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Groupoid Banach algebras, weighted composition operators on L^p-spaces and transfer operators

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Abstract

We present foundations of the theory of Banach algebras associated with étale Hausdorff groupoids, which was elaborated in a more general twisted non-Hausdorff setting in [**BKM25**], [**BKM26**]. Considering the special more regular case allows us to simplify and streamline some of the constructions and arguments. Amongst the main results are description of representations and simplicity criteria for such algebras.

In addition, we introduce and study L^p-operator algebras, for $p \in [1, \infty]$, associated to local homeomorphisms through Renault-Deaconu groupoids. We apply this to algebras generated by representations of transfer operators on L^p-spaces and obtain a description of spectra of weighted composition operators generating the corresponding universal L^p-operator algebras. This provides a general framework previously studied by the author for Bernoulli shifts in [**Bar24**], and gives an L^p-generalization of the theory developed for Hilbert spaces in [**BK21**], as well as of certain C^*-algebraic results from [**BKL24**].

Furthermore, we provide a full self-contained proof of Banach-Lamperti theorem characterizing (not necessarily invertible) isometries between L^p-spaces associated to localizable measures as (generalized) weighted composition operators. The formulation of this theorem for non-invertible isometries has caused some confusion and is often incorrectly formulated already for σ-finite measures.

Contents

Introduction

Group C^*-algebras emerged from 1930s-1940s work of Murray-von Neumann, through the work of Gelfand and Naimark [**GN43**] who provided the axiomatic foundation of the theory of C^*-algebras. From then until the present day, the reduced group C^*-algebra $C^*_{\mathrm{r}}(G)$ of a group G, which is a completion of a group $*$-algebra $C_c(G)$ in the norm given by the regular representation, is both an important object of study and an indispensable tool used to study properties of groups in harmonic analysis and geometric group theory. In fact other Banach algebra completions of $C_c(G)$, such as the Banach algebra $L^1(G)$, already studied in detail by Segal [**Seg47**], or Herz algebras of p-pseudofunctions [**Her71**], which are now also called *reduced L^p-operator group algebras* [**GT15**], are standard tools in modern harmonic analysis and related fields.

In the realm of C^*-algebras the theory of group algebras was extended with huge success to transformation groups, groupoids and even more general structures and actions. One of the many crucial results showing importance of such constructions is that all classifiable simple C^*-algebras are reduced C^*-algebras of (twisted) étale locally compact Hausdorff groupoids [**Li20**]. Perhaps one of the most important class of étale groupoids, apart from transformation groupoids modelling group actions, are the so-called *Renault-Deaconu groupoids*, which are associated to irreversible dynamical systems implemented by *local homeomorphisms*. Renault [**Ren80**, III.2] associated such a groupoid to the topological Bernoulli shift on $\{1, \dots, n\}^N$ to model Cuntz algebra $\mathcal{O}_n$. Deaconu [**Dea95**] considered general covering maps on compact spaces and called the associated groupoid C^*-algebras crossed products by endomorphisms. Nowadays C^*-algebras associated to irreversible maps are often recognized as Exel's crossed products or crossed products by *transfer operators*, [**Exe03$_1$**], [**EV06**], [**Kwa17**], [**BKL24**]. The crucial role of transfer operators in these constructions provides a direct link to thermodynamical formalism and ergodic theory. It should also be noted that Arzumanian and Vershik [**AV78**] were perhaps first to study an operator algebra associated to an irreversible dynamical system, which they defined concretely as the C^*-algebra generated by some *weighted composition operators* on the Hilbert space $L^2(\mu)$. The fact that such C^*-algebras are modeled by Renault-Deaconu groupoids was explored by the author and his supervisor in [**BK21**] where the combination of C^*-algebraic analysis and thermodynamical formalism arguments lead to a *dynamical description of spectra* of the corresponding weighted composition operators.

In the 2010s, through a series of lectures and preprints, cf. [**Phi12**], [**Phi13a**], [**Phi13b**], Phillips initiated a program of generalizing important C^*-algebraic constructions and results to Banach algebras, which he called *L^p-operator algebras*. By definition these are

Banach algebras that can be isometrically represented on L^p-spaces $L^p(\mu)$ for an arbitrary measure μ, and fixed $p \in [1, \infty)$. For $p = 2$ these are operator algebras that include C^*-algebras. Phillips's program has noticeably grown in recent years, see the survey paper [**Gar21**] and [**GT15**], [**GL17**], [**BP19**] [**CoR19**], [**GT20**], [**GT22**], [**AO22**], [**HO23**], [**CGT24**], [**CMR25**]. For $p \neq 2$ the proofs usually require different techniques, and one of the most important tools used in this case is Banach-Lamperti theorem which states that (invertible) isometries on L^p-spaces are weighted compositions operators [**Ban32**], [**Lam58**], [**FJ03**]. This can sometimes serve as a replacement of the involution and the C^*-equality available only for $p = 2$. On the other hand, for $p \neq 2$, the more rigid geometry of L^p-spaces allows for stronger rigidity results for L^p-operator algebras than those for C^*-algebras, see [**GT22**], [**HO23**], [**CGT24**].

In this thesis we discuss fundamental results concerning representation theory and ideal structure of complex Banach algebras associated to a locally compact Hausdorff groupoid $\mathcal{G}$. Thus in general we are interested in Banach algebra completions of the convolution algebra $C_c(\mathcal{G})$. The main motivation and applications come from groupoid L^p-operator algebras (and their symmetrised versions) [**GL17**], [**AO22**], [**HO23**], [**CGT24**], but there is a merit in considering more general Banach algebras completions. For instance, the notion of Banach algebra crossed product for group actions seems to be well established and enjoys considerable interest [**DDW11**], [**JST12**], [**JT16**], [**BK24**], but this is not an L^p-operator algebra. Thus it is desirable to have a theory groupoid Banach algebras that cover Banach algebra crossed products of discrete transformation groups. In fact algebras of étale groupoids can be viewed as crossed products of inverse semigroup actions and this viewpoint plays a crucial role in this thesis. As in [**BK24**], to cover both L^p-operator algebras and their symmetrised versions we will often consider groupoid L^P-operator algebras where $P \subseteq [1, \infty]$ is a non-empty set of parameters. The main two fundamental results that we present and discuss in detail are:

- Disintegration-integration theorem for representations;
- Dynamical criteria for the intersection property and simplicity.

These two results are in essence the main goals of the papers [**BKM25**], [**BKM26**] where a more general context is considered. Namely, the groupoids there are not necessarily Hausdorff, they are equipped with twists and the scalar field is either $\mathbb{R}$ or $\mathbb{C}$. In the present dissertation we decided to discuss the more regular case, where the groupoids are Hausdorff, there are no twists, and the spaces are complex. There are several reasons for this. Firstly, the author contributed to [**BKM25**], [**BKM26**] mainly in this case. Secondly, in this setup some technicalities can be avoided and theory can be presented in a slightly different, more accessible way, which may be useful for a number of readers. Thirdly, for many applications and in particular for that we present here Hausdorff groupoids are enough.

Our disintegration-integration theorem (Section 5.3) gives a bijective correspondence between representations of a groupoid Banach algebra and covariant representations of the corresponding inverse semigroup action. Applying this to representations on L^p-spaces we establish a clean hierarchy between regular and full L^p-operator algebras for different

$p \in [1,\infty]$ (Theorem 5.33). This is summarized in Figure 1, where $F^p(\mathcal{G})$ and $F^p_{\mathrm{r}}(\mathcal{G})$ are the full and reduced L^p-operator algebra of $\mathcal{G}$, $F_I(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_I}$ is the completion in Hahn's I-norm, $C^*(\mathcal{G})$ and $C^*_{\mathrm{r}}(\mathcal{G})$ are standard C^*-algebras associated to $\mathcal{G}$. The downward arrows are contractive homomorphisms extending identities on $C_c(\mathcal{G})$. The algebras in the middle column are Banach $*$-algebras, the horizontal anti-isomorphisms are given by the involution in $C_c(\mathcal{G})$, $1/p + 1/q = 1$. For $p = 1,\infty$ the full and reduced groupoid algebras always coincide.

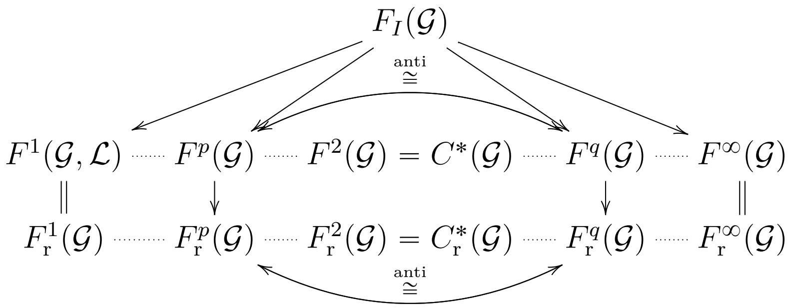

Figure 1. Hierarchy of groupoid Banach algebras

As another application of disintegration theorem, we give a geometric description of representations of $F^p(\mathcal{G})$ on spaces $L^p(\mu)$, where μ is a localizable measure, via the so-called spatial partial isometries. These are special kind of weighted composition operators on $L^p(\mu)$. This in particular gives a natural bijective correspondence between non-degenerate representations of $F^p(\mathcal{G})$ and $F^{p'}(\mathcal{G})$ for any $p, p' \in (1,\infty)$ (Theorem 5.38), which is quite surprising as in many cases there is no non-zero continuous homomorphism from $F^p(\mathcal{G})$ to $F^{p'}(\mathcal{G})$. Also this shows that in the constructions of L^p-analogues of Cuntz or graph algebras, [**Phi12**], [**CoR19**], [**CMR25**], the use of spatial partial isometries is not an assumption, in fact it is forced by the relations, or more precisely by the groupoid model (see Section 6.4). We emphasize that the disintegration theorem developed here is different than the one in [**GL17**], which gives measure theoretical disintegration of representations of $F^p(\mathcal{G})$, for $p \in (1,\infty)$, on L^p-spaces associated with standard Borel measures.

