerate representation and $S\in\mathcal{B}(H)$ (which is necessarily a partial isometry) are such that

$$S\pi(a)S^*=\pi(\mathcal{V}(a))\quad\text{ and }\quad S^*\pi(a)S=\pi(\mathcal{H}(a))\ \text{ for every } a\in A.$$

The *crossed product of the interaction* $(\mathcal{V},\mathcal{H})$ is a C^*-algebra $C^*(A,\mathcal{V},\mathcal{H})$ generated by $i_A(A)$ and s where (i_A,s) is a universal covariant representation of $(\mathcal{V},\mathcal{H})$.

It is shown in [Kwa14', Subsection 2.2] that the GNS-correspondences $X_{\mathcal{V}}$ and $X_{\mathcal{H}}$ are (mutually inverse) Hilbert bimodules and that the (generalized) correspondence $\mathfrak{X}$ is in fact isomorphic to $X_{\mathcal{H}}$. This leads to the following isomorphisms of C^*-algebras:

$$C^*(A,\mathcal{V},\mathcal{H})\cong C^*(A,\mathcal{H})\cong C^*(A,\mathcal{V})\cong C^*(A,\mathfrak{X})\cong A\rtimes_{X_{\mathcal{H}}}\mathbb{Z}\cong A\rtimes_{X_{\mathcal{V}}}\mathbb{Z}.$$

The maps $\mathcal{H}:\mathcal{V}(A)\to\mathcal{H}(A)$ and $\mathcal{V}:\mathcal{H}(A)\to\mathcal{V}(A)$ are mutually inverse isomorphisms. Identifying the spectra of the hereditary subalgebras $\mathcal{V}(A)$ and $\mathcal{H}(A)$ of A with open subsets of the spectrum $\widehat{A}$ of A we get a partial homeomorphism $\widehat{\mathcal{H}}:\widehat{\mathcal{H}(A)}\to\widehat{\mathcal{V}(A)}$ of $\widehat{A}$, where $\widehat{\mathcal{H}}([\pi])=[\pi\circ\mathcal{H}]$. The following definition is consistent with Definition 2.20.

Definition 3.14 ([Kwa14]). We say that a partial homeomorphism φ of a topological space M, i.e. a homeomorphism whose domain Δ and range $\varphi(\Delta)$ are open subsets of M, is *topologically free* if for any $n>0$ the set of fixed points for φ^n (on its natural domain) has empty interior. A set V is φ-*invariant* if $\varphi(V\cap\Delta)=V\cap\varphi(\Delta)$. If there are no nontrivial closed invariant sets, then φ is called *minimal*, and φ is said to be (residually) *free*, if it is topologically free on every closed invariant set (in the Hausdorff space case this amounts to requiring that φ has no periodic points).

Using the results of [Kwa14], see Theorem 5.4 below, and general K-theoretical considerations in [Kat04] one can get the following:

Theorem 3.15 (Theorems 2.20, 2.24 in [Kwa14']). *Let $(\mathcal{V}, \mathcal{H})$ be a corner interaction.*

i) If the map $\widehat{\mathcal{H}}$ is topologically free, then every covariant representation $(\pi(A), S)$ of $(\mathcal{V}, \mathcal{H})$, with π injective, generates a C^*-algebra naturally isomorphic to $C^*(A, \mathcal{V}, \mathcal{H})$. If in addition, $\widehat{\mathcal{H}}$ is minimal, equivalently if there are no nontrivial ideals I in A such that $\mathcal{H}(I) = \mathcal{H}(1)I\mathcal{H}(1)$, then $C^*(A, \mathcal{V}, \mathcal{H})$ is simple.

ii) If the map $\widehat{\mathcal{H}}$ is free, then $J \mapsto \widehat{J \cap A}$ is an isomorphism from the lattice of ideals in $C^*(A, \mathcal{V}, \mathcal{H})$ onto the lattice of open $\widehat{\mathcal{H}}$-invariant subsets of $\widehat{A}$.

iv) The following sequence is exact:

$$\begin{array}{ccccc} K_0(\mathcal{V}(A)) & \xrightarrow{K_0(\iota)-K_0(\mathcal{H})} & K_0(A) & \xrightarrow{K_0(i_A)} & K_0(C^*(A,\mathcal{V},\mathcal{H})) \\ \uparrow & & & & \downarrow \\ K_1(C^*(A,\mathcal{V},\mathcal{H})) & \xleftarrow{K_0(i_A)_*} & K_1(A) & \xleftarrow{K_0(\iota)-K_0(\mathcal{H})} & K_1(\mathcal{V}(A)) \end{array} \quad (13)$$

3.3.1 Graph algebras as crossed products by corner interactions

Let $E = (E^0, E^1, r, s)$ be a finite graph, that is let E^0 and E^1 be finite sets. Let $\{s_e : e \in E^1\}$ and $\{p_v : v \in E^1\}$ be the generators of the C^*-algebra $C^*(E)$, cf. paragraph 3.2.1. In [Kwa14'] a different (older) convention concerning graph was used - the roles of maps r and s are exchanged from the point of view of the conventions used in [Kwa17]. For consistency of presentation we will stick here to the (later) convention of [Kwa17]. Finiteness of E^0 implies that the algebra $C^*(E)$ is unital. Let us consider the operator

$$s := \sum_{e \in E^1} \frac{1}{\sqrt{|s^{-1}(s(e))|}} \, s_e.$$

Then s is a partial isometry which defines via formulas $\mathcal{V}(a) = sas^*$, $\mathcal{H}(a) = s^*as$ a corner interaction on $C^*(E)$. We note that $\mathcal{H}$ is a noncommutative Perron-Frobenius operator $\mathcal{L}_\lambda$ where $\lambda_e := |s^{-1}(s(e))|^{-1}$, $e \in E^1$. The pair $(\mathcal{V}, \mathcal{H})$ does not restrict to an interaction on $\mathcal{D}_E$ because $\mathcal{V}$ does not preserve $\mathcal{D}_E$. In essence, see [Kwa14', Remark 3.10], the smallest C^*-algebra containing $\mathcal{D}_E$ and invariant with respect to $\mathcal{V}$ is the fixed-point algebra $\mathcal{F}_E$ for the gauge circle action. The pair $(\mathcal{V}, \mathcal{H})$ is a corner interaction on $\mathcal{F}_E$ and

$$C^*(E) \cong C^*(\mathcal{F}_E, \mathcal{V}, \mathcal{H}),$$

[Kwa14', Proposition 3.2]. Depending on E properties of s and the pair $(\mathcal{V}, \mathcal{H})$ were described in [Kwa14', Propositions 3.5, 3.7, Corollary 3.6]. In general non of the maps $\mathcal{V}$, $\mathcal{H}$ is multiplicative. The partial homeomorphism dual to the system $(\mathcal{F}_E, \mathcal{V}, \mathcal{H})$ is described in [Kwa14', Theorem 3.9]: the space $\widehat{\mathcal{F}_E}$ is a quotient of the path space, and $\widehat{\mathcal{V}} = \widehat{\mathcal{H}}^{-1}$ is a factor of a Markov shift. Using this description it is relatively easy to deduce, see [Kwa14', Theorem 3.19] that:

i) $\widehat{\mathcal{H}}$ is topologically free $\Longleftrightarrow$ every cycle in E has an entry (E satisfied condition (L) [KPRR97]),

ii) $\widehat{\mathcal{H}}$ is free $\Longleftrightarrow$ the graph E satisfies condition (K) [KPRR97].

This theorem in particular says that item (i) in Theorem 3.15 when applied to the interaction coming for the graph E is equivalent to *Cuntz-Krieger uniqueness theorem*.

Isomorphism $\mathcal{H} : \mathcal{V}(A) \to \mathcal{H}(A)$ induces isomorphism $K_0(\mathcal{H})$ between subgroups $K_0(\mathcal{V}(\mathcal{F}_E))$, $K_0(\mathcal{H}(\mathcal{F}_E))$ of $K_0(\mathcal{F}_E)$, which is described in [Kwa14', Proposition 3.22]. Since $\mathcal{F}_E$ is approximately finite, we have $K_1(\mathcal{F}_E) = K_1(\mathcal{V}(\mathcal{F}_E)) = 0$ and therefore the exact sequence (13) gives

$$K_1(C^*(E)) \cong \ker(K_0(\iota) - K_0(\mathcal{H})), \qquad K_0(C^*(E)) \cong K_0(\mathcal{F}_E)/\mathrm{im}(K_0(\iota) - K_0(\mathcal{H})).$$

On this basis, the K-theory of graph algebras, originally calculated in [RS04], was calculated in [Kwa14', Corollary 3.22] in a new way which emphasizes the role of Markov shifts.

4 Unification of Cuntz-Pimsner algebras and Doplicher-Roberts algebras

It was noticed in [DPZ98], [FMR03] that a C^*-correspondence X (with injective left action) gives rise not only to Pimsner's algebra $\mathcal{O}_X$ but also to a *Doplicher-Roberts* algebra $\mathcal{DR}(X)$. The algebra $\mathcal{DR}(X)$ is a particular example of a general construction $\mathcal{DR}(\mathcal{T})$ considered in [DR89] in the context of abstract C^*-categories $\mathcal{T}$ with tensoring (such a structure arises naturally in the duality theory of noncommutative groups and in quantum field theory). Pictorially speaking, the algebras $\mathcal{O}_X$, $\mathcal{DR}(X)$ and $\mathcal{DR}(\mathcal{T})$ are build respectively from the spaces of generalized compact operators, adjointable operators and abstract morphisms:

$$\mathcal{K}(X^{\otimes m}, X^{\otimes n}), \qquad \mathcal{L}(X^{\otimes m}, X^{\otimes n}), \qquad \mathcal{T}(n,m), \qquad n,m \in \mathbb{N}.$$

These spaces are "glued together" with the help of "tensoring by identity operators". The objective of [Kwa13] is a systematic analysis of relationship between these constructions. On the basis of a newly introduced C^*-algebras $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ a general theory which unifies both relative Cuntz-Pimsner algebras and constructions of Doplicher-Roberts type is developed. The results of [Kwa13], which we discuss in this section, not only provide a general framework for the aforementioned construction, but also they shed new light on the structure of Cuntz-Pimsner algebras themselves and include significantly broader class of C^*-algebras (cf. Remark 4.9).

In this section we present a universal definition of the C^*-algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ (Definition 4.6) and its relationship with relative Cuntz-Pimsner algebras $\mathcal{O}(X,J)$ (Example 4.8). We present an explicit construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ on the basis of an elaborated "matrix calculus" (Theorem 4.11). We clarify the relationship between $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ and Doplicher-Roberts algebras (Remark 4.10, Theorem 4.12). We describe the structure of $\mathbb{T}$-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ (Theorems 4.16, 4.19). These results generalize and strengthen known results for relative Cuntz-Pimsner algebras (cf. Remark 4.17). In subsection 4.4, we discuss criteria for embeddings of relative Cuntz-Pimsner algebras into relative Doplicher-Roberts algebras and criteria for faithfulness of their representations.

4.1 Basic definitions and facts

A C^*-*category*, cf. [GLR85], is a small category $\mathcal{T} = \{\mathcal{T}(\sigma,\rho)\}_{\sigma,\rho\in \mathrm{Ob}(\mathcal{T})}$ where the sets of morphisms $\mathcal{T}(\sigma,\rho)$ are complex Banach spaces such that $\|ab\| \leqslant \|a\|\cdot\|b\|$, for $a \in \mathcal{T}(\tau,\sigma)$, $b \in \mathcal{T}(\sigma,\rho)$, equipped with an anti-linear involutive contravariant functor $^* : \mathcal{T} \to \mathcal{T}$ such that $a \in \mathcal{T}(\tau,\sigma)$ implies that $a^* \in \mathcal{T}(\sigma,\tau)$ and $\|a^*a\| = \|a\|^2$. Then $\mathcal{T}(\sigma,\sigma)$ is a (unital) C^*-algebra and we require that for every $a \in \mathcal{T}(\tau,\sigma)$ the element a^*a is positive in $\mathcal{T}(\sigma,\sigma)$. In [Kwa13] a more general notion is considered. It is called a C^*-*precategory* and its definition differs from the definition of a C^*-category only with that we do not require existence of identity morphisms. Thus C^*-categories are categorial generalizations of unital C^*-algebras and C^*-precategories are are generalizations of arbitrary (not necessarily unital) C^*-algebras. In particular, we may treat C^*-algebras as C^*-precategories with exactly one object. The notion of a C^*-precategory is inevitable when considering ideals in C^*-categories.

