Topology for
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Definition 1. Let X be a nonempty set. The function $\rho : X \times X \to \mathbb{R}$ is a *metric* on X, if the following conditions hold

(M1) $\forall_{x,y \in X} \quad \rho(x, y) = 0 \Longleftrightarrow x = y,$ (*nondegeneracy*)

(M2) $\forall_{x,y \in X} \quad \rho(x, y) = \rho(y, x)$ (*symmetry*)

(M3) $\forall_{x,y,z \in X} \quad \rho(x, y) \leq \rho(x, z) + \rho(z, y)$ (*triangle inequality*).

The pair (X, ρ) is called then a *metric space*.

Definition 2. If (X, ρ) is a metric space, $a \in X$ and $r > 0$, then the set

$$K(a, r) = \{x \in X : \rho(x, a) < r\}$$

is called an *open ball* with center in a and radius r.

Ex 1. Show that the following functions are metrics in plane $\mathbb{R}^2$:

$$\rho_1(p, q) = \sqrt{(p_1 - q_1)^2 + (p_2 - q_2)^2} \qquad (\textit{euclidean metric}),$$

$$\rho_2(p, q) = |p_1 - q_1| + |p_2 - q_2| \qquad (\textit{city, or taxi, metric}),$$

$$\rho_3(p, q) = \max\{|p_1 - q_1|, |p_2 - q_2|\} \qquad (\textit{maximum metric}),$$

$$\rho_4(p, q) = \max\{|p_1 - q_1|, 2|p_2 - q_2|\} \qquad (\textit{mutated maximum metric}),$$

where $p = (p_1, p_2)$, $q = (q_1, q_2)$. Draw open balls in the spaces $(\mathbb{R}^2, \rho_i)$, $i = 1, 2, 3, 4$.

Ex 2. Check whether functions f and g given by $f(p, q) = |p_1 - q_1|$ and $g(p, q) = (p_1 - q_1)^2 + (p_2 - q_2)^2$ are metrics on $\mathbb{R}^2$.

Ex 3. Prove that an arbitrary nonempty set X together with the function ρ_5 given by

$$\rho_5(x, y) = \begin{cases} 0, & \text{gdy} \quad x = y, \\ 1, & \text{gdy} \quad x \neq y, \end{cases}$$

form a metirc space (this metric space is called *discrete*). Describe the open balls in that space.

Definition 3. Let X be a nonempty set and let τ be a family of subsets of X ($\tau \subset 2^X$) satysfying the following conditions:

(O1) $\emptyset \in \tau$ i $X \in \tau$,

(O2) if $U_1, U_2 \in \tau$, then $U_1 \cap U_2 \in \tau$,

(O3) if $\{U_i\}_{i \in I} \subset \tau$, then $\bigcup_{i \in I} U_i \in \tau$, ($I$ is an arbitrary set).

Then we say that τ is a *topology* on X, the pair (X, τ) is a *topological space*, and members of τ are called *open sets* (that is if a subset $A \subset X$ belongs τ we say it is open).

Ex 4. Let (X, ρ) be a metric space. Prove that the family

$$\tau = \{A \subset X : \forall_{x \in A} \, \exists_{r > 0} \, K(x, r) \subset A\}$$

is a topology on X. This topology is called a *topology of the metric space* (X, ρ).

Definition 4. Let (X, ρ) be a metric space, and let τ be the topology defined in Ex. 4. This topology is called a *topology of the metric space* (X, ρ), or also a *topology induced by the metric* ρ.

Definition 5. We say that two metrics on a set X are *equivalent* if and only if they induce the same topology on X.

Ex 5. Find out which of the given subsets of the euclidean plane $(\mathbb{R}^2, \rho_1)$ are open:

a) a strip $\{p = (p_1, p_2) : a < p_1 < b\}$; $a, b \in \mathbb{R}$,

b) a singleton $\{p\}$; $p \in \mathbb{R}^2$,

c) an open interval $\{tp + (t-1)q : t \in (0,1)\}$; $p, q \in \mathbb{R}^2$,

d) the set of points with with rational coordinates $\{p = (p_1, p_2) : p_1, p_2 \in \mathbb{Q}\}$,

d) interior of a circle $\{p = (p_1, p_2) : \sqrt{(p_1 - a_1)^2 + (p_2 - a_2)^2} < r\}$; $(a_1, a_2) \in \mathbb{R}^2$, $r > 0$.

(recall that metric induces a topology, and open sets are the ones belonging to topology)

Ex 6. Describe the topology in the discrete space (cf. Ex.3).

Ex 7. Show that in any metric space the open ball is open.

Ex 8. Which of the metrics ρ_i, $i = 1, ..., 5$ (defined in Ex.1 and Ex.3) on the plane $\mathbb{R}^2$ are equivalent.

Definition 6. Let (X, τ) be a topological space. We say that a subset $A \subset X$ is *closed* if and only if its complement $X \setminus A$ is open.

Ex 9. Find out which of the given subsets of the euclidean plane $(\mathbb{R}^2, \rho_1)$ are closed:

a) $\{p = (p_1, p_2) : a \leq p_1 \leq b\}$,

b) $\{p = (p_1, p_2) : a \cdot p_1 + b \cdot p_2 \leq c\}$,

c) $\{p = (p_1, p_2) : a \cdot p_1 + b \cdot p_2 = c\}$.

Ex 10. Describe the closed sets in the discrete spaces.

Ex 11. Let (X, τ) be a topological space. Show that for any $\mathcal{A} \subset X$ there exists a maximal open subset of A and a minimal closed set containing A. These are called the *interior* and the *closure* of A respectively.

Definition 7. Let X be a nonempty set. A function $\mathrm{Cl} : 2^X \to 2^X$ satysfing the conditions

(CO1) $\mathrm{Cl}\,\emptyset = \emptyset$,

(CO2) $A \subset \mathrm{Cl}\,A$,

(CO3) $\mathrm{Cl}\,\mathrm{Cl}\,A = \mathrm{Cl}\,A$,

(CO4) $\mathrm{Cl}\,(A \cup B) = \mathrm{Cl}\,A \cup \mathrm{Cl}\,B$.

is called a *closure operation* (2^X denotes the set of all subsets of X).