To study the ideal structure of the considered algebras we introduce a general notion of a *reduced groupoid Banach algebra*. This is a Banach algebra completion $F_{\mathcal{R}}(\mathcal{G})$ of $C_c(\mathcal{G})$, which contains as a Banach subalgebra the algebra $C_0(X)$ of continuous and vanishing at infinity functions on the unit space X of $\mathcal{G}$. The property of "being reduced" can be phrased equivalently in terms of faithfulness of a canonical *conditional expectation* or in terms of existence of the so-called *j-map*, which is an injective contractive map from a completion of $C_c(\mathcal{G})$ to $C_0(\mathcal{G})$. The j-map allows us to treat abstract elements in the Banach algebras as functions on the groupoid $\mathcal{G}$. A key condition in the study of the ideal structure of C^*-algebras associated to discrete group actions and their generalizations is *topological freeness*, see [**ZM68**], [**KT90**], [**Tom92**], [**AS93**], [**KM21**], [**KM22**]. In the context of Hausdorff étale groupoids topological freeness is also known under the name *effectiveness* [**Ren08**], [**KM21**], [**KM22**]. For any reduced groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$, we show (see Section

6.2) that $\mathcal{G}$ is topologically free if and only if $C_0(X)$ is maximal abelian subalgebra in $F_{\mathcal{R}}(\mathcal{G})$, and if this holds, then every non-zero (closed two-sided) ideal in $F_{\mathcal{R}}(\mathcal{G})$ has a non-zero intersection with $C_0(X)$. The latter property is called *the intersection property* or *detection of ideals*. Simplicity of $F_{\mathcal{R}}(\mathcal{G})$ is equivalent to the intersection property and minimality of $\mathcal{G}$. Hence we conclude that in the presence of topological freeness minimality of $\mathcal{G}$ is equivalent to simplicity of $F_{\mathcal{R}}(\mathcal{G})$. This generalizes the corresponding results for Banach algebra crossed product from [**BK24**]. This also applies to the reduced L^p-operator algebras $F^p_r(\mathcal{G})$ and so solves in positive the problem posed in [**GL17**, Problem 8.2]. Moreover, when the full $F^p(\mathcal{G})$ and reduced $F^p_r(\mathcal{G})$ algebras coincide, which is automatic when $p \in \{1, \infty\}$ and holds for all $p \in [1, \infty]$ when $\mathcal{G}$ is amenable, then $F^p(\mathcal{G})$ is simple if and only if $\mathcal{G}$ is topologically free and minimal.

We illustrate the above results on L^p-operator algebras associated to Renault-Deaconu groupoids (Section 6.3). Outside of the C^*-setting such algebras so far have been considered only in the context of graph algebras, see [**CoR19**], [**CMR25**], [**BKM26**]. In addition, in this thesis we discuss the following two new issues, which perhaps deserve to be published in the form of two further articles:

- We characterize weighted composition operators and present the full proof and the correct formulation of general Lamperti's theorem for L^p-spaces associated with localizable measures;
- We introduce L^p-operator algebras associated to transfer operators and describe the spectra of weighted composition operators that generate these algebras.

Up to an isometric isomorphism, every L^p-space is of the form $L^p(\mu)$ for a localizable measure μ and for such measures existence of Radon-Nikodym derivative can be characterized. This allows to formulate the Banach-Lamperti characterization of invertible isometries on arbitrary L^p-spaces with $p \neq 2$. This was proved recently for $p \in (1, \infty)\backslash\{2\}$ in [**GT22**] where a "point-free" approach to L^p-spaces is used. However, the characterization of non-invertible isometries between L^p-spaces already for σ-finite measures causes considerable confusion in the literature. We give (in Chapter 2) a right formulation and self-contained, detailed proof of this theorem, which seems not to appear in the literature in this level of generality. In addition, we characterize in general bounded weighted composition operators on L^p-spaces, where the composition operator is given by a set map, rather than a point map. This part of the thesis is of independent interest and can be read independently of the remaining material.

Finally in the last chapter (Chapter 7) we generalize a main part of [**BK21**] from Hilbert spaces to L^p-spaces. We apply the theory developed in this thesis to model L^p-operator algebras associated with transfer operators, which can be viewed as L^p-analogues of Exel's crossed products [**Exe03**$_1$], [**EV06**]. Also combining this with some operator theory techniques and formulas from thermodynamical formalism we obtain description of weighted composition operators that generate these algebras. This gives a general theory covering the particular example worked out in [**Bar24**]. The spectral theory in L^p-operator algebras is much more subtle than the one in C^*-algebras. However, for instance when the underlying local homeomorphism generates a simple graph algebra, the spectrum is independent of the representation (see Theorem 7.54 and Conjecture 7.51).

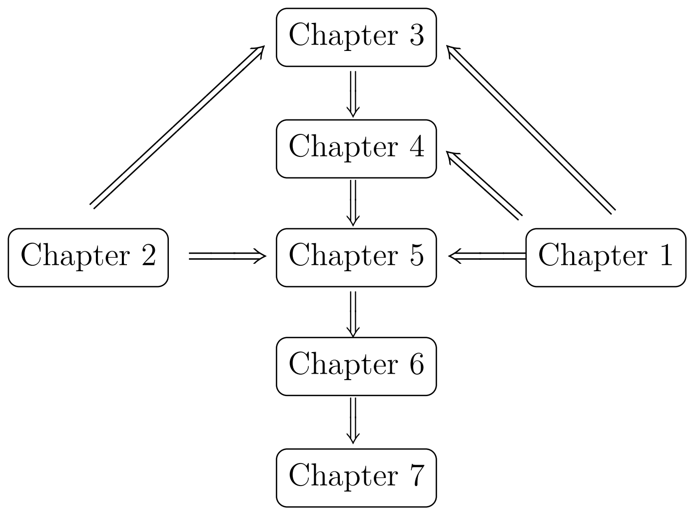

FIGURE 2. Dependence of chapters

Structure of the dissertation: Schematic relationships between chapters are presented in Figure 2. Chapter 1 is preliminary and contains the well known to experts definitions, constructions and results concerning étale groupoids and inverse semigroups. In particular, we recall the correspondence between étale groupoids and inverse semigroup actions, which is given by a construction of the transformation groupoid.

The main result of Chapter 2 is a general characterization of isometries between L^p-spaces associated to localizable measures, as (generalized) weighted composition operators, see Theorem 2.44, Theorem 2.46, and Corollary 2.51. Such results are usually called Lamperti or Banach-Lamperti theorems and are well known to experts in special cases — for (σ-)finite measures [**Lam58**], [**FJ03**] or for invertible isometries and $p \in (1,\infty)\backslash\{2\}$ [**GT22**]. For the sake of completeness, we give a full and detailed account on localizable measures, Radon-Nikodym derivatives and weighted composition operators associated to set morphisms.

In Chapter 3 we use Banach-Lamperti theorems from Chapter 2 to characterize representations of the algebra $C_0(X)$ on L^p-spaces (Theorem 3.11) and identify Moore-Penrose partial isometries with spatial partial isometries for $p \in [1,\infty]\backslash\{2\}$ (Theorem 3.26). This chapter is based on [**BKM25**, Section 2].

In Chapter 4 we give a self-contained and simplified presentation of the theory of covariant representations for inverse semigroup actions on Banach algebras, introduced in [**BKM25**]. The main result is a general universal description of the associated Banach algebra crossed product, that allows disintegration of all representations of the crossed product (Theorem 4.31). Such a disintegration was studied in [**BKM25**] only for actions on spaces, using groupoid models. However, considering crossed products of inverse semigroup actions on noncommutative Banach spaces might be of interest. We report on these constructions and results in [**BK26**].

Chapter 5 introduces inverse semigroup graded groupoid Banach algebras, which are shown to be isomorphic to inverse semigroup crossed products. This leads to Disintegration-Integration Theorem for representation of such algebras (Theorem 5.21). This theorem is used to establish relationship between L^p-groupoid algebras summarized in Figure 1 and characterization of their representations in terms of weighted shift operators (Theorems 5.33, 5.38). These are the main results of [**BKM25**] (in the untwisted Hausdorff case).

Chapter 6 discusses reduced groupoid Banach algebras, and relationship between intersection property for such algebras and topological freeness of the groupoid. Here the main results are Theorems 6.31, 6.33. They imply efficient simplicity criteria. These results form a part of [**BKM26**]. As a new application we obtain characterization of the intersection property and simplicity for L^p-operator algebras associated to Renault-Deaconu groupoids (Theorems 6.41, 6.46), which generalizes the corresponding results for C^*-algebras, see [**BKL24**].

In Chapter 7 we introduce L^p-operator algebra crossed product for transfer operators of local homeomorphisms and identify them with groupoid algebras (Theorem 7.20). Using this groupoid model, analysis of Riesz projections, direct computations and formulas for the spectral radius for transfer operators, we obtain a dynamical description of the universal weighted composition operators generating the crossed product (Theorem 7.48). This generalizes the corresponding result for C^*-algebras [**BK21**] and gives efficient estimates for the spectrum of operators satisfying certain relations involving the transfer operator (cf. Conjecture 7.51 and Theorem 7.54).

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CHAPTER 1

Étale groupoids and inverse semigroup actions

An action of a discrete group G on a topological space X is simply a group homomorphism $G \to \mathrm{Homeo}(X)$ from G to the group of homeomorphisms of X. This can be viewed as a dynamical system with discrete time and evolution happening via global symmetries. There are two natural ways of generalizing this to evolution via local symmetries of X. One may consider either inverse semigroup actions by partial homeomorphisms of X or étale groupoids with unit space X. It is well known that these two generalizations are in essence equivalent, see [**Pat99**], [**Exe08**]. In this preliminary chapter we recall this correspondence, for convenience of the reader and to settle notation. This will be crucial in the sequel.

1.1. Inverse semigroups

Let S be a semigroup (i.e. a set with an associative multiplication). An element $t \in S$ is called *partially invertible* if there is an element $t^* \in S$ such that

$$t = tt^*t \quad \text{and} \quad t^* = t^*tt^*.$$

Then t^*t and tt^* are idempotents, and we call t^* a *generalized inverse* for t.

Definition 1.1. A semigroup S is an *inverse semigroup* if every element in S has a unique generalized inverse.

Remark 1.2. A semigroup S is called *regular* if every element in S has (a not necessarily unique) generalized inverse in S. By [**Law98**, Theorem 1.1.3] a regular semigroup S is an inverse semigroup if and only if the semigroup of idempotents

$$E(S) := \{e \in S : e^2 = e\}$$

is commutative. Hence uniqueness of a generalized inverse is strictly tied to commutativity of the corresponding range and source idempotents.

Let S be an inverse semigroup. Then the map $t \mapsto t^*$ is an anti-multiplicative involution, and any semigroup homomorphism between two inverse semigroups is automatically $*$-preserving. Idempotents in S do not only commute but we also have the following equalities

$$E(S) = \{e \in S : e^2 = e\} = \{tt^* : t \in S\} = \{t^*t : t \in S\} \tag{1.1}$$

see for instance [**Pat99**, Proposition 2.1.1]. There is a natural partial order on S where for any $s, t \in S$ we write $s \leqslant t$ if one of the following equivalent conditions hold:

(1) $s = ts^*s$,
(2) $s = ss^*t$,

(3) $s = te$ for some $e \in E(S)$,
(4) $s = et$ for some $e \in E(S)$,

see [**Law98**, Theorem 1.4.6]. For $e, f \in E(S)$, $e \leqslant f$ is equivalent to $ef = e$. If S has a unit 1 then $e \in E(S)$ if and only if $e \leqslant 1$.

Example 1.3. A group G is an inverse semigroup with a unique idempotent, which then is necessarily a unit, cf. [**Law98**, Proposition 1.4.4].