Definition 4.1 (Definition 2.4 in [Kwa13])**.** An ideal in a C^*-precategory $\mathcal{T}$ is a family $\mathcal{K} = \{\mathcal{K}(\sigma,\rho)\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{T})}$ where $\mathcal{K}(\sigma,\rho)$ is a closed linear subspace of $\mathcal{T}(\sigma,\rho)$ and

$$\mathcal{T}(\tau,\sigma)\mathcal{K}(\sigma,\rho) \subseteq \mathcal{K}(\tau,\rho), \qquad \mathcal{K}(\tau,\sigma)\mathcal{T}(\sigma,\rho) \subseteq \mathcal{K}(\tau,\rho), \qquad \sigma,\rho,\tau \in \mathrm{Ob}(\mathcal{T}).$$

Then $\mathcal{K}$ is automatically a C^*-precategory.

If $\mathcal{K}$ is an ideal in $\mathcal{T}$ then $\mathcal{K}(\sigma,\sigma)$ is an ideal in $\mathcal{T}(\sigma,\sigma)$ for every $\sigma \in \mathrm{Ob}(\mathcal{T})$. Ideals $\mathcal{K}(\sigma,\sigma)$ determine $\mathcal{K}$ uniquely by [Kwa13, Theorem 2.6]. For every ideal $\mathcal{K}$ there is an *annihilator*, which by definition is the ideal $\mathcal{K}^\perp$ such that $\mathcal{K}^\perp(\sigma,\sigma) = \mathcal{K}(\sigma,\sigma)^\perp$ for every $\sigma \in \mathrm{Ob}(\mathcal{T})$ [Kwa13, Proposition 2.7]. There is also a *quotient* C^**-precategory* $\mathcal{T}/\mathcal{K}$ where $\mathcal{T}/\mathcal{K}(\sigma,\rho) := \mathcal{T}(\sigma,\rho)/\mathcal{K}(\sigma,\rho)$ for every $\sigma,\rho \in \mathrm{Ob}(\mathcal{T})$ [Kwa13, Proposition 2.10]. By a *homomorphism* between C^*-precategories we mean a covariant "functor" which preserves the involution and the linear structure of spaces of morphisms [Kwa13, Definition 2.8]. *Representation of a* C^**-precategory* $\mathcal{T}$ in a C^*-algebra B is a family $\{\pi_{\sigma,\rho}\}_{\sigma,\rho\in\mathrm{Ob}(\mathcal{T})}$ of linear operators $\pi_{\sigma,\rho} : \mathcal{T}(\sigma,\rho) \to B$ such that $\pi_{\sigma,\rho}(a)^* = \pi_{\rho,\sigma}(a^*)$ and $\pi_{\sigma,\tau}(ab) = \pi_{\sigma,\rho}(a)\pi_{\rho,\tau}(b)$, $a \in \mathcal{T}(\sigma,\rho)$, $b \in \mathcal{T}(\rho,\tau)$ [Kwa13, Definition 2.11].

The main object of study in [Kwa13] is a C^*-precategory with right tensoring over the semigroup $\mathbb{N} = \{0,1,2,...\}$.

Definition 4.2 (Definition 3.1 in [Kwa13])**.** A *right tensor* C^**-precategory* is a C^*-precategory $\mathcal{T} = \{\mathcal{T}(n,m)\}_{n,m\in\mathbb{N}}$ equipped with an endomorphism $\otimes 1 : \mathcal{T} \to \mathcal{T}$ that maps n to $n+1$:

$$\otimes 1 : \mathcal{T}(n,m) \to \mathcal{T}(n+1,m+1), \qquad n,m \in \mathbb{N}.$$

Instead of $\otimes 1(a)$ we write $a \otimes 1$, $a \in \mathcal{T}(n,m)$. By iteration of $\otimes 1$ we get a semigroup $\{\otimes 1^k\}_{k\in\mathbb{N}}$ endomorphisms $\otimes 1^k : \mathcal{T} \to \mathcal{T}$ where $\otimes 1^k : \mathcal{T}(n,m) \to \mathcal{T}(n+k,m+k)$. By definition $\otimes 1^0 := id$.

Definition 4.3 (Definition 3.6 in [Kwa13]). Let $\mathcal{K}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$. We say that a representation $\{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ is a *right tensor representation* if it satisfies

$$\pi_{n,m}(a)\pi_{m+k,l}(b) = \pi_{n+k,l}((a \otimes 1^k)\, b) \tag{14}$$

for all $a \in \mathcal{K}(n,m)$ and $b \in \mathcal{K}(m+k,l)$, $k,l,m,n \in \mathbb{N}$. We emphasize that the right hand side of (14) makes sense because $\mathcal{K}$ is an ideal in $\mathcal{T}$ (we do not assume that the right tensoring preserves $\mathcal{K}$).

Examples 4.4 (Right tensor C^*-precategory $\mathcal{T}_X$ of a C^*-correspondence X.). A C^*-correspondence X gives rise to a C^*-precategory $\mathcal{T}_X$ where

$$\mathcal{T}_X(n,m) := \begin{cases} \mathcal{K}(X^{\otimes m}, X^{\otimes n}), & \text{if } n = 0 \text{ or } m = 0, \\ \mathcal{L}(X^{\otimes m}, X^{\otimes n}), & \text{if } n, m \geqslant 1. \end{cases}$$

Here $X^{\otimes n} := X \otimes_A \cdots \otimes_A X$ is the n-th tensor power of X and $X^{\otimes 0} := A$. There is a natural right tensoring defined on $\mathcal{T}_X$. For $n > 0, m > 0$ we put

$$\mathcal{L}(X^{\otimes m}, X^{\otimes n}) \ni a \longmapsto a \otimes 1 := a \otimes 1_X \in \mathcal{L}(X^{\otimes (m+1)}, X^{\otimes (n+1)}),$$

where 1_X is the identity in $\mathcal{L}(X)$. If $n = 0$ ot $m = 0$, we define right tensoring using the identifications $\mathcal{K}(A, X^{\otimes k}) = X^{\otimes k}$, $\mathcal{K}(X^{\otimes k}, A) = \widetilde{X}^{\otimes k}$, $k \in \mathbb{N}$, see [Kwa13, Example 3.2]. Then

$$\mathcal{K}_X := \{\mathcal{K}(X^{\otimes m}, X^{\otimes n})\}_{m,n\in\mathbb{N}}$$

is an ideal in $\mathcal{T}_X$. Every ideal in $\mathcal{K}_X$ is of the form $\mathcal{K}_X(J) := \{\mathcal{K}(X^{\otimes m}, X^{\otimes n}J)\}_{n,m\in\mathbb{N}}$ where J is an ideal in A, [Kwa13, Proposition 2.17]. Relations $\pi_{0,0} = \pi$, $\pi_{1,0} = t$ give a bijective correspondence between representations (π, t) of the C^*-correspondence X and right tensor representations $\{\pi_{n,m}\}_{m,n\in\mathbb{N}}$ of the ideal $\mathcal{K}_X$ [Kwa13, Proposition 3.13].

Examples 4.5 (Right tensor C^*-precategory of an endomorphism α). Let $\alpha : A \to A$ be an endomorphism. The family $\mathcal{T}_\alpha = \{\alpha^n(A)A\alpha^m(A)\}_{n,m\in\mathbb{N}}$ with operations inherited from A is a C^*-precategory that we equip with a right tensoring $\alpha^n(A)A\alpha^m(A) \ni a \longrightarrow a \otimes 1 = \alpha(a) \in \alpha^{n+1}(A)A\alpha^{m+1}(A)$. Then $\mathcal{T}_\alpha$ is a right tensor C^*-precategory which is isomorphic with the right tensor C^*-precategory $\mathcal{T}_{E_\alpha} = \mathcal{K}_{E_\alpha}$ associated to the C^*-correspondence E_α. In particular, $\mathcal{T}_{E_\alpha} = \mathcal{K}_{E_\alpha}$ and every ideal in $\mathcal{T}_\alpha$ is of the from $\mathcal{J} = \{\alpha^n(A)J\alpha^m(A)\}_{n,m\in\mathbb{N}}$ for an ideal J in A.

Example 4.4 and theory of relative Cuntz-Pimsner algebras motivates the following definitions, see [Kwa13, Subsection 3.2]. The preimage and kernel of the functor $\otimes 1$ are defined in a natural manner, see [Kwa13, Section 2].

Definition 4.6 ([Kwa13]). Let $\mathcal{K}$ be an ideal in a right tensor C^*-precategory $\mathcal{T}$. Let $\mathcal{J}$ be an ideal in $J(\mathcal{K}) := (\otimes 1)^{-1}(\mathcal{K}) \cap \mathcal{K}$. We say that a right tensor representation $\{\pi_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ $\mathcal{J}$*-covariant* (or *coisometric* on $\mathcal{J}$) if

$$\pi_{n,m}(a) = \pi_{n+1,m+1}(a \otimes 1), \qquad \text{for all} \quad a \in \mathcal{J}(n,m),\ n,m \in \mathbb{N}.$$

We denote by $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) := C^*(\{\iota_{n,m}(\mathcal{K}(n,m))\}_{n,m\in\mathbb{N}})$ the C^*-algebra generated by a universal $\mathcal{J}$-covariant right tensor representation $\iota = \{\iota_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$. We call $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ the C^**-algebra associated to* $\mathcal{K}$ *relative to* $\mathcal{J}$.

Remark 4.7. The algebra $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ is equipped with the circle action $\gamma:\mathbb{T}\to\mathrm{Aut}(\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}))$ determined by $\gamma_z(\iota_{n,m}(a)) = z^{n-m}\iota_{n,m}(a)$, $a\in\mathcal{K}(n,m)$, $z\in\mathbb{T}$. An example preceding [Kwa13, Theorem 5.3] shows that every circle action an C^*-algebra can be realized in this way. More specifically, for any Fell bundle $\{B_n\}_{n\in\mathbb{Z}}$ we have a natural C^*-precategory $\mathcal{T}$ and a $\mathbb{T}$-equivariant isomorphism $\mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{T})\cong C^*(\mathcal{B})$.

Examples 4.8. Let X be a C^*-correspondence. Then $\mathcal{J}(\mathcal{K}_X) := (\otimes 1)^{-1}(\mathcal{K}_X)\cap\mathcal{K}_X = \mathcal{K}_X(J(X)) = \{\mathcal{K}(X^{\otimes m}, X^{\otimes n}J(X))\}_{n,m\in\mathbb{N}}$ and for every $J\lhd J(X)$ we have

$$\mathcal{O}(J,X)\cong\mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X,\mathcal{J}),$$

where $\mathcal{J}=\mathcal{K}_X(J)=\{\mathcal{K}(X^{\otimes m},X^{\otimes n}J)\}_{n,m\in\mathbb{N}}$. This motivates the following definition [Kwa13, Definition 8.7]: for every ideal J in A, we define the *relative Doplicher-Roberts algebra of* X relative to J as the C^*-algebra

$$\mathcal{DR}(J,X) := \mathcal{O}_{\mathcal{T}_X}(\mathcal{T}_X,\mathcal{T}_X(J)),$$

where $\mathcal{T}_X(J) := \{\mathcal{L}_J(X^{\otimes m},X^{\otimes n})\}_{n,m\in\mathbb{N}}\cap\mathcal{T}_X$[3]. In this notation, algebras considered in [FMR03], [DPZ98] are of the form $\mathcal{DR}(A,X)$.

Remark 4.9. Isomorphism $\mathcal{O}(J,X)\cong\mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X,\mathcal{J})$ is equivariant with respect to gauge circle actions. It is well known that the gauge action on relative Cuntz-Pimsner algebras is semisaturated (Fell bundles of spectral subspaces is semisaturated). In light of Remark 4.7 we see that the discussed formalism unifies Cuntz-Pimsner algebras and C^*-algebras associated to arbitrary Fell bundles over $\mathbb{Z}$.

Remark 4.10. The *Doplicher-Roberts algebra* $\mathcal{DR}(\mathcal{T})$ *of a right tensor C^*-category* $\mathcal{T}$ is defined to be the completion of the algebraic direct sum $\bigoplus_{k\in\mathbb{Z}}\mathcal{DR}^{(k)}(\mathcal{T})$ of spaces given by direct limits

$$\mathcal{DR}^{(k)}(\mathcal{T}) := \varinjlim \mathcal{T}(r,r+k),$$

with operations inherited from $\mathcal{T}$, in the unique C^*-norm for which the automorphic action defined by the grading is isometric [DR89, p. 179]. It follows from the construction that we will present below that $\mathcal{DR}(\mathcal{T})\cong\mathcal{O}_{\mathcal{T}}(\mathcal{T},\mathcal{T})$.

4.2 Construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$

Let us fix a C^*-precategory $\mathcal{T}$, an ideal $\mathcal{K}$ in $\mathcal{T}$ and an ideal $\mathcal{J}$ in $\mathcal{J}(\mathcal{K})$. Spectral subspaces of the gauge action on $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ forms a Fell bundle $\{\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})\}_{k\in\mathbb{Z}}$ and $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) = \overline{\bigoplus_{k\in\mathbb{Z}}\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})}$. In [Kwa13, Subsection 4.2] an explicit construction of the $*$-algebra $\bigoplus_{k\in\mathbb{Z}}\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})$ and norm formulas in spectral subspaces $\mathcal{O}_{\mathcal{T}}^{(k)}(\mathcal{K},\mathcal{J})$ are given.