Example 1.4. Let X be a non-empty set. A *partial bijection* on X is a bijection $\theta : U \to \theta(U)$ between two subsets $U, \theta(U) \subseteq X$. For two partial bijections $\theta : U \to \theta(U)$, $\vartheta : V \to \vartheta(V)$ we define their composition as the usual composition on the largest domain that it makes sense

$$\theta \circ \vartheta : \vartheta^{-1}(U \cap \vartheta(V)) \to \theta(U \cap \vartheta(V)).$$

Then the set $S := \mathrm{PBij}(X)$ of all partial bijections on X with the above composition is an inverse semigroup. In this semigroup, $\theta^* = \theta^{-1} : \theta(U) \to U$ is the inverse to θ, and idempotents $E(S) = \{\theta : \theta = id|_U \text{ for some open } U \subseteq X\}$ may be identified with subsets of X. Moreover, $\theta \leqslant \vartheta$ if and only if θ is the restriction of the map ϑ to the domain of θ.

Remark 1.5. The above is generic as every inverse semigroup S can be identified with an inverse subsemigroup of $\mathrm{PBij}(X)$ for some X. Namely, for every inverse semigroup S we have an injective semigroup homomorphism $h : S \to \mathrm{PBij}(S)$, where for any $t \in S$ the corresponding partial bijection is $h_t : t^*tS \to tt^*S$ where $h_t(s) := ts$, see [**Pat99**, Proposition 2.1.3], [**Law98**, Proposition 1.5.1].

Example 1.6. Let X be a topological space. A *partial homeomorphism* on X is a homeomorphism between two open subsets of X. The set $\mathrm{PHomeo}(X)$ of all partial homeomorphisms on X is an inverse subsemigroup of $\mathrm{PBij}(X)$.

Definition 1.7. An *action of an inverse semigroup S on a topological space X* is a semigroup homomorphism $\theta : S \to \mathrm{PHomeo}(X)$ which is "approximately" unital. Namely, θ can be identified with a family of homeomorphisms $\theta_t : X_{t^*} \to X_t$ between two open subsets X_{t^*}, $X_t \subseteq X$ such that

(1) $\theta_t \circ \theta_s = \theta_{ts}$ (as partial maps) for all $s, t \in S$
(2) $\bigcup_{t \in S} X_t = X$.

If S is unital, then (2) means that the homomorphism θ is unital, i.e. $\theta_1 = \mathrm{id}|_X$. If S has a *zero*, i.e., an element $0 \in S$ such that $0s = s0 = 0$ for any $s \in S$, then we will also require that $X_0 = \varnothing$ and θ_0 is the empty map.

Remark 1.8. It readily follows from the axioms of an inverse semigroup action $\theta : S \to \mathrm{PHomeo}(X)$ that for all $s, t \in S$ and $e \in E(S)$ we have

$$X_{t^*} = X_{t^*t}, \qquad X_t = X_{tt^*}, \qquad \theta_e = \mathrm{id}|_{X_e}, \qquad s \leqslant t \Longrightarrow X_{s^*} \subseteq X_{t^*} \text{ and } \theta_t|_{X_{s^*}} = \theta_t.$$

In particular, condition (2) in Definition 1.7 is equivalent to $\bigcup_{e \in E(S)} X_e = X$.

Example 1.9. If the inverse semigroup is a group G, then an action of G as an inverse semigroup is a group action in the usual sense, that is, a group homomorphism $\theta : G \to \mathrm{Homeo}(X)$.

1.2. Étale groupoids

A good introduction to étale groupoids is [**Sim20**]. A *groupoid* is a small category $\mathcal{G}$ in which every arrow is invertible. As it is customary, we identify $\mathcal{G}$ with its set of arrows and in particular we identify object with the identity arrows. We will denote the *unit space* of $\mathcal{G}$ by $X \subseteq \mathcal{G}$. Thus we have the range and domain maps $r, d : \mathcal{G} \to X \subseteq \mathcal{G}$ that map arrows to their final and initial objects, respectively:

$$r(\gamma) = \gamma\gamma^{-1}, \qquad d(\gamma) = \gamma^{-1}\gamma. \tag{1.2}$$

By convention we compose arrows from right to left, so the set of composable arrows is

$$\mathcal{G}^2 = \{(\eta, \gamma) \in \mathcal{G} \times \mathcal{G} : d(\eta) = r(\gamma)\}.$$

DEFINITION 1.10. A *topological groupoid* is a groupoid $\mathcal{G}$ equipped with a topology that makes the composition and inversion of arrows continuous.

Continuity of composition means that the map $\mathcal{G}^2 \ni (\eta, \gamma) \to \eta\gamma \in \mathcal{G}$ is continuous, where $\mathcal{G}^2$ is equipped with the product topology inherited from $\mathcal{G} \times \mathcal{G}$. In a number of sources, including [**Sim20**], it is additionally assumed that in the topological groupoid the range and domain maps $r, d : \mathcal{G} \to X \subseteq \mathcal{G}$ are continuous, but this follows from (1.2):

LEMMA 1.11. *In every topological groupoid $\mathcal{G}$ the range and source maps $r, d : \mathcal{G} \to X \subseteq \mathcal{G}$ are continuous.*

PROOF. Let (γ_n) be a net in $\mathcal{G}$ converging to $\gamma \in \mathcal{G}$. Then (γ_n^{-1}) converges to $\gamma^{-1} \in \mathcal{G}$ and so $(\gamma_n, \gamma_n^{-1})$ is a net in $\mathcal{G}^2$ converging to $(\gamma, \gamma^{-1}) \in \mathcal{G}^2$. Hence $r(\gamma_n) = \gamma_n\gamma_n^{-1} \to \gamma\gamma^{-1} = r(\gamma)$. Similarly one shows that $s(\gamma_n) \to s(\gamma)$. □

Here we will consider topological groupoids with a much stronger property that the range and domain maps are local homeomorphisms. Recall that a continuous map between topological spaces is a *local homeomorphism* if it is open and locally injective. This is a notion of a local isomorphism or in other words *étale morphism* in the category of topological spaces.

DEFINITION 1.12. An *étale groupoid* is a topological groupoid $\mathcal{G}$ such that the range and source maps $r, d : \mathcal{G} \to X$ are local homeomorphisms, i.e., locally injective and open. In general, a *bisection* of topological groupoid $\mathcal{G}$ is a subset $U \subseteq \mathcal{G}$ such that $r|_U$ and $s|_U$ are partial homeomorphisms of $\mathcal{G}$, i.e. U, $r(U)$, $d(U)$ are open in $\mathcal{G}$ and $r|_U : U \to r(U)$, $d|_U : U \to d(U)$ are homeomorphisms.

REMARK 1.13. Clearly, a topological groupoid is étale if and only if it can be covered by bisections if and only if it has a topological basis consisting of bisections (as an open subset of a bisection is a bisection). Also, since $r(\gamma) = d(\gamma^{-1})$ and $\gamma \mapsto \gamma^{-1}$ is a homeomorphism of $\mathcal{G}$ for any topological groupoid $\mathcal{G}$, we see that r is locally injective or open if and only if d has that property. Local injectivity of r (equivalently of d) is equivalent to openness of the unit space X in the groupoid $\mathcal{G}$, see [**Tho10**, Lemma 2.1], Hence for any topological groupoid $\mathcal{G}$ the following conditions are equivalent:

(1) $\mathcal{G}$ is étale;

(2) $r : \mathcal{G} \to \mathcal{G}$ is an open map ($r : \mathcal{G} \to X$ is an open map and X is open set in $\mathcal{G}$);
(3) $d : \mathcal{G} \to \mathcal{G}$ is an open map ($d : \mathcal{G} \to X$ is an open map and X is open set in $\mathcal{G}$).

Local injectivity (openness of X) implies that for any $x \in X$ the fibers $r^{-1}(x)$ and $d^{-1}(x)$ are discrete subsets of $\mathcal{G}$, cf. [**Ren80**, I.2.7(i)].

In any topological groupoid $\mathcal{G}$ the set of bisections

$$\mathrm{Bis}(\mathcal{G}) := \{U \subseteq \mathcal{G} : U \text{ is a bisection of } \mathcal{G}\}$$

forms an inverse semigroup with the multiplication and generalized inverse given by

$$(1.3) \qquad U \cdot V := \{\gamma\eta : \gamma \in U,\ \eta \in V \text{ are composable}\}, \qquad U^* = U^{-1} := \{\gamma^{-1} : \gamma \in U\}$$

for $U, V \in \mathrm{Bis}(\mathcal{G})$. Idempotents of this semigroup can be identified with the topology of X:

$$E(\mathrm{Bis}(\mathcal{G})) = \{U \subseteq X : U \text{ is open in } X\},$$

and the unit space $X \in \mathrm{Bis}(\mathcal{G})$ is the unit in the semigroup $\mathrm{Bis}(\mathcal{G})$. Moreover, for each $U \in \mathrm{Bis}(\mathcal{G})$ we have a homeomorphism

$$\theta_U := r \circ d|_U^{-1} : d(U) \to r(U),$$

and these homeomorphisms define an inverse semigroup action $\theta : \mathrm{Bis}(\mathcal{G}) \to \mathrm{PHomeo}(X)$ on the unit space X, see [**Exe08**, Proposition 5.3].

Definition 1.14. We refer to the above inverse semigroup action $\theta : \mathrm{Bis}(\mathcal{G}) \to \mathrm{PHomeo}(X)$ as the *canonical action of* $\mathrm{Bis}(\mathcal{G})$ *on* X.

Remark 1.15. The inverse semigroup $\mathrm{Bis}(\mathcal{G})$ acts not only on X but also on $\mathcal{G}$ itself. Indeed, for each $U \in \mathrm{Bis}(\mathcal{G})$ we have a homeomorphism

$$\tilde{\theta}_U : r^{-1}(d(U)) \to r^{-1}(r(U)) \text{ given by } \tilde{\theta}_U(\gamma) = d|_U^{-1}(r(\gamma))\gamma.$$

In other words, $\tilde{\theta}_U(\gamma) = \eta\gamma$ where η is the unique element in U that can be composed with γ. Using this description one sees that $\tilde{\theta} : \mathrm{Bis}(\mathcal{G}) \to \mathrm{PHomeo}(\mathcal{G})$ is an inverse semigroup action. This *extended action* restricts to the canonical action θ in the sense that

$$\theta = r \circ \tilde{\theta}|_X.$$

We will refer to θ the *canonical action of* $\mathrm{Bis}(\mathcal{G})$ *on* $\mathcal{G}$.

Example 1.16. Étale groupoids with a single unit are nothing but discrete groups. Indeed, if the unit space $X = \{1\}$ is a singleton, then all arrows are composable and invertible, so that $\mathcal{G}$ forms a group with the neutral element being 1. Local injectivity of $r : \mathcal{G} \to \{1\}$ forces $\mathcal{G}$ to be discrete. Conversely, a discrete group G may be viewed as a groupoid with discrete topology, a single object $1 \in G$ and arrows being the elements of G. As non-empty bisections are singletones $\{g\}$, $g \in G$, we have a natural isomorphism

$$\mathrm{Bis}(G) \cong G \cup \{0\},$$

where $G \cup \{0\}$ is an inverse semigroup obtained from G by adding zero to it.

Example 1.17. To any action $\theta : G \to \mathrm{Homeo}(X)$ of a discrete group G on a topological space X there is a naturally associated groupoid, called the *transformation groupoid* of θ. It is $\mathcal{G} := G \times X$ equipped with the product topology; the unit space $\{1\} \times X$ is identified with X; the range and source maps are given by $r(g,x) = \theta_g(x)$, $d(g,x) = x$; and, the composition and inversion are given by

$$(h, \theta_g(x)) \cdot (g,x) := (hg, x), \qquad (g,x)^{-1} = (g^{-1}, \theta_g(x)),$$

where $g, h \in G$ and $x \in X$. Note that, we have a natural injective unital semigroup homomorphism $G \ni g \mapsto \{g\} \times X \subseteq \mathrm{Bis}(\mathcal{G})$. In particular, the family of bisections $S := \{\{g\} \times X\}_{g \in G}$ forms an inverse subsemigroup of $\mathrm{Bis}(\mathcal{G})$ which is isomorphic to G. Moreover, the action of S coincides with the action of G in the sense that: $\theta_{\{g\}\times X} = \theta_g$ for $g \in G$.

Example 1.17 suggests that information about the structure of an étale groupoid $\mathcal{G}$ is contained in the structure of the canonical action of the inverse semigroup $\mathrm{Bis}(\mathcal{G})$. This is indeed true but for that we need to generalize the construction of the transformation groupoid that we now discuss.

1.3. Transformation groupoids

Let us fix an inverse semigroup action $\theta : S \to \mathrm{PHomeo}(X)$ of an inverse semigroup S on a topological space X, see Definition 1.7. We recall here the construction of the associated *transformation groupoid* from [**Pat99**, p. 140], see also [**KM21**, 2.1] or [**Exe08**, Section 4]. We first construct a large auxiliary groupoid based on the subset

$$S \ltimes X := \{(t,x) : x \in X_{t^*}, t \in S\},$$

of the product space $S \times X$, where we declare that (s,y) and (t,x) are composable only if $y = \theta_t(x)$, which forces that $x \in X_{(st)^*}$, and we put $(s, \theta_t(x)) \cdot (t,x) := (s \cdot t, x)$ for $x \in X_{(st)^*}$. Then $S \ltimes X$ is an étale groupoid with inverse $(t,x)^{-1} = (t^*, \theta_t(x))$ and, range and domain maps given by $r(t,x) = (tt^*, h_t(x))$, $d(t,x) = (t^*t, x)$. Its unit space is $\bigcup_{e \in E(S)} \{e\} \times X_e$ and $(e,x) \mapsto x$ maps it onto X (surjectivity is exactly condition (2) in Definition 1.7). This surjective map is typically far from being injective and we would like to have a groupoid with the unit space that can be identified with X. Thus for all $e, f \in E(S)$ we would like to identify (e,x) and (f,x) whenever $x \in X_{ef}$. It turns out that this generates a very well behaved groupoid congruence $\sim_\theta$ on $S \ltimes X$. Namely, we put

$$(t,x) \sim_\theta (t',x') \overset{def}{\iff} x = x' \text{ and } x \in X_{v^*} \text{ for some } v \leqslant t, t'.$$

This is an equivalence relation on $S \ltimes X$ and in fact a groupoid congruence. This means that denoting the equivalence class of (t,x) by $[t,x]$, the quotient space

$$S \ltimes_\theta X := S \ltimes X / \sim_\theta = \{[t,x] : x \in X_{t^*}, t \in S\}$$

is a groupoid with composition law

$$[s, \theta_t(x)] \cdot [t,x] = [s \cdot t, x], \qquad x \in X_{(st)^*}, t, s \in S,$$

and inversion $[t,x]^{-1} = [t^*, \theta_t(x)]$ for $x \in X_{t^*}$, $t \in S$. In fact when equipped with the quotient topology this groupoid is étale. Indeed, one checks that for each $t \in S$, the set

$$U_t := \{[t,x] : x \in X_{t^*}\}$$

is open in $S \ltimes_\theta X$ and the map $U_t \ni [t,x] \to x \in X_{t^*}$ yields a homeomorphism $U_t \cong X_{t^*}$. Accordingly, the map $\bigcup_{e\in E(S)}\{e\} \times X_e \ni [e,x] \mapsto x \ni X$ allows us to identify the unit space of $S \ltimes_\theta X$ with X. Then the range and domain maps $r, d : S \ltimes_\theta X \to X$ are given by

$$r([t,x]) = \theta_t(x) \quad \text{and} \quad d([t,x]) = x,$$

and so they are local homeomorphisms, as they restrict to homeomorphisms $r : U_t \to X_t$ and $d : U_t \to X_{t^*}$ for $t \in S$. In particular, $\{U_t\}_{t\in S}$ are bisections that cover $S \ltimes_\theta X$.

DEFINITION 1.18. The étale groupoid $S \ltimes_\theta X$ constructed above is called the *transformation groupoid* of the inverse semigroup action $\theta : S \to \mathrm{PHomeo}(X)$.

REMARK 1.19. In [**Exe08**, Section 4] it is called the groupoid of germs, but we believe this term should be reserved for the different construction that is used for instance in [**Ren08**].

For any transformation groupoid $S \ltimes_\theta X$ the map $S \ni t \mapsto U_t \in \mathrm{Bis}(S \ltimes_\theta X)$ is a semigroup homomorphism. Composing it with the canonical homomorphism $\mathrm{Bis}(S \ltimes_\theta X) \to \mathrm{PHomeo}(X)$ we recover the action θ. Composing it with the canonical homomorphism $\mathrm{Bis}(S \ltimes_\theta X) \to \mathrm{PHomeo}(S \ltimes_\theta X)$, we get an extension $\tilde{\theta}$ of θ where

$$\tilde{\theta}_t : \{[s,x] : x \in h_s^{-1}(X_{t^*})\} \to \{[s,x] : x \in h_s^{-1}(X_t)\}$$

is given by $\tilde{\theta}_t([s,x]) := [ts,x]$. In this way every inverse semigroup action can be viewed as coming from an étale groupoid, and then it can be extended to an action on this groupoid, see Remark 1.15.

Conversely, every étale groupoid $\mathcal{G}$ comes from an inverse semigroup action, and to this end one use any inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ which is *wide* in the sense that S covers $\mathcal{G}$ and $U \cap V$ is a union of bisections in S for all $U, V \in S$.

PROPOSITION 1.20 (see [**Exe08**, Proposition 5.4], [**KM21**, Proposition 2.2]). *For any étale groupoid $\mathcal{G}$ and any inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ acting canonically on X we have a continuous open groupoid homomorphism*

$$S \ltimes X \ni [U,x] \mapsto (d|_U)^{-1}(x) \in \mathcal{G}.$$

This map is an isomorphism $S \ltimes X \cong \mathcal{G}$ if and only if S is wide.

1.4. Hausdorffness

From now on we will always assume that

X is a locally compact Hausdorff space.

Thus every étale groupoid $\mathcal{G}$ with unit space X is automatically locally compact and locally Hausdorff but not necessary (globally) Hausdorff. The following is the well known criterion for when $\mathcal{G}$ is Hausdorff, see for instance [**Sim20**, Lemma 2.3.2].

Lemma 1.21. *An étale groupoid $\mathcal{G}$ with unit space X is Hausdorff if and only if X is a closed subset of $\mathcal{G}$.*

In this thesis we will focus on Hausdorff étale groupoids. For such groupoids the unit space X is both closed and open in $\mathcal{G}$. Unfortunately, many naturally occurring étale groupoids are non-Hausdorff. It can be explained by the fact that étale groupoids are transformation groupoids for inverse semigroups actions, so they are quotients $S \ltimes_\theta X = S \ltimes X/\sim_\theta$ of Hausdorff groupoids $S \ltimes X$. However, a quotient of a Hausdorff space is often non-Hausdorff. For the sake of completeness, we recall a condition characterizing when $S \ltimes_\theta X$ is Hausdorff, see [**KM21**, Lemma 2.6].

Lemma 1.22. *The transformation groupoid $S \ltimes_\theta X$ for an inverse semigroup action $\theta : S \to \mathrm{PHomeo}(X)$ is Hausdorff if and only if for each $t \in S$ the union $X_{1,t} := \bigcup_{e \leqslant t, e \in E(S)} X_e$ is a closed subset of X_t.*

Example 1.23. When $S = G$ is a group, then the transformation groupoid $S \ltimes_\theta X = G \times X$ discussed in Example 1.17 is obviously Hausdorff. In this case, $E(S) = \{1\}$ and hence $X_{1,g}$ is empty if $g \neq 1$ and $X_{1,g} = X = X_1$ if $g = 1$.

Example 1.24. Let X be a locally compact Hausdorff space and let $\varphi : X \to X$ be a local homeomorphism i.e., any point of $x \in X$ has an open neighborhood $U \subseteq X$ such that $\varphi(U)$ is open and $\varphi|_U : U \to \varphi(U)$ is a homomorphism. We define

$$\mathcal{G} := \{(x, n, y) \in X \times \mathbb{Z} \times X : \varphi^k(x) = \varphi^l(y) \text{ for some } k, l \in \mathbb{N}_0 \text{ such that } n = k - l\}.$$

Then setting

$$r(x, n, y) := x,\ d(x, n, y) := y,\ (x, n, y)(y, m, z) := (x, n + m, z),$$

we obtain a groupoid structure on $\mathcal{G}$. For open sets $U, V \subseteq X$, we define sets $Z(U, m, n, V) := \{(x, m-n, y) \in X \times \mathbb{Z} \times X : \varphi^m(x) = \varphi^n(y)\}$, for all $m, n \in \mathbb{N}_0$. Then the sets $Z(U, m, n, V)$ form a basis of locally compact Hausdorff topology on $\mathcal{G}$. Moreover, $\mathcal{G}$ equipped with such a topology is étale. This groupoid is called the *Renault-Deaconu groupoid* corresponding to φ, see, for instance, [**Ren00**].

Example 1.25. Perhaps the most popular example of a non-Hausdorff étale groupoid (or a non-Hausdorff manifold) is the so-called *line with two origins*. It can be viewed as the following transformation groupoid. Let $S = \{1, g, \infty\}$ be the abelian unital inverse semigroup where $g^2 = 1$ and ∞ is an absorber ($\infty \cdot t = \infty$). Put $X_g = X_1 = \mathbb{R}$ and $X_\infty = \mathbb{R}\backslash\{0\}$, and let θ_t be identity for any $t \in S$. Then $S \ltimes_\theta \mathbb{R}$ is obtained from $\mathbb{R} \times \{1, g\} \cup (\mathbb{R}\backslash\{0\}) \times \{\infty\}$ by identifying the point $(x, 1)$, (x, g) and (x, ∞) for every $x \in \mathbb{R}\backslash\{0\}$. An equivalent description of the space is to take $\mathbb{R}$ and replace the origin 0 with two origins 0 and 0_g. The subspace $\mathbb{R}$ retains its usual Euclidean topology, and a local base of open neighborhoods of 0_g is formed by the sets $(-\varepsilon, \varepsilon)\backslash\{0\} \cup \{0_g\}$. As a groupoid it is the unit space $\mathbb{R}$ with one non-unit arrow 0_g such that $0_g \cdot 0_g = 0$.