More specifically, let $\mathcal{M}_{\mathcal{T}}$ be the set of all infinite matrices $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ where $a_{n,m}\in\mathcal{T}(n,m)$. We denote by $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ the subset of $\mathcal{M}_{\mathcal{T}}$ consisting of matrices $\{a_{n,m}\}_{n,m\in\mathbb{N}}$ such that

$$a_{n,m}\in\mathcal{K}(n,m),\qquad n,\,m\in\mathbb{N},$$

[3] $\mathcal{L}_I(X,Y)$ denotes the space of elements $a\in\mathcal{L}(X,Y)$ satisfying the equivalent conditions $a(X)\subset YI$ and $a^*(Y)\subset XI$

and there is at most finite number of elements $a_{n,m}$ which are non-zero. We define the addition, multiplication by scalars, and involution on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ in a quite natural manner: for $a = \{a_{n,m}\}_{n,m\in\mathbb{N}}$ and $b = \{b_{n,m}\}_{n,m\in\mathbb{N}}$ we put

$$(a+b)_{n,m} := a_{n,m} + b_{n,m}, \tag{15}$$

$$(\lambda a)_{n,m} := \lambda a_{n,m} \tag{16}$$

$$(a^*)_{n,m} := a^*_{m,n}. \tag{17}$$

A convolution multiplication "$\star$" on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is more involved. We set

$$a \star b = a \cdot \sum_{k=0}^{\infty} \Lambda^k(b) + \sum_{k=1}^{\infty} \Lambda^k(a) \cdot b \tag{18}$$

where "$\cdot$" stands for the standard multiplication of matrices and mapping $\Lambda : \mathcal{M}_{\mathcal{T}}(\mathcal{K}) \to \mathcal{M}_{\mathcal{T}}$ is defined to act as follows

$$\Lambda(a) = \begin{pmatrix} 0 & 0 & 0 & 0 & \cdots \\ 0 & a_{0,0}\otimes 1 & a_{0,1}\otimes 1 & a_{0,2}\otimes 1 & \cdots \\ 0 & a_{1,0}\otimes 1 & a_{1,1}\otimes 1 & a_{1,2}\otimes 1 & \cdots \\ 0 & a_{2,0}\otimes 1 & a_{2,1}\otimes 1 & a_{2,2}\otimes 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{19}$$

that is $\Lambda(a)_{n,m} = a_{n-1,m-1} \otimes 1$, for $n, m > 1$, and $\Lambda(a)_{n,m} = 0$ otherwise.

Let $q_{\mathcal{J}} : \mathcal{K} \to \mathcal{K}/\mathcal{J}$ be the quotient homomorphism. Combining [Kwa13, Proposition 4.10 and Theorem 4.12] we get:

Theorem 4.11 ([Kwa13])**.** *The set $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ with operations* (15)*,* (16)*,* (17)*,* (18) *becomes an algebra with involution. The formula*

$$\|a\|_{\mathcal{J}} = \sum_{k\in\mathbb{Z}} \lim_{r\to\infty} \max\left\{ \max_{s=0,\dots,r-1} \left\{ \Big\| q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geqslant 0}}^{s} a_{i+k,i} \otimes 1^{s-i} \Big) \Big\| \right\}, \quad \Big\| \sum_{\substack{i=0,\\ i+k\geqslant 0}}^{r} (a_{i+k,i} \otimes 1^{r-i}) \Big\| \right\}$$

defines a submultiplicative seminorm on $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ such that for the enveloping C^*-algebra of the quotient *-algebra $\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}$ we have

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) \cong \overline{\mathcal{M}_{\mathcal{T}}(\mathcal{K})/\|\cdot\|_{\mathcal{J}}}.$$

In order to get a description of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ in the spirit of Doplicher-Roberts algebra, we consider the following subspaces of $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$. For every $r \in \mathbb{N}$, $k \in \mathbb{Z}$, $r + k \geqslant 0$, we put

$$\mathcal{M}(r+k, r) := \{\{a_{n,m}\}_{m,n\in\mathbb{N}} \in \mathcal{M}_{\mathcal{T}}(\mathcal{K}) : \ a_{n,m} \neq 0 \Longrightarrow \ n - m = k, \ m \leqslant r\}.$$

Hence an element $a \in \mathcal{M}_{\mathcal{T}}(\mathcal{K})$ is in $\mathcal{M}(r+k, r)$ iff it is of the form

$$\begin{pmatrix} & a_{0,k} & & 0 \\ & & \ddots & \\ & & & a_{r,r+k} \\ 0 & & & \end{pmatrix}, \text{ if } \ k \geqslant 0, \text{ or } \quad \begin{pmatrix} & & & 0 \\ a_{-k,0} & & & \\ & \ddots & & \\ 0 & & a_{r,r+k} & \end{pmatrix}, \text{ if } \ k \leqslant 0.$$

Combining [Kwa13, Theorem 4.11 and Proposition 5.1] we get:

Theorem 4.12 ([Kwa13])**.** *For every $r \in \mathbb{N}$, $k \in \mathbb{Z}$, $r + k \geqslant 0$, the formula*

$$\|a\|^{\mathcal{J}}_{r+k,r} := \max\left\{ \max_{s=0,\dots,r-1} \left\{ \|q_{\mathcal{J}}\Big(\sum_{\substack{i=0,\\ i+k\geqslant 0}}^{s} a_{i+k,i} \otimes 1^{s-i} \Big)\| \right\}, \Big\| \sum_{\substack{i=0,\\ i+k\geqslant 0}}^{r} (a_{i+k,i} \otimes 1^{r-i}) \Big\| \right\}$$

defines a seminorm on $\mathcal{M}(r+k,r)$ such that the family of quotinet spaces

$$\mathcal{K}_{\mathcal{J}} := \{\mathcal{M}(n,m)/\|\cdot\|^{\mathcal{J}}_{n,m}\}_{n,m\in\mathbb{N}}$$

with operations inhereited from $\mathcal{M}_{\mathcal{T}}(\mathcal{K})$ forms a right tensor C^*-precategory with a right tensoring $\otimes_{\mathcal{J}} 1$ induced by the inclusions $\mathcal{M}(m,n) \subset \mathcal{M}(m+1,n+1)$, $m,n \in \mathbb{N}$. Moreover,

$$\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J}) \cong \mathcal{DR}(\mathcal{K}_{\mathcal{J}}).$$

Remark 4.13. The above construction of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ has a number of important consequences. For instance, it gives an explicit formulas for norms of elements in the spectral subspaces $\mathcal{O}^{(k)}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$ [Kwa13, Theorem 4.10] and for the kernel of the universal representation $\iota = \{\iota_{n,m}\}_{n,m\in\mathbb{N}}$ of $\mathcal{K}$ in $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$. In particular, ι is injective if and only if $\mathcal{J} \subset (\ker \otimes 1)^{\perp}$ [Kwa13, Corollary 4.15].

4.3 Ideal structure of $\mathcal{O}_{\mathcal{T}}(\mathcal{K},\mathcal{J})$

We will discuss briefly the results of sections 6 and 7 in [Kwa13].

Definition 4.14 (Definitions 6.1, 6.6 in [Kwa13])**.** We say that an ideal $\mathcal{N}$ in a right tensor C^*-precategory $\mathcal{T}$ is *invariant* if $\mathcal{N} \otimes 1 \subseteq \mathcal{N}$. Then the quotient C^*-precategory $\mathcal{T}/\mathcal{N}$ is in a natural manner a *quotinet right tensor C^*-precategory.* More generally, if $\mathcal{K}$ is an ideal in $\mathcal{T}$ we say that an ideal $\mathcal{N}$ is *$\mathcal{K}$-invariant* if

$$(\mathcal{N}(n,m) \otimes 1)\mathcal{K}(m+1,l) \subseteq \mathcal{N}(n+1,l), \qquad n,m,l \in \mathbb{N}.$$

If $\mathcal{J}$ is an ideal in $\mathcal{T}$, we say that $\mathcal{N}$ is *$\mathcal{J}$-saturated* if $\mathcal{J} \cap \otimes 1^{-1}(\mathcal{N}) \subseteq \mathcal{N}$. In general, we denote by $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ the smallest $\mathcal{J}$-saturated ideal containing $\mathcal{N}$ and we call it *$\mathcal{J}$-saturation* of the ideal $\mathcal{N}$.

Examples 4.15. If $\mathcal{T}_\alpha = \{\alpha^n(A)A\alpha^m(A)\}_{n,m\in\mathbb{N}}$ is the right tensor C^*-precategory associated to an endomorphism $\alpha : A \to A$ and $\mathcal{J} = \{\alpha^n(A)J\alpha^m(A)\}_{n,m\in\mathbb{N}}$ is an ideal in $\mathcal{T}_\alpha$, cf. Example 4.5, then $\mathcal{J}$-saturated and invariant ideals in $\mathcal{T}_\alpha$ correspond to J-invariant ideals in (A,α), cf. Definition 2.10.

Theorem 4.16 (Structural theorem [Kwa13])**.** *Let $\mathcal{K}$, $\mathcal{J}$ and $\mathcal{N}$ be ideals in a right tensor C^*-precategory $\mathcal{T}$ such that $\mathcal{J} \subseteq J(\mathcal{K})$ and $\mathcal{N} \subseteq \mathcal{K}$ is $\mathcal{K}$-invariant. The formula*

$$\mathcal{O}(\mathcal{N}) = \overline{\operatorname{span}}\{\iota_{n,m}(a) : a \in \mathcal{N}(n,m),\ n,m \in \mathbb{N}\} \subseteq \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$$

defines a $\mathbb{T}$-invariant ideal in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ and we have natural isomorphisms

$$\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}}(\mathcal{N}, \mathcal{J} \cap \mathcal{N}), \qquad \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})/\mathcal{O}(\mathcal{N}) \cong \mathcal{O}_{\mathcal{T}/\widetilde{\mathcal{N}}}(\mathcal{K}/\widetilde{\mathcal{N}}, \mathcal{J}/\widetilde{\mathcal{N}}), \tag{20}$$

where $\widetilde{\mathcal{N}}$ is an arbitrary ideal in $\mathcal{T}$ such that $\mathcal{N} = \widetilde{\mathcal{N}} \cap \mathcal{K}$. Moreover, $\mathcal{S}_{\mathcal{J}}(\mathcal{N}) = \iota^{-1}(\mathcal{O}(\mathcal{N}))$ and hence in the right-hand sides of (20) *the ideals $\mathcal{N}$ and $\widetilde{\mathcal{N}}$ can be replaced by their $\mathcal{J}$-saturations $\mathcal{S}_{\mathcal{J}}(\mathcal{N})$ and $\mathcal{S}_{\mathcal{J}}(\widetilde{\mathcal{N}})$, respectively.*

Remark 4.17. Applying Theorem 4.16 to the objects from Example 4.4 one obtains the following result [Kwa13, Theorem 6.20]: Let X be a C^*-correspondence over A. Let $I \lhd A$ be an X-invariant ideal, i.e. $IX \subseteq XI$, and let J be an ideal in $J(X)$. Let $\mathcal{O}(I)$ be the ideal in $\mathcal{O}(J, X)$ generated by the image of I. We have natural isomorphisms

$$\mathcal{O}(I) \cong \mathcal{O}_X(I, J \cap I), \qquad \mathcal{O}(J, X)/\mathcal{O}(I) \cong \mathcal{O}(J/I, X/XI).$$

where

$$\mathcal{O}_X(I, J) := \mathcal{O}_{\mathcal{T}_X}(\mathcal{K}_X(I), \mathcal{K}_X(J))$$

is a *generalization of relative Cuntz-Pimsner algebra* [Kwa13, Definition 6.17]. Furthermore, the algebras $\mathcal{O}_X(I, J)$ and $\mathcal{O}(J \cap I, XI)$ are Morita-Rieffel equivalent. In general, if $IX \neq XI$, these algebras are not isomorphic. Accordingly, when applied to C^*-correspondences Theorem 4.16 gives a generalization of the analogous results from [FMR03], [Kat07], as well as a more detailed description of gauge invariant ideals.

Remark 4.18. Applying Theorem 4.16 to the zero ideal $\mathcal{N} = \{0\}$ we get that the kernel $\mathcal{R}_{\mathcal{J}} := \ker \iota$ of the universal representation $\iota : \mathcal{K} \to \mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ is invariant $\mathcal{J}$-saturation of the zero ideal and by passing to the quotient C^*-precategory $\mathcal{T}/\mathcal{R}_{\mathcal{J}}$ we may always assume that the universal representation ι is injective (equivalently that $\mathcal{J} \subseteq (\ker \otimes 1)^{\perp}$), [Kwa13, Theorem 6.11].