CHAPTER 2

Generalized weighted composition operators on L^p-spaces

In this chapter we present characterizations and basic facts about weighted shift operators and isometries on L^p-spaces $L^p(\mu)$ for $p \in [1, \infty]$ and a localizable measure μ. A central role here is played by Banach-Lamperti theorem, which is well knwon when the measure μ is finite or σ-finite and $p < \infty$. However, as we explain already for σ-finite measures there are some errors and misunderstandings in the literature. Moreover, some of the results, such as our general Lamperti Theorem (Theorem 2.43) do not appear in the literature at the level of generality we present. We need this in the sequel if we want to have results without any kind of separability assumptions, which in addition cause some technical inconveniences.

We give full proofs and the presentation is self-contained. In particular, we tried to give a concise but comprehensive introduction to the theory of localizable measure spaces. We also believe that the present chapter could be of independent interest.

2.1. Prologue: History of Banach-Lamperti theorem

In 1932 in his famous book, see [**Ban32**, page 178], Banach states that invertible isometries on Lebesgue spaces $L^p[0, 1]$, where $1 \leqslant p < \infty$ and $p \neq 2$, are all weighted composition operators of the form

$$T\xi(t) = \Big(\frac{d(\mu \circ \varphi)}{d\mu}\Big)^{\frac{1}{p}}(t)\xi(\varphi(t)), \qquad \xi \in L^p[0, 1],\ t \in [0, 1], \tag{2.1}$$

where $\varphi : [0, 1] \to [0, 1]$ is a Borel isomorphism, μ is the Lebesgue measure, and $\frac{d(\mu\circ\varphi)}{d\mu}$ is the relevant Radon-Nikodym derivative. Banach does not include a proof and only states that the proof will appear in an article that was to be published in Studia Mathematica. But this never happened. Also it is obvious that as stated the statement is incorrect, as one can multiply the above operator by any function with modulus one to get an isometry, which is not exactly of the prescribed form. Namely, (2.1) should be replaced with

$$T\xi(t) = h(t)\xi(\varphi(t)) \quad \text{where} \quad |h|^p = \frac{d(\mu \circ \varphi)}{d\mu}$$

and h is measurable map. Funnily enough, in the same place Banach gives the full proof and correctly describes invertible isometries on small L^p-spaces ℓ^p, for $p \in [1, \infty)\backslash\{2\}$. A difficult step in the characterization of isometries on Lebesgue spaces $L^p[0, 1]$ (that Banach ultimately did not take) is the transition from set maps to point maps, that for instance Royden and Kan attribute to Sikorski, see [**Roy73**, Proposition 3 on page 397], [**Kan78**, Remark 4.2]. For discrete spaces ℓ^p this is easy, because point maps can be treated as maps on singletones.

In 1958 Lamperti [**Lam58**], used the above shortcomings as one of his motivations for supplying characterization of arbitrary (not necessarily invertible) isometries on an L^p-space $L^p(\mu)$ where $p \neq 2$ and (Ω, Σ, μ) is an arbitrary σ-finite measure space. The first important step (which was already noticed by Banach [**Ban32**, page 178]) is that isometries on such spaces preserve disjointness (they map functions with disjoint supports to functions with disjoint supports). Lamperti deduced this from Clarkson inequalities [**Cla36**]. The second, conceptually important step was that Lamperti completely ignored point maps and formulated everything in terms of set maps $\Phi : \Sigma \to \Sigma$, which finally led to introducing the generalized composition operators T_Φ that we construct in the full generality in Proposition 2.29 below. Lamperti's claim was that an operator $T : L^p(\mu) \to L^p(\mu)$, where $1 \leqslant p \neq 2$, is an isometry (not necessarily invertible) if and only if

$$T\xi = hT_\Phi \xi$$

where hT_Φ is a generalized weighted composition operator associated to a set monomorphism $\Phi : \Sigma \to \Sigma$ and $h : \Omega \to \mathbb{C}$ is a measurable map satisfying

$$|h|^p = \frac{d\mu \circ \Phi^{-1}}{d\mu|_{\Phi(\Sigma)}} \quad \text{on the set } \Phi(\Omega), \tag{2.2}$$

where $\mu \circ \Phi^{-1}(\Phi(A)) = \mu(A)$ is a well defined measure on the σ-algebra $\Phi(\Sigma)$ of subsets of $\Phi(\Omega)$. If $\Phi(A) = \phi^{-1}(A)$ comes from an measurable isomorphism $\varphi : \Omega \to \Omega$, then T_Φ is a composition operator with φ and $\frac{d\mu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma)}} = \frac{d(\mu\circ\varphi)}{d\mu}$, and so the above description reduces to the one given by Banach. However, in general, for non-invertible isometries condition (2.2), given by Lamperti, is incorrect. The reason is that by definition the Radon-Nikodym derivative $\frac{d\mu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma)}}$ is $\Phi(\Sigma)$-measurable, while $|h|^p$ need not be. Concrete simple examples where this fails on $L^p[0,1]$ are given in [**FJ03**, Example 3.2.6] and they are attributed to Grząślewicz, cf. also [**Grz85**]. Therefore, the authors, aware of this defect, replace condition (2.2) with

$$\mathbb{E}\left(|h|^p \mid \Phi(\Sigma)\right) = \frac{d\mu \circ \Phi^{-1}}{d\mu|_{\Phi(\Sigma)}}, \tag{2.3}$$

where $\mathbb{E}\left(|h|^p \mid \Phi(\Sigma)\right)$ is a conditional expectation of $|h|^p : \Phi(\Omega) \to \mathbb{C}$ with respect to the σ-subalgebra $\Phi(\Sigma)$, see for instance [**Grz85**], [**FJ03**]. But this is not the end of the story. It appeared to us that condition (2.3) still needs to be corrected when μ is not finite. The reason is that even when μ is σ-finite, its restriction $\mu|_{\Phi(\Sigma)}$ may not. In this case it may happen the Radon-Nikodym derivative $\frac{d\mu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma)}}$ does not exist. We give a simple example on $L^p(\mathbb{R})$, see Example 2.13 below. Thus it turns out that already for σ-finite measures condition (2.3) should be replaced with

$$\mu(A) = \int_{\Phi(A)} |h|^p d\mu, \qquad \text{for all } A \in \Sigma. \tag{2.4}$$

This last condition can be readily adapted to describe isometries for all localizable measures, cf. (2.11) below.

2.2. Localizable measures and Radon-Nikodym theorem

Let (Ω, Σ, μ) be a measure space. Hence Σ is a σ-algebra of subsets of Ω and $\mu : \Sigma \to [0, \infty]$ is a measure. We say that μ is *semifinite* if for any $B \in \Sigma$ with $\mu(B) > 0$ there exists $A \in \Sigma$ with $0 < \mu(A) < \infty$ and $A \subseteq B$. In general, for any measure μ putting

$$\mu_0(A) := \sup\{\mu(B) : B \subseteq A,\, \mu(B) < \infty\}, \qquad A \in \Sigma, \tag{2.5}$$

defines a semifinite measure $\mu_0 \leqslant \mu$. Then $\mu = \mu_0 + \mu_\infty$ where the measure $\mu_\infty := \mu - \mu_0$ is degenerate in the sense that it attains values in $\{0, \infty\}$. In particular μ is semifinite if and only if $\mu = \mu_0$. Obviously, every σ-finite measure is semifinite. In fact this is true for decomposable measures. In what follows we use $\bigsqcup$ or $\sqcup$ to denote a disjoint union of sets.

Definition 2.1. We say that a measure $\mu : \Sigma \to [0, \infty]$ is *decomposable* (or *strictly localizable*) if $\Omega = \bigsqcup_{i \in I} \Omega_i$ for some $\{\Omega_i\}_{i \in I} \subseteq \Sigma$ such that $\mu(\Omega_i) < \infty$, for all $i \in I$ and

(D1) $\mu(A) = \sum_{i \in I} \mu(A \cap \Omega_i)$ for all $A \in \Sigma$;
(D2) If $A \subseteq \Omega$ and $A \cap \Omega_i \in \Sigma$ for all $i \in I$ then $A \in \Sigma$.

Remark 2.2. Condition (D1) implies that decomposable measures are semifinite. If $I := \mathbb{N}$, then conditions (D1), (D2) above are automatic, and hence every σ-finite measure is decomposable. A decomposable measure is nothing but an appropriately defined direct sum of finite measures. In particular, a counting measure on any power set is decomposable. Another natural class of decomposable measures is given by Haar measures on locally compact groups, cf. for instance [**Fol99**, Exercise 9 in subsection 11.1]. In general, when appropriately defined all Radon measures on locally compact Hausdorff spaces are decomposable, see [**BGL22**, Theorem 6.2].

For sets $A, B \in \Sigma$ we write $A \overset{\mu}{\subseteq} B$ if A is *essentially contained* in $B \in \Sigma$ that is $\mu(A \backslash B) = 0$. We write $A \overset{\mu}{=} B$ if A and B are *essentially equal*, that is $A \overset{\mu}{\subseteq} B$ and $B \overset{\mu}{\subseteq} A$. We say that a family $\mathcal{A} \subseteq \Sigma$ of measurable sets has an *essential supremum* (also called *essential union*) if there is a measurable set $\sup \mathcal{A} \in \Sigma$ such that $A \overset{\mu}{\subseteq} \sup \mathcal{A}$ for all $A \in \mathcal{A}$, and $\sup \mathcal{A} \overset{\mu}{\subseteq} B$ for every $B \in \Sigma$ such that $A \overset{\mu}{\subseteq} B$ for all $A \in \mathcal{A}$. The set $\sup \mathcal{A}$, if it exists, is essentially unique. If there is a risk of confusion, we will add μ to the adjective "essential".

Example 2.3. When $\mathcal{A}$ is countable then $\sup \mathcal{A} = \bigcup_{A \in \mathcal{A}} A$. When $\mathcal{A}$ consists μ-null sets then $\sup \mathcal{A} = \varnothing$.

Every (σ-additive) measure is totally additive with respect to essential suprema.

Lemma 2.4. *For any family $\mathcal{A} \subseteq \Sigma$ of pairwise essentially disjoint sets which has a μ-essential supremum* $\sup \mathcal{A} \in \Sigma$ *we have* $\mu(\sup \mathcal{A}) = \sum_{A \in \mathcal{A}} \mu(A)$.

Proof. If $\sum_{A\in\mathcal{A}}\mu(A)<\infty$ then there is a countable subset $\mathcal{A}_0\subseteq\mathcal{A}$ such that $\mu(A)=0$ for $A\in\mathcal{A}\backslash\mathcal{A}_0$. Then $\sup\mathcal{A}_0=\bigsqcup_{A\in\mathcal{A}_0}A=\sup\mathcal{A}$. Hence $\mu(\sup\mathcal{A})=\mu(\sup\mathcal{A}_0)=\sum_{A\in\mathcal{A}_0}\mu(A)=\sum_{A\in\mathcal{A}}\mu(A)$. If $\sum_{A\in\mathcal{A}}\mu(A)=\infty$ then there is a countable subset $\mathcal{A}_0\subseteq\mathcal{A}$ such that $\sum_{A\in\mathcal{A}_0}\mu(A)=\infty$ and since $\bigsqcup_{A\in\mathcal{A}_0}A\overset{\mu}{\subseteq}\sup\mathcal{A}$ we get $\infty=\sum_{A\in\mathcal{A}_0}\mu(A)=\mu(\bigsqcup_{A\in\mathcal{A}_0})\leqslant\mu(\sup\mathcal{A})$. ☐

The above notions become much clearer when passing to the quotient $[\Sigma]:=\Sigma/\overset{\mu}{=}$ which is a σ-complete Boolean algebra with operation induces from the σ-algebra Σ. Then a family $\mathcal{A}\subseteq\Sigma$ has an essential supremum $\sup\mathcal{A}$ if and only if $[\mathcal{A}]=\{[A]:A\in\mathcal{A}\}$ has a supremum in $[\Sigma]$ and then $\sup[\mathcal{A}]=[\sup\mathcal{A}]$.

Definition 2.5. We say that a measure $\mu:\Sigma\to[0,\infty]$ is *Dedekind complete* if every family $\mathcal{A}\subseteq\Sigma$ has the essential supremum (equivalently the Boolean algebra $[\Sigma]$ is Dedekind complete). If in addition, μ is semifinite, μ is called *localizable* (or *Maharam*).

Remark 2.6. It is well known that every decomposable measure is localizable (for the sake of completeness we give a proof below). For examples of decomposable measures that are not localizable, see [**Fr78**, Section 5] or [**Fr02**, 216E].

Lemma 2.7. *Every decomposable measure is localizable.*

Proof. Let (Ω,Σ,μ) be a decomposable measure space and choose sets $\{\Omega_i\}_{i\in I}$ forming the decomposition into sets with finite measure satisfying (D1), (D2). We only need to show that any family $\mathcal{A}\subseteq\Sigma$ has the essential supremum in Σ. To this end, put $\mathcal{F}:=\{F\in\Sigma:\mu(F\cap A)=0\text{ for all }A\in\mathcal{A}\}$ and note that $\mathcal{F}$ is closed under countable unions and measurable subsets. For each $i\in I$ put $\gamma_i:=\sup\{\mu(F\cap\Omega_i):F\in\mathcal{F}\}=\sup\{\mu(F):F\in\mathcal{F}\cap 2^{\Omega_i}\}$ and choose $\{F_n^{(i)}\}_{n\in\mathbb{N}}\subseteq\mathcal{F}\cap 2^{\Omega_i}$ such that $\lim_{n\to\infty}\mu(F_n^{(i)}\cap\Omega_i)=\gamma_i$. Then $F_i:=\bigcup_{n\in\mathbb{N}}F_n^{(i)}\in\mathcal{F}$ is a subset of Ω_i such that $\mu(F_i)=\gamma_i$. Putting $F:=\bigcup_{i\in I}F_i$ we have $F\cap\Omega_i=F_i$ for every $i\in I$ and hence $F\in\Sigma$ is measurable by (D2). We prove that $S:=\Omega\backslash F\in\Sigma$ is the essential supremum for $\mathcal{A}$. For any $A\in\mathcal{A}$, using (D1), we have

$$\mu(A\backslash S)=\mu(A\cap F)=\sum_{i\in I}\mu(A\cap F\cap\Omega_i)=\sum_{i\in I}\mu(A\cap F_i)=0.$$

Hence $A\overset{\mu}{\subseteq}S$ for every $A\in\mathcal{A}$. Let $P\in\Sigma$ be any other set with this property. This means that $D:=\Omega\backslash P\in\mathcal{F}$ ($\mu(A\cap D)=\mu(A\backslash P)=0$ for every $A\in\mathcal{A}$). Thus $F\cup D\in\mathcal{F}$. Therefore for each $i\in I$ we get

$$\mu((F\cup D)\cap\Omega_i)\leqslant\gamma_i=\mu(F_i)=\mu(F\cap\Omega_i),$$

which by monotonicity implies that $\mu((F\cup D)\cap\Omega_i)=\mu(F\cap\Omega_i)$. This in turn is equivalent to $\mu(D\backslash F\cap\Omega_i)=0$ as in general we have $\mu(F\cap\Omega_i)+\mu(D\backslash F\cap\Omega_i)=\mu((F\cup D)\cap\Omega_i)$, and $\mu(\Omega_i)<\infty$. Hence by (D1) we get $\mu(D\backslash F)=\sum_{i\in I}\mu(D\backslash F\cap\Omega_i)=0$ and so

$$\mu(S\backslash P)=\mu(S\cap(\Omega\backslash P))=\mu(S\cap D)=\mu((\Omega\backslash F)\cap D)=\mu(D\backslash F)=0.$$

Thus $S\overset{\mu}{\subseteq}P$. This proves that S is the essential supremum of $\mathcal{A}$ as claimed. ☐

The existence of essential suprema of sets is equivalent to the existence of essential suprema of measurable functions. Let us make it precise.

Definition 2.8. Let $\{f_i\}_{i\in I}$ be a family of measurable functions $f_i : \Omega \to \overline{\mathbb{R}}$ taking values in the extended real line $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, \infty\}$. We say that $\{f_i\}_{i\in I}$ has an *essential supremum* if there exists a measurable function $f : \Omega \to \overline{\mathbb{R}}$ such that

(1) $f_i \leqslant f$ μ-almost everywhere for all $i \in I$;
(2) if $g : \Omega \to \overline{\mathbb{R}}$ is a measurable function satisfying (1) then $f \leqslant g$ μ-almost everywhere.

If such a function exists it is unique up to equality μ-almost everywhere. When there is no risk of confusion we will denote it by $\sup_{i\in I} f_i$.

Lemma 2.9. *A measure space (Ω, Σ, μ) is Dedekind complete (any family of measurable sets has the essential supremum) if and only if any family of measurable functions on Ω with values in $\overline{\mathbb{R}}$ has the essential supremum.*

Proof. "$\Longleftarrow$". Assume that $\mathcal{A}$ is an any family of measurable sets and let $\mathbb{1}_{\mathcal{A}} := \sup_{A\in\mathcal{A}} \mathbb{1}_A$ be the essential supremum of the family $\{\mathbb{1}_A\}_{A\in\mathcal{A}}$. Up to a μ-null set, the function $\mathbb{1}_{\mathcal{A}}$ takes values in $\{0, 1\}$. Since $\mathbb{1}_{\mathcal{A}}$ is measurable, the set $S := \{\omega \in \Omega : \mathbb{1}_{\mathcal{A}}(\omega) = 1\}$ is measurable. Moreover, $\mathbb{1}_{\mathcal{A}} \cdot \mathbb{1}_A = \mathbb{1}_A$ μ-almost everywhere for any $A \in \mathcal{A}$. Hence $\mu(A\backslash S) = 0$. Suppose that $B \in \Sigma$ is a set such that $\mu(A\backslash B) = 0$ for all $A \in \mathcal{A}$. Then $\mathbb{1}_A \leqslant \mathbb{1}_B$ μ-almost everywhere. Since $\mathbb{1}_{\mathcal{A}}$ is the supremum of family $\{\mathbb{1}_A\}_{A\in\mathcal{A}}$, it implies that $\mathbb{1}_{\mathcal{A}} \leqslant \mathbb{1}_B$ and therefore $\mu(S\backslash B) = 0$. Thus S is the essential supremum $\sup \mathcal{A}$ of $\mathcal{A}$.

"$\Longrightarrow$". Let $\{f_i\}_{i\in I}$ be a family of measurable functions and define sets $A_{i,r} := \{\omega : f_i(\omega) \geqslant r\}$, $i \in I$, $r \in \mathbb{Q}$. For any $r \in \mathbb{Q}$ there exists essential supremum B_r of the family $\{A_{i,r}\}_{i\in I}$. Note that $\mu(B_s\backslash B_r) = 0$ if $r \leqslant s$ and put $C_r := \bigcup_{r<s} B_s$. Then $\mu(C_r\backslash C_s) = 0$ if $r < s$. Define $g_r(\omega) := r$ if $\omega \in C_r$ and $g_r(\omega) := -\infty$ otherwise. Then the function $g(\omega) = \sup\{g_r(\omega) : r \in \mathbb{Q}\}$ is measurable as a supremum of a countable family of measurable functions. We show that $f_i \leqslant g$ for all $i \in I$. Let $G_i := \{\omega : g(\omega) \leqslant f_i\}$. If $\omega \in G_i$ then there are $r, s \in \mathbb{Q}$ such that $g(\omega) < r < s < f_i(\omega)$ and then $\omega \in \bigcup_{r<s} A_{\alpha,s}\backslash C_r$. Since $B_s \subseteq C_r$ for $r < s$, we have $G_i \subseteq \bigcup_{r<s} A_{\alpha,s}\backslash C_r \subseteq \bigcup_{r<s} A_{\alpha,s}\backslash B_s$. Thus G_i is contained in countable sum of μ-null sets $A_{\alpha,s}\backslash B_s$ so $\mu(G_i) = 0$. Hence $f_i \leqslant g$ almost everywhere for all $i \in I$. Suppose that h is another measurable function such that $f_i \leqslant h$ for all $i \in I$. We show that the set $H := \{\omega : g(\omega) > h(\omega)\}$ has zero measure. By definition of g we obtain that $g(\omega) > h(\omega)$ if and only if there exists $r \in \mathbb{Q}$ such that $\omega \in C_r$ and $r > h(\omega)$. Therefore

$$H = \bigcup_{r\in\mathbb{Q}} C_r \cap \{\omega : r > h(\omega)\} = \bigcup_{r\in\mathbb{Q}} \bigcup_{r<s} B_s \cap \{\omega : h(\omega) < r\}.$$

It follows from the definition of $A_{i,s}$ and $f_i \leqslant h$ that $\mu(A_{i,s}\backslash\{\omega : h(\omega) < r\}) = 0$. It implies that $B_s\backslash\{\omega : h(\omega) < r\}$ is also supremum of the family $\{A_{i,s}\}_{i\in I}$. Thus $\mu(B_s \cap \{\omega : h(\omega) < r\}) = 0$ and $\mu(H) = 0$ as a countable union of such sets. □

We will use the above lemma to turn quasi-functions to functions.

Definition 2.10. A *quasi-function* on a measure space (Ω, Σ, μ) is a family of measurable functions $\{f_A\}_{A\in\mathcal{F}_\mu}$ (with values in $\mathbb{C}$ or $\overline{\mathbb{R}}$) indexed by the family $\mathcal{F}_\mu := \{A \in \Sigma :$

$\mu(A) < \infty\}$ of sets with finite measures and such that f_A vanishes μ-almost everywhere outside A, and f_A agrees with f_B μ-almost everywhere on $A \cap B$, for all $A, B \in \mathcal{F}_\mu$. If there is a measurable function f on Ω such that $f \cdot \mathbb{1}_A = f_A$ μ-almost everywhere for every $A \in \mathcal{F}_\mu$, we say that the *quasi-function* $\{f_A\}_{A\in\mathcal{F}_\mu}$ *comes from the function* f.