Let $\mathrm{Ideal}_{\mathcal{J}}(\mathcal{K})$ denote the lattice of $\mathcal{K}$-invariant and $\mathcal{J}$-saturated ideals in $\mathcal{K}$ and let $\mathrm{Ideal}^{\mathbb{T}}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$ be the lattice of $\mathbb{T}$-invariant ideals in $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$. By the above theorem we have a natural embedding

$$\mathrm{Ideal}_{\mathcal{J}}(\mathcal{K}) \hookrightarrow \mathrm{Ideal}^{\mathbb{T}}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})).$$

In general (as already example of crossed products by endomorphisms shows) this embedding is not surjective. In order to get a complete description of the lattice $\mathrm{Ideal}^{\mathbb{T}}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$ one may employ the description of $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ as the Doplicher-Roberts algebra $\mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ for the C^*-precategory $\mathcal{K}_{\mathcal{J}}$, see Theorem 4.12. Using the identification $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}) = \mathcal{DR}(\mathcal{K}_{\mathcal{J}})$ a number of necessary and sufficient conditions for a representation π of the ideal $\mathcal{K}$ to generate a C^*-algebra isomorphic to $\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J})$ are given in [Kwa13, Theorem 7.1]. This leads to an isomorphism

$$\mathrm{Ideal}_{\mathcal{K}_{\mathcal{J}}}(\mathcal{K}_{\mathcal{J}}) \cong \mathrm{Ideal}^{\mathbb{T}}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$$

[Kwa13, Theorem 7.6]. Applying the above isomorphism to relative Cuntz-Pimsner algebras $\mathcal{O}(J, X)$, see [Kwa13, Theorem 7.6], one obtains new insight into description of $\mathbb{T}$-invariant ideals in $\mathcal{O}(J, X)$ by means of pairs of ideals in A, obtained previously by Katsura [Kat07]. In general, we also have the following result:

Theorem 4.19 (Theorem 7.8 in [Kwa13])**.** *If* $\mathcal{K} \subseteq \mathcal{J} + \ker \otimes 1$*, then* $\mathrm{Ideal}_{\mathcal{J}}(\mathcal{K}) \cong \mathrm{Ideal}^{\mathbb{T}}(\mathcal{O}_{\mathcal{T}}(\mathcal{K}, \mathcal{J}))$.

4.4 Embeddings of Cuntz-Pimsner algebras into Doplicher-Roberts algebras

The relationship between the Pimsner algebra $\mathcal{O}_X$ and the Doplicher-Roberts algebra $\mathcal{DR}(X)$ associated to a C^*-correspondence X (under the assumption that the left action of A on X is injective) was studied in [DPZ98], [FMR03]. It was noticed that we have natural embeddings $\mathcal{O}_X \subseteq \mathcal{DR}(X) \subseteq \mathcal{O}_X^{**}$, cf. [DPZ98, Proposition 3.9], and that under certain circumstances injective representations of the C^*-correspondence X integrate to a faithful representation of $\mathcal{DR}(X)$, cf. [FMR03, Theorem 6.6].

[Kwa13, Section 8] provides necessary and sufficient conditions under which algebras of the form $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_1, \mathcal{J}_1)$ embed in a natural way in algebras of the form $\mathcal{O}_{\mathcal{T}}(\mathcal{K}_2, \mathcal{J}_2)$. For instance, for any C^*-correspondence X the relative Cuntz-Pimsner algebra $\mathcal{O}(J \cap J(X), X)$ embeds into the relative Doplicher-Roberts algebra $\mathcal{DR}(J, X)$ if and only if $R_J = R_{J\cap J(X)}$, which holds automatically if the left action on X is injective, [Kwa13, Corollary 8.8]. Conditions implying that a representation of X extends a faithful representation of $\mathcal{DR}(J, X)$ are given in [Kwa13, Theorem 9.4]. It is a far reaching generalization of [FMR03, Theorem 6.6].

5 Cuntz-Pimsner algebras for product systems associated to product systems over Ore semigroups

Let P be a semigroup with identity $e \in P$. We assume it is left cancellative (that is $rp = rq$ implies $p = q$ for $p, q, r \in P$). Fowler [Fow02] following closely the path marked by Pimsner [Pim97] defined the C^*-algebra $\mathcal{O}_X$ for any product systems of C^*-correspondences $X = \bigsqcup_{p\in P} X_p$. One of the drawbacks of this definition is that, as in the case of original Pimsner algebras, it can happen that $\mathcal{O}_X = \{0\}$ even though the system X is non-trivial. Another drawback is that Fowler's definition does not take into account additional relations that occur in the Fock representation of X. To remedy those issues, Sims and Yeend [SY10] proposed a definition of another C^*-algebra $\mathcal{NO}_X$ which they call Cuntz-Nica-Pimsner algebra. It is defined only in the case when P is a positive cone in a *quasi-lattice ordered group* (G, P) [Nic92]. In essence, all research concerning C^*-algebras associated to product systems so far concerned exclusively the algebra $\mathcal{NO}_X$. However, as shown in [CLSV11] this algebra does not always have desired properties. Moreover, the assumption that P is a positive cone excludes, for instance, the case when P is a group. Therefore, the theory of algebras $\mathcal{NO}_X$ does not embrace the theory of group actions. The problem of a general definition of an analogue of Cuntz-Pimsner algebra for product systems seems to be very difficult.

In this section we discuss results of the paper [KS16]. An important result is showing that under certain natural assumptions Fowler's algebra $\mathcal{O}_X$ is an appropriate object to study, which models a number of interesting examples and the structure of which can be explored by construction of Doplicher-Roberts type and Fell bundles (Theorem 5.1, subsection 5.3). As it is known [AS93], [AL94] (cf. Theorems 2.21, 3.15) topological freeness is a fundamental dynamical property which allows one to prove uniqueness theorems and study simplicity and ideal structure of the associated algebras. Therefore it is of fundamental importance to find an analogue of this property for product systems. For crossed products by single Hilbert bimodules the relevant theory was elaborated in [Kwa14] (Theorem 5.4). A further generalization of this work to product systems was done in [KS16] and is presented in Definition 5.8. It allows to prove a general uniqueness theorem (Theorem 5.10) and simplicity criterion (Theorem 5.11). These results give non-trivial applications to Fell bundles, twisted (semi)group crossed products, topological P-graphs, semigroup versions of Exel's crossed product and to the new Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$ associated to an affine semigroup (subsection 5.3).

5.1 Construction of $\mathcal{O}_X$ in the spirit of Doplicher and Roberts

In [KS16] we work under the assumption that P is a *semigroup of Ore type*, that is $sP \cap tP \neq \varnothing$ for every $s, t \in P$. This assumption is equivalent to the requirement that P with the natural quasi-order:

$$p \leqslant q \overset{def}{\Longleftrightarrow} pr = q \quad \text{for some } r \in P$$

is a directed set. Then there is a semigroup homomorphism $\iota : P \to G(P)$ into the *enveloping group* $G(P)$ of P such that $G(P) = \iota(P)\iota(P)^{-1}$. This homomorphism is injective if and only if P is (right) cancellative, [KS16, Proposition 2.2]. Amongst examples of semigroups of Ore type are all left cancellative abelian semigroups and all groups. The assumption of left cancellativity allows us to write $p^{-1}q := r$, when $pr = q$ (this equality determines r uniquely).

Let $X = \bigsqcup_{p\in P} X_p$ be a product system over P. We assume that X is *regular*, which means that for every $p \in P$ the left action of $A = X_e$ on the C^*-correspondence X_p is injective and acts by generalized compacts [KS16, Definition 3.1]. The family

$$\mathcal{K}_X := \{\mathcal{K}(X_q, X_p)\}_{p,q\in P}$$

forms in a natural way a C^*-precategory. Regularity of X implies that $\mathcal{K}_X$ has also the structure of right tensoring. Namely, for any $r \in P$ we have linear maps $\otimes 1_r : \mathcal{K}(X_q, X_p) \to \mathcal{K}(X_{qr}, X_{pr})$, $p, q \in P$, where

$$(T \otimes 1_r)(xy) := (Tx)y \qquad x \in X_q,\ y \in X_r,\ T \in \mathcal{K}(X_q, X_p)$$

and $q \neq e$. For $q = e$, $\otimes 1_r$ is defined by using the identification $\mathcal{K}(X_q, X_p) = X_p$. We have the following relations

$$(T \otimes 1_r)^* = (T^*) \otimes 1_r, \qquad (T \otimes 1_r) \otimes 1_s) = T \otimes 1_{rs},$$

$$(T \otimes 1_r)(S \otimes 1_r) = (TS) \otimes 1_r, \qquad T \in \mathcal{K}(X_p, X_q),\ S \in \mathcal{K}(X_s, X_p).$$

We call the pair $(\mathcal{K}_X, \{\otimes 1_r\}_{r\in P})$ the right tensor C^*-precategory associated to X [KS16, Definition 3.4].

Theorem 5.1 (Theorem 3.8 in [KS16])**.** *Let X be a regular product system over a semigroup P of Ore type and let $G(P)$ be the enveloping group of P. For every $\iota(p)\iota(q)^{-1} \in G(P)$ we define*

$$B_{\iota(p)\iota(q)^{-1}} := \varinjlim \mathcal{K}(X_{qr}, X_{pr})$$

to be the Banach space direct limit of the directed system $(\{\mathcal{K}(X_{qr}, X_{pr})\}_{r\in P}, \{\otimes 1_s\}_{s\in P})$. The family $\mathcal{B}_X = \{B_g\}_{g\in G(P)}$ with operations inherited from $\mathcal{K}_X$ is Fell bundle over $G(P)$ and we have a canonical isomorphism

$$\mathcal{O}_X \cong C^*(\mathcal{B}_X) = C^*(\{B_g\}_{g\in G(P)})$$

where $C^*(\mathcal{B}_X)$ is the cross-sectional C^*-algebra. Moreover,

i) the universal representation $j_X : X \to \mathcal{O}_X$ is injective;

ii) $\mathcal{O}_X$ has a natural grading $\{(\mathcal{O}_X)_g\}_{g\in G(P)}$ over $G(P)$ where

$$(\mathcal{O}_X)_g = \overline{\operatorname{span}}\{j_X(x)j_X(y)^* : x \in X_p, y \in X_q,\ \iota(p)\iota(q)^{-1} = g\}; \tag{21}$$

iii) for any injective covariant representation ψ of X the integrated representation Π_ψ of $\mathcal{O}_X$ is isometric on every subspace $(\mathcal{O}_X)_g$, $g \in G(P)$, and hence it is an isomorphism on the C^*-subalgebra $(\mathcal{O}_X)_e$ of $\mathcal{O}_X$.

Remark 5.2. The paper [KS16] was posted on arXiv at the end of 2013 (arXiv:1312.7472). At the beginning of 2015 (arXiv:1502.07768) the paper [AM15] was posted, where the authors independently obtained a similar description of the algebra $\mathcal{O}_X$, under the additional assumption that homomorphisms defining left actions on the fibers X_p are nondegenerate.

The above theorem is a powerful tool for studying the structure of the algebra $\mathcal{O}_X$. It also allows us to define a reduced version of the algebra $\mathcal{O}_X$, see [KS16, Remark 3.9].

Definition 5.3 ([KS16])**.** The *reduced Cuntz-Pimsner algebra* of the regular product system X is the reduced cross-sectional C^*-algebra

$$\mathcal{O}_X^r := C_r^*(\mathcal{B}_X)$$

of the Fell bundle $\mathcal{B}_X = \{B_g\}_{g\in G(P)}$ constructed in Theorem 5.1.

5.2 Topological freeness for product systems

Rieffel proved that if X is a Morita-Rieffel Hilbert A-B-bimodule, then the induced representation functor factorises to a homeomorphism $[X\text{-Ind}] : \widehat{B} \to \widehat{A}$ between the spectra of algebras A and B, see, for instance, [RW98]. If X is an arbitrary Hilbert bimodule over A, then the spaces $\langle X, X\rangle_A$ i ${}_A\langle X, X\rangle$ are ideals in A and we may treat X as an equivalence $\langle X, X\rangle_A$-${}_A\langle X, X\rangle$-bimodule. Accordingly, we may treat $[X\text{-Ind}] : \widehat{\langle X, X\rangle_A} \to \widehat{{}_A\langle X, X\rangle}$ as a partial homeomorphism of $\widehat{A}$. The results of [Kwa14] can be summarized as follows:

Theorem 5.4 ([Kwa14]). *Let X be a Hilbert bimodule over A and let $[X\text{-Ind}]$ be the dual partial homeomorphism of $\widehat{A}$.*

i) If $[X\text{-Ind}]$ is topologically free, then every non-zero ideal I in $A \rtimes_X \mathbb{Z}$ has a non-zero intersection with A (A detects the ideals in $A \rtimes_X \mathbb{Z}$).

ii) If $[X\text{-Ind}]$ is free, then $J \mapsto \widehat{J \cap A}$ is a bijection between the ideals in $A \rtimes_X \mathbb{Z}$ and open $[X\text{-Ind}]$-invariant subsets of $\widehat{A}$ (A separates the ideals $A \rtimes_X \mathbb{Z}$, equivalentely every ideal in $A \rtimes_X \mathbb{Z}$ is $\mathbb{T}$-invariant).

iii) If $X\text{-Ind}$ is topologically free and minimal, then $A \rtimes_X \mathbb{Z}$ is simple.