PROPOSITION 2.11. *Let* (Ω, Σ, μ) *be a Dedekind complete measure space. Every quasi-function* $\{f_A\}_{A\in\mathcal{R}}$ *(with values in* $\mathbb{C}$ *or* $\overline{\mathbb{R}}$*) comes from a function* f *(with values in* $\mathbb{C}$ *or* $\overline{\mathbb{R}}$*). If* μ *is localizable (semifinite) this function is unique up to equality* μ*-almost everywhere (if the functions* f_A *are real valued, then* $f = \sup_{A\in\mathcal{F}_\mu} f_A$*).*

PROOF. Assume first that the functions in $\{f_A\}_{A\in\mathcal{F}_\mu}$ take values in $\overline{\mathbb{R}}$. By Lemma 2.9 we may put $f := \sup_{A\in\mathcal{F}_\mu} f_A$. Let $A \in \mathcal{F}_\mu$. The inequality $f_A \leqslant f$ holds μ-almost everywhere on Ω by construction. To show that $f \leqslant f_A$ μ-almost everywhere on A, consider $g : \Omega \to \overline{\mathbb{R}}$ such that $g = f_A$ on A and $g = f$ on $\Omega \backslash A$. For any $B \in \mathcal{F}_\mu$, using properties of a quasi-function, we get that $f_B \leqslant g$ μ-almost everywhere. Since f is an essential supremum we get $f \leqslant g$ μ-almost everywhere. In particular, $f \leqslant f_A$ μ-almost everywhere on A. Hence f is the desired function. When μ is localizable then f has to be given by the supremum $\sup_{A\in\mathcal{F}_\mu} f_A$. Indeed, if f is any measurable function such that $f \cdot \mathbb{1}_A = f_A$ μ-almost everywhere for all $A \in \mathcal{F}_\mu$, then by definition of supremum $\sup_{A\in\mathcal{F}_\mu} f_A \leqslant f$ μ-almost everywhere. Thus if this is not the equality, then we have $\sup_{A\in\mathcal{F}_\mu} f_A > f$ on a set $A_0 \in \Sigma_\mu$ such that $\mu(A_0) > 0$. Since μ is semifinite, we may assume that $A_0 \in \mathcal{F}_\mu$ and then on this set we have get

$$f_{A_0} \leqslant \sup_{A\in\mathcal{F}_\mu} f_A < f = f_{A_0},$$

a contradiction.

If functions in $\{f_A\}_{A\in\mathcal{F}_\mu}$ take values in $\mathbb{R}$, then modyfing f on a μ-null set we may assume that f also takes values in $\mathbb{R}$. When $\{f_A\}_{A\in\mathcal{F}_\mu}$ are $\mathbb{C}$-valued, we may apply the previous step to real $\{\mathrm{Re} f_A\}_{A\in\mathcal{F}_\mu}$ and imaginary $\{\mathrm{Im} f_A\}_{A\in\mathcal{F}_\mu}$ parts to get $\mathbb{R}$ valued functions f_1 and f_2 such that $f := f_1 + f_2$ is the desired $\mathbb{C}$-valued function. ☐

PROPOSITION 2.12 (Monotone convergence theorem for nets). *Let* (Ω, Σ, μ) *be a measure space and* $\{f_\alpha\}_{\alpha\in I}$ *be an increasing net of real valued measurable functions, that is, for any* $\alpha, \beta \in I$ *with* $\alpha \leqslant \beta$ *we have* $f_\alpha \leqslant f_\beta$ μ*-almost everywhere. If the essential supremum* $\sup_{\alpha\in I} f_\alpha$ *exists then*

$$\sup_{\alpha\in I} \int_\Omega f_\alpha d\mu = \int_\Omega \sup_{\alpha\in I} f_\alpha d\mu.$$

PROOF. Let $M := \sup_{\alpha\in I} \int_\Omega f_\alpha d\mu$ and $f := \sup_{\alpha\in I} f_\alpha$. This is clear that $\int_\Omega f_\alpha d\mu \leqslant \int_\Omega f d\mu$ and hence $M \leqslant \int_{\alpha\in I} f d\mu$. This implies that if $M = \infty$ the desired equality is trivially satisfied. Suppose that $M < \infty$ and take arbitrary monotone subsequence $\{f_{\alpha_n}\}_{n\in\mathbb{N}}$. By monotone convergence theorem we have $M = \lim_{n\to\infty} \int_\Omega f_{\alpha_n} d\mu$. Therefore it is sufficient to show that $\lim_{n\to\infty} f_{\alpha_n} = f$ almost everywhere. The inequality $\lim_{n\to\infty} f_{\alpha_n} \leqslant f$ is clear. To show that $\lim_{n\to\infty} f_{\alpha_n} \geqslant f$ it is sufficient to show that $f_\alpha \leqslant \lim_{n\to\infty} f_{\alpha_n}$ almost everywhere for all $\alpha \in I$. Suppose that there exists $\alpha_0 \in I$ such that the set $A = \{\omega \in \Omega : \lim_{n\to\infty} f_{\alpha_n} <$

$f_{\alpha_0}\}$ has a non-zero measure. Then for $\varepsilon := \int_A f_{\alpha_0} d\mu - \int_A \lim_{n\to\infty} f_{\alpha_n} d\mu$ we have $\varepsilon > 0$. Let $N \in \mathbb{N}$, $\beta \in I$ be such that $\int_\Omega f_{\alpha_N} d\mu > M - \varepsilon$ and α_N, $\alpha_0 \leqslant \beta$. Then

$$\begin{aligned} M - \varepsilon &< \int_\Omega f_{\alpha_N} d\mu = \int_A f_{\alpha_N} d\mu + \int_{\Omega\setminus A} f_{\alpha_N} d\mu \leqslant \int_A \lim_{n\to\infty} f_{\alpha_n} d\mu + \int_{\Omega\setminus A} f_{\alpha_N} d\mu \\ &< \int_A f_{\alpha_0} d\mu + \int_{\Omega\setminus A} f_{\alpha_N} - \varepsilon \leqslant \int_A f_\beta d\mu + \int_{\Omega\setminus A} f_\beta d\mu - \varepsilon = \int_\Omega f_\beta d\mu - \varepsilon \\ &\leqslant M - \varepsilon \end{aligned}$$

so $M - \varepsilon < M - \varepsilon$ which is contradiction. Thus $\lim_{n\to\infty} f_{\alpha_n} = f$ almost everywhere. $\square$

We use the above results to deduces a Radon-Nikodym theorem for localizable measures from its classical version for finite measures.

Definition 2.13. Let μ and ν be measures on a measurable space (Ω, Σ). By a *Radon-Nikodym derivative* of ν with respect to μ we mean a measurable function $\frac{d\nu}{d\mu} : \Omega \to [0, \infty]$ such that for any $A \in \Sigma$ we have

$$\nu(A) = \int_A \frac{d\nu}{d\mu}\, d\mu. \tag{2.6}$$

If we only know that (2.6) holds for $A \in \Sigma$ with $\mu(A) < \infty$, then we call $\frac{d\nu}{d\mu}$ a *weak Radon-Nikodym derivative.*

Remark 2.14. If $\frac{d\nu}{d\mu}$ is a weak Radon-Nikodym derivative, then in fact (2.6) holds for all $A \in \Sigma$ which are μ-σ-finite (because the two sides of the equality are "σ-additive"). Thus if μ is σ-finite, then the weak Radon-Nikodym derivative is just the Radon-Nikodym derivative. However, as the classical Saks' example, cf. [**Sak37**, page 36], shows, in general the Radon-Nikodym derivative may not exist, even though the weak one does.

Example 2.15 (Saks' example). Let μ and ν be the counting and Lebesgue measure on the σ-algebra $\mathcal{B}([0,1])$ of Borel sets on $[0,1]$. Since $\mu(A) < \infty$ is equivalent to A being finite, it implies that $\frac{d\nu}{d\mu} \equiv 0$ is a weak Radon-Nikodym derivative. In fact, it is uniquely determined as if $\frac{d\nu}{d\mu}$ satisfies (2.6) for finite A, then for each $x \in [0,1]$ we have $\frac{d\nu}{d\mu}(x) = \int_{\{x\}} \frac{d\nu}{d\mu} d\mu = \nu(\{x\}) = 0$. However, $\frac{d\nu}{d\mu} \equiv 0$ is not a Radon-Nikodym derivative, as for instance we have $\nu([0,1]) = 1 \neq 0 = \int_{[0,1]} \frac{d\nu}{d\mu}\, d\mu$.

The above example shows in particular that weak Radon-Nikodym derivatives can not be used to change measures in the process of integration. Nevertheless for genuine Radon-Nikodym derivatives, whenever they exist, the standard argument works and yields:

Lemma 2.16 (Change of measures). *Let μ and ν be measures on a measurable space (Ω, Σ) and suppose that a Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ exists. Then a measurable function $f : \Omega \to \mathbb{C}$ is ν-integrable if and only if $f \cdot \frac{d\nu}{d\mu}$ is μ-integrable, and if this holds*

$$\int_\Omega f\, d\nu = \int_\Omega f \cdot \frac{d\nu}{d\mu}\, d\mu.$$

The above equality holds for any positive measurable (not necessarily integrable) $f : \Omega \to [0, \infty)$ (both sides are simultaneously finite or infinite).

An obvious necessary condition for the existence of a weak Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ is that ν is *absolutely continuous* with respect to μ, i.e. $\nu(A) = 0$ whenever $\mu(A) = 0$. We denote this by $\nu \ll \mu$. It is well known that for σ-finite measures the Radon-Nikodym derivative exists if and only if $\nu \ll \mu$. In fact, it is only important that μ is σ-finite, see [**Hal50**, VI.31(7)]. This is also true when μ is merely Dedekind complete, but then in general we only get a weak Radon-Nikodym derivative. For Radon-Nikodym derivative we need to assume ν is μ-semifinite:

Definition 2.17. Let ν, μ be measures on (Ω, Σ). The measure ν is *μ-semifinite* if every $A \in \Sigma$ with $\nu(A) > 0$ contains a measurable subset $B \subseteq A$ with $\nu(B) > 0$ and $\mu(B) < \infty$.

Remark 2.18. A measure μ is semifinite if and only if it is μ-semifinite. In general, ν is μ-semifinite if and only if $\nu(A) = \sup\{\nu(B) : B \subseteq A,\ \mu(B) < \infty\}$ for any $A \in \Sigma$. If μ and ν are *equivalent*, that is $\nu \ll \mu$ and $\mu \ll \nu$, and μ is semifinite then ν is μ-semifinite. Indeed, if $\nu(A) > 0$, then $\mu(A) > 0$, because $\nu \ll \mu$. Hence by semifiniteness of μ there is $B \subseteq A$ such that $0 < \mu(B) < \infty$, which in particular implies $\nu(B) > 0$ because $\mu \ll \nu$,

Remark 2.19. If μ is σ-finite then any measure ν defined on the same σ-algebra Σ is automatically μ-semifinite. Indeed, let $\{\Omega_n\}_{n\in\mathbb{N}} \subseteq \Sigma$ be a sequence of pairwise disjoint sets such that $\Omega = \bigsqcup_{n\in\mathbb{N}} \Omega_n$ and $\mu(\Omega_n) < \infty$ for all $n \in \mathbb{N}$. By σ-additivity of ν, for any $A \in \Sigma$ with $\nu(A) > 0$ we must have $\nu(A \cap X_n) > 0$ for some $n \in \mathbb{N}$. Hence $B := A \cap X_n$ witnessess μ-semifiniteness for A.