Remark 5.5. In light of recent results of [KM], see [KM, Theorems 9.11, 9.12, 9.14], if A is separable or contains an essential ideal of Type I, then the implications in items i), ii), iii) are in fact equivalences.

The proof of Theorem 5.4, see [Kwa14], is written in the language of Fell bundles (over $\mathbb{Z}$). Therefore with a bit of work it can be generalized to the case of arbitrary Fell bundles over a discrete group G. This was done in [AA]. In [KS16] the authors undertook a more ambitious project, the aim of which was to prove analogous theorem for product systems over semigroups.

The very first problem is a definition of the object dual to a single C^*-correspondence. In [KS16] it was solved by introducing a notion of a multivalued function dual to a homomorphism. Let $\alpha : A \to B$ be a homomorphism between two C^*-algebras. A *multifunction dual to α* [KS16, Definition 4.1] is the multifunction $\widehat{\alpha} : \widehat{B} \to \widehat{A}$ given by the formula

$$\widehat{\alpha}([\pi_B]) := \{[\pi_A] \in \widehat{A} : \pi_A \leqslant \pi_B \circ \alpha\}, \qquad \pi_B \in \mathrm{Irr}(B). \tag{22}$$

If X is a regular C^*-correspondence over A, that is if the left action is given by an injective homomorphism $\phi : A \to \mathcal{K}(X)$, then treating X as an equivalence $\mathcal{K}(X)$-$\langle X, X\rangle_A$-bimodule we have a homeomorphism $[X\text{-Ind}] : \widehat{\langle X, X\rangle_A} \to \widehat{\mathcal{K}(X)}$ and a multifunction $\widehat{\phi} : \widehat{\mathcal{K}(X)} \to \widehat{A}$. The *multifunction $\widehat{X}$ dual to the C^*-correspondence* X [KS16, Definition 4.4] is the composition of multifunctions

$$\widehat{X} := \widehat{\phi} \circ [X\text{-Ind}].$$

Remark 5.6. Definition of a dual to a homomorphism $\alpha : A \to B$ can be improved by taking into account in (22) the multiplicities of subrepresentations π_A of $\pi_B \circ \alpha$. That would lead to a notion of a dual graph (in place of a dual multifunction). The habilitant is currently working with Eduard Ortega and Toke Carlsen on elaboration of such a theory of dual graphs for single C^*-correspondences. Nonetheless, the concept of multifunction already gives sharp results in a number of examples. Moreover, for product systems the condition we are about to introduce is already sufficiently complicated even for multifunctions.

Let $X = \bigsqcup_{p \in P} X_p$ be a product system over P.

Definition 5.7 (Definicja 4.8 w [KS16]). The family $\widehat{X} := \{\widehat{X}_p\}_{p\in P}$ of multifunctions dual to C^*-correspondences X_p, $p \in P$, is called the *semigroup dual* to the regular product system X (this family indeed forms a semigroup with composition of multifunctions).

Only beacuse of Remark 5.6, we call the following analogue of topological freeness for X the topological aperiodicity.

Definition 5.8 (Definition 5.3 in [KS16]). We say that a regular product system X, or the dual semigroup $\{\widehat{X}_p\}_{p\in P}$, is *topologically aperiodic* if for each nonempty open set $U \subseteq \widehat{A}$, each finite set $F \subseteq P$ and element $q \in P$ such that $\iota(q) \neq \iota(p)$ for $p \in F$, there exists a $[\pi] \in U$ such that for some enumeration of elements of $F = \{p_1, ..., p_n\}$ and some elements $s_1, ..., s_n \in P$ with $q \leqslant s_1 \leqslant ... \leqslant s_n$ and $p_i \leqslant s_i$ we have

$$[\pi] \notin \widehat{X}_{q^{-1}s_i}(\widehat{X}^{-1}_{p_i^{-1}s_i}([\pi])) \qquad \text{for all} \;\; i = 1, ..., n. \tag{23}$$

Remark 5.9. In particular cases the above condition can be considerably simplified, see [KS16, Proposition 5.5]. For instance, if $\{X^{\otimes n}\}_{n\in\mathbb{N}}$ is a product system arising from a single regular C^*-correspondence X, the topological aperiodicity is equivalent to the condition that for each $n > 0$ the set

$$F_n = \{[\pi] \in \widehat{A} : \pi \in \widehat{X}^n([\pi])\}$$

has empty interior.

Theorem 5.10 (Theorem 5.6 in [KS16]). *Let X be a regular product system and let $\Lambda : \mathcal{O}_X \to \mathcal{O}_X^r$ be the canonical epimorphism (the regular representation). If X is topologically aperiodic, then for every injective covariant representation ψ of X there exists an epimorphism $\pi_\psi : C^*(\psi(X)) \to \mathcal{O}_X^r$ such that the diagram*

$$\begin{array}{ccccc} \mathcal{O}_X & \longrightarrow & C^*(\psi(X)) & \xrightarrow{\pi_\psi} & \mathcal{O}_X^r \end{array}, \qquad \mathcal{O}_X \xrightarrow{\Lambda} \mathcal{O}_X^r \tag{24}$$

commutes. In particular, if Λ is an isomorphism which holds automatically, for instance, when the group $G(P)$ is amenable, then

$$\mathcal{O}_X \cong C^*(\psi(X))$$

for every injective covariant representation ψ of the product system X.

Theorem 5.11 (Theorem 5.10 in [KS16]). *Suppose that X is a regular topologically aperiodic product system. The algebra $\mathcal{O}_X^r$ is simple if and only if X is minimal, in the sense that for every $J \lhd A$ we have*

$$\forall_{p\in P} \;\; \{a \in A : \langle X_p, aX_p\rangle_p \subseteq J\} = J \quad \Longrightarrow \quad J = \{0\} \;\; \text{or } J = A.$$

5.3 Applications and examples

Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a Fell bundle over an arbitrary discrete group G. Suppose that $\mathcal{B}$ is *saturated*, that is $B_g B_{g^{-1}} = B_e$ for every $g \in G$. Then every fiber B_g is in a natural way an equivalence bimodule over $A = B_e$. We may treat $\mathcal{B}$ as a product system over $P := G$. The semigroup dual to the so-understood system coincides with the dual group $\widehat{\mathcal{B}} = \{\widehat{B}_g\}_{g\in G}$ of homeomorphisms of $\widehat{A}$ induced by the bimodules B_g: $\widehat{B}_g = [B_g\text{-Ind}]$, $g \in G$. Topological aperiodicity of $\mathcal{B}$ coincides with *topological freeness* of $\widehat{\mathcal{B}}$. We recall that the group action $\widehat{\mathcal{B}}$ is *topologically free*, if for every finite $F \subseteq G\setminus\{e\}$ the union of fixed points for $\widehat{B}_g$, $g \in F$, has empty interior, cf. [AS93]. This gives the following result, cf. [KS16, Corollary 6.5].

Theorem 5.12. *Let $\{B_g\}_{g\in G}$ be a saturated Fell bundle over a discrete group G and let $\Lambda : C^*(\mathcal{B}) \to C^*_r(\mathcal{B})$ be the canonical epimorphism (the regular representation).*

i) If the action $\{\widehat{B}_g\}_{g\in G}$ is topologically free, then for every ideal I in $C^*(\mathcal{B})$ such that $I \cap B_e \neq \{0\}$ we have $I \subseteq \ker\Lambda$.

ii) If the action $\{\widehat{B}_g\}_{g\in G}$ is topologically free and has no nontrivial open invariant sets, then the C^*-algebra $C^*_r(\{B_g\}_{g\in G})$ is simple.

Remark 5.13. If $\alpha : G \to \mathrm{Aut}(A)$ is a group action $\mathcal{B}_\alpha := \{E_{\alpha_{g^{-1}}}\}_{g\in G}$ is the associated Fell bundle, then $C^*(\mathcal{B}_\alpha) \cong A \rtimes_\alpha G$ and $C^*_r(\mathcal{B}_\alpha) = A \rtimes^r_\alpha G$. Hence Theorem 5.12 implies the results of Archbold and Spielberg [AS93] for classical crossed products. Twisting multiplication in $\mathcal{B}_\alpha$ we may apply Theorem 5.12 to twisted crossed products, which already gives a new application.

Remark 5.14. More generaly we may consider a regular product system $X = \bigsqcup_{p\in P} X_p$ over a semigroup P of Ore type such that every fiber X_p, $p \in P$ is a Hilbert bimodule over $A := X_e$. Let $\mathcal{B}_X = \{B_g\}_{g\in G(P)}$ be the associated Fell bundle constructed in Theorem 5.1. By describing the dual (partial) action $\{\widehat{B}_g\}_{g\in G(P)}$ on $\widehat{A} = \widehat{B_e}$ it is shown in [KS16, Proposition 6.4] that topological freeness of $\{\widehat{B}_g\}_{g\in G(P)}$ implies topological aperiodicity of X. Thus, in this case topological aperiodicity is (at least formally) a weaker condition.

5.3.1 Twisted crossed products by semigroups of endomorphisms

Let $\alpha : P \to \mathrm{End}(A)$ be an action of a semigroup P by extendible endomorphisms of A and let ω be a $\mathbb{T}$-valued cocycle over P, that is $\omega : P \times P \to \mathbb{T}$ is such that $\omega(p,q)\omega(pq,r) = \omega(p,qr)\omega(q,r)$ for $p,q,r \in P$. Fowler in [Fow02, Definicja 3.1] associated to the quadruple (A,α,P,ω) the *twisted semigroup crossed product* $A \times_{\alpha,\omega} P$. It is the C^*-algebra generated by $\{i_A(a)i_P(s) : a \in A, s \in P\}$, where (i_A, i_P) is a universal representation of (A,P,α,ω), so that $i_A : A \to A \times_{\alpha,\omega} P$ is a homomorphism and $\{i_P(p) : p \in P\}$ are isometries in $M(A \times_{\alpha,\omega} P)$ such that

$$i_P(p)i_P(q) = \omega(p,q)i_P(pq) \quad \text{and} \quad i_P(p)i_A(a)i_P(p)^* = i_A(\alpha_p(a)),$$

for all $p,q \in P$ and $a \in A$. The family of C^*-correspondences $\{E_{\alpha_p}\}_{p\in P}$ form in a natural way a product system over the semigroup P^{op} opposite to P. Fowler proved in [Fow02, Proposition 3.4] that twisting multiplication in this product by the cocycle ω we get a product system $X = \bigsqcup_{p\in P^{op}} X_p$ such that

$$A \rtimes_{\alpha,\omega} P \cong \mathcal{O}_X.$$

The system X is regular if and only if every endomorphism α_p is injective (this is also a necessary condition for the homomorphism i_A to be injective). Let us assume that every endomorphism α_p is injective and that P^{op} is an Ore semigroup. The results of [KS16] imply that under these assumptions $A \rtimes_{\alpha,\omega} P$ has good properties. In particular, A embeds into $A \rtimes_{\alpha,\omega} P$ and we may define the *reduced twisted semigroup crossed product* $A \times^r_{\alpha,\omega} P$:

$$A \rtimes^r_{\alpha,\omega} P := \mathcal{O}^r_X.$$

Identifying the dual group $\widehat{X} = \{\widehat{X_p}\}_{p\in P^{op}}$ with the semigroup of multifunctions $\{\widehat{\alpha_p}\}_{p\in P^{op}}$ dual to endomorphisms α_p, $p \in P$, cf. [KS16, Lemma 6.7], Theorem 5.10 gives *completely new results for semigroup crossed products*, cf. [KS16, Proposition 6.9].

5.3.2 Topological P-graphs

The notion of a topological graph of higher rank [Yee07] can be readily generalized to a notion of a topological P-graph, see [KS16, Definition 6.14]: instead of the semigroup $\mathbb{N}^k$ the authors of [KS16] consider an arbitrary semigroup P of Ore type. Thus a *topological P-graph* is a pair (Λ, d) where the "space of paths" Λ is a small category equipped with a topology of a locally compact Hausdorff space such that composition of morphisms is continuous and open, the target map r is continuous and the source map s is a local homeomorphism. The path length map $d\colon \Lambda \to P$ is a continuous functor which satisfies a factorisation rule for paths. In [KS16, Subsection 6.4] a natural product system $X = \bigsqcup_{p\in P} X_p$ is associated to (Λ, d) and characterisation of regularity of X is given. This leads to the notion of a regular topological P-graph (Λ, d). For such (Λ, d) the algebras $C^*(\Lambda, d)$ and $C^*_r(\Lambda, d)$ are defined as $\mathcal{O}_X$ and $\mathcal{O}^r_X$, respectively. When $P = \mathbb{N}^k$, the algebra $C^*(\Lambda, d)$ coincides with the C^*-algebra of the topological higher-rank graph [Yee07]. Topological aperiodicity and minimality of X are characterised in terms of (Λ, d) in [KS16, Proposition 6.17].

In the case $P = \mathbb{N}^k$ topological aperiodicity of (Λ, d) is easier to verify, but also it is a stronger condition than aperiodicity introduced in [Yee07]; the reason for this is explained in Remark 5.6. In the case when the target map r is injective, the two conditions do coincide. In this case $C^*(\Lambda, d)$ can be interpreted as a semigroup crossed product by transfer operators. In particular, if $P = \mathbb{N}$ and $r = id$, then $C^*(\Lambda, d)$ is isomorphic to Exel crossed product, where the transfer operator corresponds to the local homeomorphism s. Then topological aperiodicity is equivalent to topological freeness of the map s [KS16, Example 6.13].

5.3.3 Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$

The so-called new Cuntz algebra $\mathcal{Q}_{\mathbb{N}}$ associated to the "$ax + b$"-semigroup over $\mathbb{N}$, introduced in [Cun08], can be realized as a semigroup crossed product of Exel's type, where $P = \mathbb{N}^\times$ is a multiplicative semigroup and the coefficient algebra is $C(\mathbb{T})$, [HLS12]. By a direct analysis Cuntz showed that $\mathcal{Q}_{\mathbb{N}}$ is simple and purely infinite. Corollary 5.11 gives simplicity of $\mathcal{Q}_{\mathbb{N}}$ in a natural immediate way. Corollary 5.11 was used in [HSS] to prove simplicity of a twisted version of the algebra $\mathcal{Q}_{\mathbb{N}}$.

6 Ideal structure and pure infiniteness of C^*-algebras associated to Fell bundles

Kirchberg-Philips classification [Kir00], [Phi00] of simple purely infinite C^*-algebras gave a strong impulse for research related with classification program of C^*-algebras and in particular non-simple purely infinite C^*-algebras. Additional impetus was given by a discovery of relationships between pure infiniteness and boundary and paradoxical actions [LS96], [JR00], [RS12]. In this section, based on the results of [KS17], we clarify these relationships and present pure infiniteness criteria which may be applied to C^*-algebras considered in previous sections.

More specifically, the dilation of product systems to Fell bundles described in Theorem 5.1 is a strong tool allowing to study the semigroup version of Cuntz-Pimsner algebra as a cross-sectional C^*-algebra of a Fell bundle. It is important as the theory of Fell bundles over discrete groups is well developed [Exel]. We recall that such C^*-algebras model in a direct way twisted versions of classical crossed products, cf. Remark 5.13, and also versions for such crossed products for partial actions. In this section we complement the theory of Fell bundles with general statements describing the ideal structure, including primitive ideal space (Theorem 6.4), and pure infiniteness criteria (Theorems 6.6, 6.10) for the reduced cross-sectional C^*-algebras. This, in particular, gives a full set of tools to study the C^*-algebras associated with product systems.

It should be emphasized that already for classical crossed products by group actions the relationships between the known criteria for pure infiniteness [LS96], [JR00], [RS12] were not completely understood. In [KS17] these criteria are not only generalized to Fell bundles, but in fact they are unified and improved – weakened (cf. Remarks 6.8, 6.11). In particular, the aforementioned relationships are clarified. The achieved results are optimal, for instance, when applied to graph algebras (Remarks 6.5, 6.12) and they lead to completely new results for semigroup crossed products (subsection 6.2.1).

6.1 Ideal structure of $C_r^*(\mathcal{B})$

Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a Fell bundle over a discrete group G. An *ideal in the Fell bundle* $\mathcal{B}$ is a family $\mathcal{J} = \{J_g\}_{g\in G}$ where J_g is a closed linear subspace of B_g such that $J_g B_g \subseteq J_g$ and $B_g J_g \subseteq J_g$ for every $g \in G$. Let

$$\mathrm{Ideal}^{\mathcal{B}}(B_e) := \{I \lhd B_e : B_g I B_{g^{-1}} \subseteq I,\ g \in G\}$$

be the lattice of $\mathcal{B}$*-invariant ideals* in B_e. If $\mathcal{J} = \{J_g\}_{g\in G}$ is an ideal in $\mathcal{B}$ then $J_e \in \mathrm{Ideal}^{\mathcal{B}}(B_e)$. Conversely, if $I \in \mathrm{Ideal}^{\mathcal{B}}(B_e)$ then putting $J_g = B_g I$ for every $g \in G$, the family $\mathcal{J} := \{J_g\}_{g\in G}$ is an ideal in $\mathcal{B}$. Furthermore, we then have that $J := \overline{\bigoplus_{g\in G} J_g} \cong C_r^*(\mathcal{J})$ is an ideal in $C_r^*(\mathcal{B})$ generated by I and $I = A \cap J$ (we call ideal J a *graded* *ideal*). Accordingly, we have an embedding

$$\mathrm{Ideal}^{\mathcal{B}}(B_e) \hookrightarrow \mathrm{Ideal}(C_r^*(\mathcal{B})),$$

which is an isomorphism if and only if B_e separates the ideals in $C_r^*(\mathcal{B})$ (equivalently, every ideal in $C_r^*(\mathcal{B})$ is graded). Generalizing the work of Sierakowski [Sie10] for classical crossed products we also have the following equivalence, see [AA, Theorem 3.19], [KS17, Twierdzenie 3.12] (the manuscript [AA], arXive:1503.07094, appeared on arXive just before the end of work [KS17], arXiv:1505.05202, with independently achieved identical preliminary results, which led to a reformulation of Section 3 in [KS17]):

$$\begin{array}{ccc} B_e \text{ separates the ideals in } C_r^*(\mathcal{B}) & \Longleftrightarrow & \mathcal{B} \text{ is exact and has the} \\ \big(\mathrm{Ideal}^{\mathcal{B}}(B_e) \cong \mathrm{Ideal}(C_r^*(\mathcal{B}))\big) & & \text{residual intersection property} \end{array}$$

We say that $\mathcal{B}$ has the *intersection property* if B_e detects ideals in $C_r^*(\mathcal{B})$, that is when every non-zero ideal in $C_r^*(\mathcal{B})$ has a non-zero intersection with B_e. We say that $\mathcal{B}$ has the *residual intersection property*, if every quotient Fell bundle $\mathcal{B}/\mathcal{J}$ has the intersection property for every ideal $\mathcal{J}$ in $\mathcal{B}$. For any ideal $\mathcal{J}$ in $\mathcal{B}$ we have a natural sequence

$$0 \longrightarrow C_r^*(\mathcal{J}) \xrightarrow{\iota_r} C_r^*(\mathcal{B}) \xrightarrow{\kappa_r} C_r^*(\mathcal{B}/\mathcal{J}) \longrightarrow 0.$$

We say that $\mathcal{B}$ is *exact* if for every ideal $\mathcal{J}$ in $\mathcal{B}$ the above sequence is exact. If G is exact then every Fell bundle $\mathcal{B}$ over G is exact. Finding general efficient criteria for exactness of $\mathcal{B}$ requires further research. It is shown in [KS17, Proposition 3.7] that $\mathcal{B}$ is exact if and only if every Fourier ideal[4] J in $C_r^*(\mathcal{B})$ is a graded ideal. Also $\mathcal{B}$ is exact, when it is *amenable*, that is when $C^*(\mathcal{B}) = C_r^*(\mathcal{B})$.

Let us now discuss conditions implying the residual intersection property.

In the case when $\mathcal{B}$ is saturated, Theorem 5.12 implies that $\mathcal{B}$ has the intersection proprety if the dual dynamical system $\{\widehat{B}_g\}_{g\in G}$ on $\widehat{\mathcal{B}_e}$ is topologically free. For arbitrary Fell bundles this result was generalized in [AA]. This leads to a dynamical description of ideal in $C_r^*(\mathcal{B})$ assuming that $\{\widehat{B}_g\}_{g\in G}$ is *residually topologically free*, i.e every restriction of $\{\widehat{B}_g\}_{g\in G}$ to a closed invariant set if topologically free, cf. [KS17, Corollary 3.23].

The authors of [KS17] proposed a different condition which apart from the intersection property also gives some control over the positive elements of the $C_r^*(\mathcal{B})$. This is important in the study of pure infiniteness. For group actions an analogues condition appears in the work of Kishimoto [Kis81], Olesen and Pedersen [OP82], and in the context of C^*-correspondences in [MS00, Definition 5.1].

Definition 6.1 (Definition 4.1 in [KS17]). A Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$ is *aperiodic* if for each $b_t \in B_t$, $t \in G\backslash\{e\}$, and every non-zero hereditary subalgebra D of B_e

$$\inf\{\|ab_g a\| : a \in D^+,\ \|a\| = 1\} = 0. \tag{25}$$

We say that $\mathcal{B}$ is *residually aperiodic* if $\mathcal{B}/\mathcal{J}$ is aperiodic for any ideal $\mathcal{J}$ in $\mathcal{B}$.

[4] i.e. an ideal which is invariant under the projections from $C_r^*(\mathcal{B})$ onto the subspaces B_g, $g \in G$

Remark 6.2. While writing the paper [KS17] the relationship between aperiodicity and topological freeness for Fell bundles was not completely clear. Results of Olesen and Pedersen [OP82] suggest that these concepts should be equivalent, at least in the separable case. This problem has become the starting point of the article [KM], where an in-depth analysis of the relationships between various non-triviality conditions for Fell bundles has been carried out. Among other things, it is shown in [KM] that if B_e is separable or if it contains an essential ideal of Type I then

$$\text{the bundle } \mathcal{B} \text{ is aperiodic} \iff \text{the dual system } \{\widehat{B}_g\}_{g\in G} \text{ is topologically free.}$$

Moreover, these equivalent conditions are also equivalent to intersection property for $\mathcal{B}$ when $G = \mathbb{Z}$ or $G = \mathbb{Z}_n$ where n is a square free number.

Remark 6.3. When B_e is noncommutative, aperiodicity is sometimes easier to verify than topological freeness, cf. [KS17, Proposition 7.3]. Moreover, we have the following implication, cf. [KS17, Corollary 4.4], [Kwa16, Proposition 2.42]:

$$\text{the bundle } \mathcal{B} \text{ is aperiodic} \implies \begin{array}{l}\text{for every } b \in C_r^*(\mathcal{B})^+\setminus\{0\} \text{ there is } a \in B_e^+\setminus\{0\} \\ \text{such } a \precsim b \text{ (a preceds b in Cuntz sense).}\end{array} \tag{26}$$

The consequent of the above implication implies that $\mathcal{B}$ has the intersection property.

Let $\mathcal{B}$ be a Fell bundle. The dual system $\{\widehat{B}_g\}_{g\in G}$ is a partial dynamical system (it consists of partial homeomorphisms). It projects via $\widehat{B_e} \ni [\pi] \to \ker\pi \in \mathrm{Prim}(B_e)$ to a partial dynamical system $\{\check{B}_g\}_{g\in G}$ on the space of primitive ideals $\mathrm{Prim}(B_e)$, cf. [KS17, Proposition 3.16]. The orbit Gx of a point $x \in \mathrm{Prim}(B_e)$ with respect to $\{\check{B}_g\}_{g\in G}$ is defined in the obvious way, cf. [KS17, Definition 2.4]. The *quasi-orbit* $\mathcal{O}(x)$ is by definition an equivalence class of x with respect to the relation $\sim$ given by the formula

$$x \sim y \iff \overline{Gx} = \overline{Gy}.$$

We denote by $\mathcal{O}(\mathrm{Prim}(B_e))$ the *space of quasi-orbits* $\Omega/\sim$ equipped with quotient topology. The following description of the primitive ideal space is a generalization of classical results for crossed products, cf. [Gre78, Section 5].

Theorem 6.4 (Theorem 6.8 in [KS17])**.** *Let $\mathcal{B}$ be an exact Fell bundle. Suppose that $\mathcal{B}$ has the residual intersection property, which holds for instance if one of the following conditions is satisfied:*

(i) the system $(\{\widehat{D}_g\}_{g\in G}, \{\widehat{h}_g\}_{g\in G})$ dual to $\mathcal{B}$ is residually topologically free,

(ii) $\mathcal{B}$ is residually aperiodic.

Then the map $\mathcal{I}(C_r^*(\mathcal{B})) \ni J \to J \cap B_e \in \mathcal{I}^{\mathcal{B}}(B_e)$ is a lattice isomorphism. If additionally $\mathcal{B}$ is separable, then the isomorphism $\mathrm{Ideal}^{\mathcal{B}}(B_e) \cong \mathrm{Ideal}(C_r^*(\mathcal{B}))$ *induces homeomorphism*

$$\mathrm{Prim}(C_r^*(\mathcal{B})) \cong \mathcal{O}(\mathrm{Prim}(B_e)).$$

Remark 6.5. The above theorem was applied to graph algebras. This gives description of the primitive ideal space $\mathrm{Prim}(C^*(E))$ when E satisfies condition (K), see [KS17, Corollary 7.8]. Such description was originally obtained using different methods in [BHRS02].

6.2 Pure infiniteness of $C_r^*(\mathcal{B})$

As noted in (26), aperiodicity of a Fell bundle $\mathcal{B}$ gives a certain control over elements in $C_r^*(\mathcal{B})^+\backslash\{0\}$. Nevertheless, in general it may happen that $a \precsim b$ where a is properly infinite and yet b is finite. Such a situation can not occur, e.g., when a is a projection or when $C_r^*(\mathcal{B})$ simple. Therefore, in the following theorem we need assumptions implying that "the ratio of ideals in $C_r^*(\mathcal{B})$ to projections in B_e is small". One of the conditions that guarantees this is the ideal property, see page 15.

The following criterion of pure infiniteness is a generalization of the corresponding result for partially reversible endomorphisms, cf. [Kwa16, Proposition 2.46].

Theorem 6.6 (Theorem 4.10 in [KS17])**.** *Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is an exact, residually aperiodic Fell bundle such that either B_e has the ideal property or* $\mathrm{Ideal}^{\mathcal{B}}(B_e)$ *is finite. The following statements are equivalent:*

(i) Every non-zero positive element in B_e is properly infinite in $C_r^*(\mathcal{B})$.

(ii) $C_r^*(\mathcal{B})$ is purely infinite.

(iii) Every non-zero hereditary C^*-subalgebra in any quotient $C_r^*(\mathcal{B})$ contains an infinite projection.

If B_e is of real rank zero, then each of the above conditions is equivalent to

(i') Every non-zero projection in B_e is properly infinite in $C_r^*(\mathcal{B})$.

By virtue of Theorem 6.6, in order to get a pure infiniteness criterion for $C_r^*(\mathcal{B})$ it suffices to find conditions implying that elements in $B_e^+\backslash\{0\}$ are properly infinite in $C_r^*(\mathcal{B})$. The analogues of (properly and residually) infinite elements for Fell bundles are defined in [KS17, Definition 5.1].

Definition 6.7 ([KS17])**.** Let $\mathcal{B} = \{B_g\}_{g\in G}$ be a Fell bundle.

1. We call $a \in B_e^+\backslash\{0\}$ *$\mathcal{B}$-infinite* if there is $b \in B_e^+\backslash\{0\}$ such that for each $\varepsilon > 0$ there are $n, m \in \mathbb{N}$ and $t_i \in G$, $a_i \in aB_{t_i}$ for $i = 1, \dots, n+m$ such that

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \qquad b \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i \quad \text{oraz} \quad a_i^* a_j \approx_{\varepsilon/\max\{n^2,m^2\}} 0 \quad \text{for } i \neq j. \qquad (27)$$

   We call $a \in A^+\backslash\{0\}$ *residually $\mathcal{B}$-infinite* if $a + J_e$ is $\mathcal{B}/\mathcal{J}$-infinite for every ideal $\mathcal{J} = \{J_g\}_{g\in G}$ in $\mathcal{B}$ with $a \notin J_e$.

2. We call $a \in B_e^+\backslash\{0\}$ *$\mathcal{B}$-paradoxical* if for each $\varepsilon > 0$ there are $n, m \in \mathbb{N}$ and $a_i \in aB_{t_i}$, $t_i \in G$ for $i = 1, \dots, n+m$ such that

$$a \approx_\varepsilon \sum_{i=1}^{n} a_i^* a_i, \qquad a \approx_\varepsilon \sum_{i=n+1}^{n+m} a_i^* a_i \quad \text{and} \quad a_i^* a_j \approx_{\varepsilon/\max\{n^2,m^2\}} 0 \quad \text{for } i \neq j. \qquad (28)$$

We say that $a \in B_e^+\backslash\{0\}$ is *strictly $\mathcal{B}$-infinite* (resp. *strictly $\mathcal{B}$-paradoxical*) when condition (27) (resp. (28)) holds with $\varepsilon = 0$.

Remark 6.8. Let $B = \overline{\bigoplus_{g\in G} B_g}$ be an arbitrary graded C^*-algebra with gradation given by a Fell bundle $\mathcal{B} = \{B_g\}_{g\in G}$. One can show, see [KS17, Proposition 5.3], that if $a \in B_e^+\backslash\{0\}$ is $\mathcal{B}$-infinite, then it is infinite in B, and if a is $\mathcal{B}$-paradoxical, then a is properly infinite in B. By Remark 1.2, if additionally B_e separates the ideals in B, then every residually $\mathcal{B}$-infinite element is properly infinite in B. Hence both paradoxicality as well as residual infiniteness can be used to establish proper infiniteness of the corresponding elements. However paradoxicality is a stronger condition than residual $\mathcal{B}$-infiniteness:

$$a \text{ jest } \mathcal{B}\text{-paradoxical} \Longrightarrow \quad a \text{ is residually} \mathcal{B}\text{-infinite}, \tag{29}$$

and in general it is much easier to verify the latter condition.

Examples 6.9. Let $\alpha : G \to \mathrm{Aut}(A)$ be a group action on a commutative C^*-algebra $A = C_0(\Omega)$. Let $\mathcal{B}_\alpha := \{E_{\alpha_{g^{-1}}}\}_{g\in G}$ be the associated Fell bundle and let $\theta = \{\theta_g\}_{g\in G} : G \to \mathrm{Homeo}(\Omega)$ be the action dual to α. An element $a \in A^+\backslash\{0\}$ is strictly $\mathcal{B}_\alpha$-paradoxical if and only if the set

$$V = \{x \in \Omega : a(x) > 0\}$$

is *θ-paradoxical* [BT24], [RS12], that is there are open sets $V_1, ..., V_{n+m}$ and elements $t_1, ..., t_{n+m} \in G$, such that

$$V = \bigcup_{i=1}^{n} V_i = \bigcup_{i=n}^{n+m} V_i, \quad \theta_{t_i}(V_i) \subseteq V \quad \text{and} \quad \theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \varnothing \text{ for every } i \neq j.$$

The element a is strictly $\mathcal{B}_\alpha$-infinite if and only if the set $V = \{x \in \Omega : a(x) > 0\}$ is *θ-infinite* in the sense that there are open sets $V_1, ..., V_n$ and elements $t_1, ..., t_n \in G$, $n \geqslant 1$, such that

$$V = \bigcup_{i=1}^{n} V_i, \quad \overline{\bigcup_{i=1}^{n} \theta_{t_i}(V_i)} \subsetneq V \quad \text{and} \quad \theta_{t_i}(V_{t_i}) \cap \theta_{t_j}(V_{t_j}) = \varnothing \text{ for every } i \neq j.$$

In the forthcoming theorem the equivalence of positive elements in $C_r^*(\mathcal{B})$ should be understood in Cuntz sense: elements $a, b \in C_r^*(\mathcal{B})^+$ are *equivalent*, if $a \precsim b$ and $b \precsim a$ in $C_r^*(\mathcal{B})$.

Theorem 6.10 (Theorem 5.13 in [KS17])**.** *Suppose that $\mathcal{B} = \{B_g\}_{g\in G}$ is an exact, residually aperiodic Fell bundle and one of the following conditions holds:*

(i) B_e has the ideal property and every element in $B_e^+\backslash\{0\}$ is equivalent to a residually $\mathcal{B}$-infinite element;

(i') B_e is of real rank zero and every non-zero projection in B_e is Cuntz equivalent to a residually $\mathcal{B}$-infinite element;

(ii) $|\,\mathrm{Ideal}^{\mathcal{B}}(B_e)| < \infty$ and every element in $B_e^+\backslash\{0\}$ is equivalent to a residually $\mathcal{B}$-infinite element;

(ii') $\mathcal{B}$ is minimal, i.e. $|\operatorname{Ideal}^{\mathcal{B}}(B_e)| = 2$, and every element in $B_e^+\backslash\{0\}$ is equivalent to a $\mathcal{B}$-infinite element.

Then $C_r^*(\mathcal{B})$ has the ideal property and is purely infinite.

Remark 6.11. Rørdam and Sierakowski [RS12] considered paradoxical actions on totally disconnected locally compact Hausdorff spaces. For such actions condition (i') in Theorem 6.10 holds. In fact, in light of (29), Theorem 6.10 gives a stronger result than [RS12, Corollary 4.4]. Jolissaint and Robertson [JR00] introduced the notion of *n-filling actions* on a not necessarily commutative C^*-algebra A with unit. Any such action α is automatically minimal and every element in $A^+\backslash\{0\}$ is residually $\mathcal{B}_\alpha$-infinite, see [KS17, Lemma 5.12]. Hence [JR00, Theorem 1.2] and [LS96, Theorem 5] follow from Theorem 6.10, as condition (ii') is satisfied.

Remark 6.12. It is shown in [KS17, Theorem 7.9] that Theorem 6.10 applied to graph algebras is sharp. More specifically, a graph algebra $C^*(E)$ is purely infinite if and only if the corresponding Fell bundle $\mathcal{B}_E = \{B_n\}_{n\in\mathbb{Z}}$ is residually aperiodic and condition (i') in Theorem 6.10 holds.

6.2.1 Applications to semigroup crossed products

The aforementioned results were applied in [KS17, Section 8] to crossed products associated with semigroup systems (A, G^+, α, L), where G^+ is a positive cone in a totally ordered group G and for every $t \in G^+$, (A, α_t, L_t) is a regular Exel system with $\alpha_t(A)$ being a corner in A. As shown in [KS17, Proposition 8.3, 8.5] one can view such systems in a number of different but equivalent ways. They can be identified with: the semigroup of endomorphisms α, the semigroup of transfer operators, or an interaction group $\mathcal{V}$. Any such system gives rise to a Fell bundle over G and it is shown that the cross-sectional algebra of this bundle denoted by $A \rtimes_{\alpha,L} G^+$ is universal with respect to the corresponding representations of α, L, or $\mathcal{V}$, see [KS17, Theorem 8.10]. Hence the algebra $A \rtimes_{\alpha,L} G^+$ can be viewed as a crossed product associated with any of the aforementioned stuctures. In particular, $A \rtimes_{\alpha,L} G^+$ coincides with the semigroup version of Exel's crossed product studied by Larsen in [Lar10], see [KS17, Corollary 8.12]. When A is unital, $A \rtimes_{\alpha,L} G^+$ coincides with the crossed product studied in [KL09]. Theorems 6.4 and 6.10 can be translated in a natural way to the language of the system (A, G^+, α, L), see [KS17, Theorems 8.17, 8.22]. The so obtained criterion of pure infiniteness for crossed products by endomorphisms imply the main result of [OP14], see [KS17, Remark 8.23].

IV) Description of other results of scientific research

List of the remaining publications (in chronological order):

1. [KK01] A. K. Kwaśniewski, B. K. Kwaśniewski, *On q-difference equations and $\mathbb{Z}_n$-decompositions of* $\exp_q$ *function*, Adv. Appl. Clifford Algebras, **11** (2001), 39–61.

2. [KK02] A. K. Kwaśniewski, B. K. Kwaśniewski, *On trigonometric-like decompositions of functions with respect to the cyclic group of order n*, J. of Applied Analysis, **8** (2002), no. 1, 111–127.

3. [Kwa05] B. K. Kwaśniewski, *Covariance algebra of a partial dynamical system*, Cent. Eur. J. Math., **3** (2005), pp. 718–765.

4. [Kwa05'] B. K. Kwaśniewski, *Inverse limit systems associated with $\mathcal{F}_{2^n}$ zero schwarzian unimodal mappings*, Bulletin de la Societe des Sciences et des Lettres de Lodz, **55** (2005), 83–109.

5. [KL08] B. K. Kwaśniewski, A. V. Lebedev, *Reversible extensions of irreversible dynamical systems: C^*-method*, Mat. Sb. 199 (2008), no. 11, 45–74.

6. [Kwa09] B. K. Kwaśniewski, *Spectral analysis of operators generating irreversible dynamical systems* (in Polish), PhD thesis, IM PAN, 2009, Warsaw.

7. [KL09] B. K. Kwaśniewski, A. V. Lebedev, *Crossed Product of a C^*-algebra by a semigroup of endomorphisms generated by partial isometries* Integr. Equ. Oper. Theory 63 (2009), 403–425.

8. [Kwa12] B. K. Kwaśniewski, *On transfer operators for C^*-dynamical systems*, Rocky J. Math. **42**, No 3 (2012), 919–938.

9. [Kwa12'] B. K. Kwaśniewski, *C^*-algebras associated with reversible extensions of logistic maps* Mat. Sb. 203, No 10 (2012), 1448–1489.

10. [Kwa12"] B. K. Kwaśniewski, *Uniqueness property for C^*-algebras given by relations with circular symmetry* Geometric methods in physics, 303—310, Trends Math., Birkhäuser-Springer, Basel, 2013.

11. [Kwa14"] B. K. Kwaśniewski *Crossed product of a C^*-algebra by a semigroup of interactions* Demonstr. Math. **47** (2014), no. 2, 350—370.

12. [Kwa15'] B. K. Kwaśniewski *Extensions of C^*-dynamical systems to systems with complete transfer operators* Math. Notes **98** (2015), 419–428.

13. [KM] B. K. Kwaśniewski, R. Meyer, *Aperiodicity, topological freeness and more: from group actions to Fell bundles* submitted, after a positive review in Studia Math., arXiv:1611.06954.

14. [KL] B. K. Kwaśniewski, N. S. Larsen, *Nica-Toeplitz algebras associated with right tensor C^*-precategories over right LCM semigroups: Part I Uniqueness results*, submitted to J. Operator Theory, arXiv:1611.08525.

The first papers - special functions

Already as a student the habilitant co-authored the papers [KK01], [KK02] related to q-extended versions of special polynomials and hyperbolic functions. In [KK01] the authors introduce a family of α-projections $\{\Pi_k^\alpha\}_{k\in\mathbb{Z}_n}$ and study the corresponding decompositions of certain functions and formal series. For the exponential function such a decomposition gives α-hyperbolic functions of higher order $\{h_k^\alpha\}_{k\in\mathbb{Z}_n}$ and generalized de Moivre's formulas. Analysis of functions $\{h_{q,k}^\alpha\}_{k\in\mathbb{Z}_n}$ forming the decomposition of the

q-extension $\exp_q$ of the exponential function leads to a generalization of a number of formulas with combinatorial meaning. The paper [KK02] is focused on a detailed analysis of the functions $\{h^{\alpha}_{q,k}\}_{k\in\mathbb{Z}_n}$ in the case when $\alpha = 1$, also from the point of view of q-difference equations.

PhD thesis - spectral analysis of functional operators

In the PhD thesis [Kwa09] the habilitant studied certain classes of functional operators, including *weighted composition operators*, i.e. operators of the form

$$aTf(x) = a(x)f(\varphi(x)), \qquad f \in F(X), \tag{30}$$

where $F(X)$ is a certain space of functions, $\varphi : X \to X$ is a map and $a(x)$ is a number valued function. For instance, if $F(X) = L^p(X)$ where X is a compact Hausdorff space with a probability measure, $a \in A := C(X)$ and φ is a measure preserving measure, then T is an invertible isometry and $TAT^{-1} = A$, that is the map $a \longmapsto TaT^{-1}$ is an automorphism of A. The spectrum of operators satisfying the latter relations is described in [AL94]. Continuing this line of research in [Kwa09] the habilitant elaborated a theory of *abstract weighted shift operators*. These are operators of the form aT acting on a complex Banach space E such that:

1) $T \in \mathcal{B}(E)$ is a *partial isometry* in the sense of [Mbe04], i.e. T is a contraction for there exists a contraction S such that $TST = T$ and $STS = S$;[5]

2) a is an element of a commutative Banach algebra $A \subseteq \mathcal{B}(E)$ containing $1 \in \mathcal{B}(E)$;

3) We have $TAS \subseteq A$ and $ST \in A'$.

The above axioms imply that the map $a \overset{\alpha}{\longmapsto} TaS$ is an endomorphism of the Banacha algebra A. The endomorphism α induces the dual partial mapping φ on the Gelfand spectrum X of A. The objective of [Kwa09] is description of spectral properties aT, $a \in A$, in terms of ergodic properties *irreversible dynamical system* (X, φ). It is shown that if A is a functional algebra, then the spectral radius $r(aT)$ is given by a variational principle - maximum over geometric means of $|a|$ with respect to ergodic measures for (X, φ). Assuming that $A \cong C(X)$ the set $|\sigma(aT)| = \{|\lambda| : \lambda \in \sigma(aT)\}$ is described in terms of variational principles on φ-invariant closed subsets of X. If in addition φ is topologically free and E is a Hilbert space or aT acts on L^p spaces, then the spectrum $\sigma(aT)$ has a circular symmetry. This gives a complete description of the spectrum $\sigma(aT)$ in terms of (X, φ). Moreover, if X has no isolated points then $\sigma(aT)$ coincides with the essential spectrum $\sigma_{ess}(aT)$. Amongst the concrete examples discussed in detail are: operators generating the family of logistic maps, Markov shifts, expansive endomorphisms and homeomorphisms of the circle. These results have not yet been published in the form of articles.

An important new tool in [Kwa09] is a construction of a reversible extension $(\widetilde{X}, \widetilde{\varphi})$ of the initial irreversible dynamical system (X, φ). The extended system $(\widetilde{X}, \widetilde{\varphi})$ was

[5] If E is a Hilbert space, then automatically $S = T^*$, in general S is not uniquely determined by T

obtain by describing the Gelfand spectrum of the commutative Banacha algebra $B := \overline{\text{span}}\{S^n a T^n : a \in A\}$ which contains A and satisfies the relations

$$TBS \subseteq B, \qquad SBT \subseteq B.$$

A number of concrete examples of (X, φ) was analyzed in [Kwa09]. These systems include important dynamic objects such as hyperbolic attractors, irreducible continua and spaces associated with tilings. When A is a C^*-algebra acting on a Hilbert space these results were described in [Kwa12']. This paper mainly focuses on examples coming from the family of logistic maps and homeomorphisms of the circle. In the case when φ is a unimodal map with zero Schwarzian and a finite number of fixed points the system $(\widetilde{X}, \widetilde{\varphi})$ is described in [Kwa05'].

Results generalized in the reported scientific achievement

The main result of [KL08] is a description of the spectrum $\widetilde{X}$ of the C^*-algebra $B = \overline{\text{span}}\{U^{*n} a U^n : a \in A\}$ where $A \cong C(X)$ is a commutative C^*-algebra acting on a Hilbert space H and U is a partial isometry such that $UAU^* \subseteq A$, $U^*U \in Z(A) = A \cap A'$. Requirement that $U^*U \in A$, implies that the kernel of the endomorphism $\alpha(a) = UaU^*$ is complemented in A. Using this description of $(\widetilde{X}, \widetilde{\varphi})$, in [Kwa05] the habilitant defined a C^*-algebra $C^*(X, \varphi)$ as a partial crossed product [Exel94] associated to the partial homeomorphism $(\widetilde{X}, \widetilde{\varphi})$. One of the main results of [Kwa05] is the isomorphism theorem, which was used in [Kwa09]. The algebra $C^*(X, \varphi)$ is the particular case of the crossed product in the sense of Definition 2.5, and Theorem 2.21 is a far reaching generalisation of the main result of [Kwa05].

The universal reversible J-extension (B, β) of an endomorphism $\alpha : A \to A$, discussed on page 12, is a general noncommutative counterpart of the system $(\widetilde{X}, \widetilde{\varphi})$. When A is unital the system (B, β) described in [Kwa15']. This description led to the definition of the crossed product in [KL13].

The authors of [ABL11] studied the crossed product $A \rtimes_{\alpha,L} \mathbb{N}^+$ where A is a unital C^*-algebra and L is a complete transfer operator for α. The semigroup generalization $A \rtimes_{\alpha,L} G^+$ of this crossed product was introduced in [KL09]. One of the main results of [KL09] is an explicit construction of the Banach $*$-algebra $\ell^1(G, \alpha, A)$ whose enveloping C^*-algebra is $A \rtimes_{\alpha,L} G^+$, and the construction of a regular representation. In [KS17], the construction of $A \rtimes_{\alpha,L} G^+$ was generalized to non-unital case. It was also noted that the structure of the $*$-algebra $\ell^1(G, \alpha, A)$ in fact comes from a Fell bundle, cf. page 44.

The article [Kwa12"] discusses a general scheme for uniqueness theorems of the C^*-algebra defined in terms of generators and relations. Using symmetries in the relations one can reduce such a theorem to the uniqueness theorem for cross products by Hilbert bimodules, proved in [Kwa14], see Theorem 5.4. It is shown that the Cuntz-Krieger uniqueness theorem [CK80] for Cuntz-algebras $\mathcal{O}_A$ is equivalent to the corresponding uniqueness theorem for graph algebras. This result was generalized to graph algebras in [Kwa14'], cf. page 24.

The remaining scientific achievements

The remaining results are also related with the reported scientific achievement.

Structure of transfer operators: General structure of regular transfer operators on unital C^*-algebras is described in [Kwa12]. It is shown that an endomorphism admits a complete transfer oprator if and only if $\ker\alpha$ is complemented in A and $\alpha(A)$ hereditary subalgebra of A, cf. Definition 2.8. In the case A is commutative, all transfer operators are described, and necessary and sufficient conditions for existence a non-zero or a regular transfer operator are given. For any endomorphism $\alpha : B(H) \to B(H)$ all normal transfer operators are described and also the structure of singular transfer operators is discussed (an endomorphism α admits singular transfer operators if and only if its Powers index is ∞).

Semigroup crossed products by interactions: In [Kwa14”], the author considered crossed products $A \rtimes_{\mathcal{V},\mathcal{H}} G^+$, which generalize crossed products $A \rtimes_{\alpha,L} G^+$ [KL09] to the case where the pairs $(\mathcal{V}_t, \mathcal{H}_t)$, $t \in G^+$, are corner interactions, that is when non of the maps $\mathcal{V}$, $\mathcal{H}$ is necessarily multiplicative. One of the results is finding necessary and sufficient conditions for a representation of $(\mathcal{V}, \mathcal{H})$ so that it integrates to a faithful representation of $A \rtimes_{\mathcal{V},\mathcal{H}} G^+$. Also a notion of topological freeness for the pair $(\mathcal{V}, \mathcal{H})$ was introduced and a uniqueness theorem for $A \rtimes_{\alpha,L} G^+$ was proved.

Non-triviality conditions for Fell bundles: In [KM] the authors collect and improve a number of strong classical results proved by Kishimoto, Olesen, Pedersen and Rieffel for discrete group actions on C^*-algebras. They concern conditions that are sufficient and sometimes necessary for the algebra A to detect or separate the ideals in the reduced crossed product $A \rtimes_{\alpha,r} G$. The main achievement of [KM] is generalization of these theorems to Fell bundles and Hilbert bimodules. In particular, a notion of Connes spectrum for Fell bundles is introduced and a number of relationships between such conditions as: aperiodicity, topological freeness, pure infiniteness and Connes spectrum are established. See Remark 6.2 for a description of some of these results. The Connes spectrum was used to obtain characterization of aperiodicity, intersection property and the residual interesection property of Fell bundles. When the underlying group is $\mathbb{Z}$ or $\mathbb{Z}_p$ for a square free number p, it is show that the cross sectional algebra is simple if and only if the Fell bundle is minimal and outer. Apart from that effective conditions for strong pure infiniteness of the C^*-algebra $C^*_r(\mathcal{B})$ are given.

Nica-Toeplitz algebras for right tensor C^*-precategories: In [KL], the authors initiate a study of C^*-algebras associated with right tensor C^*-categories $\mathcal{L}$ over general semigroups P (for $P = \mathbb{N}$ see Definition 4.2). For any ideal $\mathcal{K}$ in $\mathcal{L}$ a Fock representation is constructed and the Toeplitz alebra is defined. Under the assumption that P is a *right LCM semigroup* (i.e. every two elements in P which have a common (right) mutliple have a least common multiple) new relations coming from Fock representation were identified. This leads to a definition of the *Nica-Toeplitz algebra* $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and the *reduced Nica-Toeplitz algebra* $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$. The authors of [KL] give necessary and sufficient conditions for $\mathcal{NT}_{\mathcal{L}}(\mathcal{K})$ and $\mathcal{NT}^r_{\mathcal{L}}(\mathcal{K})$ to coincide. The main result of [KL] is a uniqueness theorem in the spirit of the classical Coburn theorem. It generalizes all known to the authors results of this type. In particular, this clarifies relationships between uniqueness theorems for C^*-algebras associated to single C^*-correspondences, positive cones in quasi-lattice ordered groups and semigroup crossed products twisted by product system obtained previously by Fowler, Laca and Raeburn.

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