The following result is in essence due to Segal [**Seg51**], cf. also [**BGL22**, Corollary 4.5, Theorem 4.4]. We formulate it using our notion of a weak Radon-Nikodym derivative.

Theorem 2.20 (Radon-Nikodym theorem for localizable measures). *Let μ and ν be two measures on (Ω, Σ) where μ is localizable. A weak Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ exists if and only if $\nu \ll \mu$. If it exists, then $\frac{d\nu}{d\mu}$ is unique up to equality μ-almost everywhere. Moreover, $\frac{d\nu}{d\mu}$ attains finite values μ-almost everywhere if and only if ν is semifinite, and $\frac{d\nu}{d\mu}$ is the Radon-Nikodym derivative if and only if ν is μ-semifinite.*

Proof. If the weak Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ exists, then for any $A \in \Sigma$ with $\mu(A) = 0$ we have $\nu(A) = \int_A \frac{d\nu}{d\mu}\, d\mu = 0$ and so $\nu \ll \mu$. Assume $\nu \ll \mu$. Let $\mathcal{F}_\mu = \{A \in \Sigma : \mu(A) < \infty\}$. For any $A \in \mathcal{F}_\mu$, let μ_A, ν_A denote the measures μ, ν restricted to the σ-algebra $\Sigma_A := \{A \cap B : B \in \Sigma\}$. Then μ_A is a finite measure on A. Hence the basic (standard) Radon-Nikodym theorem says that the Radon-Nikodym derivative $f_A := \frac{d\nu_A}{d\mu_A} : A \to [0, \infty]$ exists and is unique up to equality μ-almost everywhere. This holds for any $A \in \mathcal{F}_\mu$. We extend functions f_A to Ω by putting $f_A \equiv 0$ outside of A. By uniqueness, f_A agrees with f_B μ-almost everywhere on $A \cap B$, for all $A \in \mathcal{F}_\mu$. Hence $\{f_A\}_{A\in\mathcal{F}_\mu}$ is a quasi-function and by Proposition 2.11 there is, unique up to equality μ-almost everywhere, measurable $f : \Omega \to [0, \infty]$ such that $f = f_A$ μ-almost everywhere on

A (in fact f is the essential supremum $\sup_{A\in\mathcal{F}_\mu} f_A$). By construction this is the desired weak Radon-Nikodym derivative (for the existence we only need that μ is Dedekind).

Now assume that the weak Radon-Nikodym derivative $f = \frac{d\nu}{d\mu}$ exists. Putting $A_\infty := f^{-1}(\infty)$, for every measurable $A \subseteq A_\infty$ we have either $\nu(A) = 0$ or $\nu(A) = \infty$ depending on whether $\mu(A) = 0$ or $\mu(A) > 0$ respectively. Thus if ν is semifinite, then we necessarily have $\nu(A_\infty) = 0$ and therefore $\mu(A_\infty) = 0$, that is $\frac{d\nu}{d\mu} < \infty$ μ-almost everywhere. Conversely, assume that $\mu(A_\infty) = 0$. Take any $A \in \Sigma$ with $\nu(A) > 0$. As ν is μ-semifinite there is $B \subseteq A$ such that $\nu(B) > 0$ and $\mu(B) < \infty$. Putting $B_n = B \cap \{\omega \in \Omega : f(\omega) \leqslant n\}$, for $n \in \mathbb{N}$, we get $\nu(B) = \lim_{n\to\infty} \nu(B_n)$ by continuity of a measure, and that $\nu(B_n) = \int_{B_n} f \leqslant n \cdot \mu(B) < \infty$ by construction. Hence $0 < \nu(B_n) < \infty$ for large enough n. Thus ν is semifinite.

Now assume that f is the (genuine) Radon-Nikodym derivative. Take any $A \in \Sigma$ with $\nu(A) > 0$. Then $\{f\mathbb{1}_B\}_{B\in\mathcal{F}_\mu, B\subseteq A}$ is an increasing net whose essential supremum is f. Thus using Proposition 2.12, we get

$$\nu(A) = \int_A f d\mu = \int_\Omega \sup_{B\in\mathcal{F}_\mu, B\subseteq A} f\mathbb{1}_B d\mu = \sup_{B\in\mathcal{F}_\mu, B\subseteq A} \int_\Omega f\mathbb{1}_B d\mu = \sup_{B\in\mathcal{F}_\mu, B\subseteq A} \nu(B).$$

Hence ν is μ-semifinite. Conversely, if ν is μ-semifinite, then for any $A \in \Sigma$, using Proposition 2.12 we get

$$\nu(A) = \sup_{B\in\mathcal{F}_\mu, B\subseteq A} \nu(B) = \sup_{B\in\mathcal{F}_\mu, B\subseteq A} \int_\Omega f\mathbb{1}_B d\mu = \int_\Omega f\mathbb{1}_A d\mu = \int_A f d\mu.$$

Thus f is the Radon-Nikodym derivative. □

Remark 2.21. If μ is σ-finite, then ν is automatically μ-semifinite, see Remark 2.19, and hence the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ exists if and only if $\nu \ll \mu$. That is, Theorem 2.20 reduces to the well known version of Radon-Nikodym theorem where semifiniteness is not explicitly present. Similarly, μ-semifinitess of ν is automatic whenever μ and ν are equivalent and μ is semifinite, see Remark 2.18. This explains why the relative semifiniteness is not explicitly present in [**GT22**].

Corollary 2.22 (cf. [**GT22**, Theorem 2.7]). *For any localizable measures μ and ν that are equivalent, the Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$ exist, attain values in $(0,\infty)$ and $(\frac{d\nu}{d\mu})^{-1} = \frac{d\mu}{d\nu}$ almost everywhere.*

Proof. The Radon-Nikodym derivatives $\frac{d\nu}{d\mu}$ and $\frac{d\mu}{d\nu}$ exist and attain values in $(0,\infty)$ by Remark 2.18 and Theorem 2.20. Moreover, for any $A \in \Sigma$, using Lemma 2.16, we have $\int_A (\frac{d\nu}{d\mu})^{-1}\, d\nu = \int_A (\frac{d\nu}{d\mu})^{-1} \frac{d\nu}{d\mu}\, d\mu = \mu(A)$. Hence $(\frac{d\nu}{d\mu})^{-1} = \frac{d\mu}{d\nu}$. □

For finite measures with $\nu \ll \mu$, Samuels [**Sam78**] constructed the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$ as a limit of conditional expectations with respect to countable decompositions of Ω. Namely, for any $t \in \mathbb{R}$ denoting by $A_t \in \Sigma$ the positive part in the Hahn decomposition for the signed measure $t\mu - \nu$, for any $A \in \Sigma$ we have:

(1) if $A \subseteq A_t$, then $\nu(A) \leqslant t\mu(A)$
(2) if $A \subseteq A_t'$, then $\nu(A) \geqslant t\mu(A)$.

These two properties essentially determine A_t, and if $s \leqslant t$, then $A_s \subseteq A_t$ essentially. For any $n, k = 0, 1, 2, \ldots$ we put

$$\Omega_{n,k} := A_{\frac{k+1}{2^n}} \backslash A_{\frac{k}{2^n}}.$$

Then $A \subseteq \Omega_{n,k}$ implies $\frac{k}{2^n}\mu(A) \leqslant \nu(A) \leqslant \frac{k+1}{2^n}\mu(A)$. Then for each $n \in \mathbb{N}$ sets $\{\Omega_{n,k}\}_{k=0}^{\infty}$ form a countable decomposition of Ω. We have the following limit in $L^1(\mu)$:

$$\frac{d\nu}{d\mu} = \lim_{n\to\infty} \sum_{k=0}^{\infty} \frac{\nu(\Omega_{n,k})}{\mu(\Omega_{n,k})} \mathbb{1}_{\Omega_{n,k}}.$$

In fact, this can be viewed, as an application of Freudenthal's spectral theorem for Riesz spaces. According to Zaanen this method wasto construct $\frac{d\nu}{d\mu}$ was used already in 1940 by Yosida. Gardella and Thiel [**GT22**] generalized this method so that it works for equivalent localizable measures, see Corollary 2.22. We now generalize it even further to get a description of the Radon-Nikodym derivative in Theorem 2.20, which resembles the construction of the Lebesgue integral.

Theorem 2.23 (Radon-Nikodym theorem for localizable measures II). *Let μ and ν be two measures on (Ω, Σ) where μ is localizable. For any $t \geqslant 0$ there is $A_t \in \Sigma$ such that*

$$\forall_{A\in\Sigma} \ A \subseteq A_t \implies \nu(A) \leqslant t\mu(A), \tag{2.7}$$

and any set with the above property is μ-essentially contained in A_t, so up to μ-null sets, A_t is the largest satisfying (2.7). *The simple functions*

$$f_n := \sum_{k=0}^{n2^n-1} \frac{k}{2^n} \mathbb{1}_{A_{\frac{k+1}{2^n}} \backslash A_{\frac{k}{2^n}}} + n\mathbb{1}_{\Omega\backslash A_n}, \qquad n \in \mathbb{N}, \tag{2.8}$$

form a monotone sequence converging to the weak Radon-Nikodym derivative $\frac{d\nu}{d\mu}$, μ-almost everywhere ($\frac{d\nu}{d\mu}$ is the genuine Radon-Nikodym derivative if and only if ν is μ-semifinite).

Proof. For each $t \in [0, \infty)$ we denote by $A_t \in \Sigma$ the μ-essential supremum of the family

$$\mathcal{A}_t := \{A \in \Sigma : \nu(B) \leqslant t\mu(B) \text{ for every measurable } B \subseteq A\},$$

(with the convention that $\sup \varnothing = \varnothing$). We claim that $A_t \in \mathcal{A}_t$. To show this note first that applying Kuratowski-Zorn lemma we may choose a maximal family $\mathcal{D}_t \subseteq \mathcal{A}_t$ consisting of essentially pairwise disjoint sets (with positive μ-measure or not). Then $A_t \overset{\mu}{=} \sup \mathcal{D}_t$ is also the μ-essential supremum of $\mathcal{D}_t$. Indeed, it is clear that $\sup \mathcal{D}_t \overset{\mu}{\subseteq} A_t$ and if we assume that $A_t \overset{\mu}{\subseteq} \sup \mathcal{D}_t$ fails, then by definition of essential supremum there must exist $A \in \mathcal{A}_t$ with $\mu(A \backslash \sup \mathcal{D}_t) > 0$. But then $D := A \backslash \sup \mathcal{D}_t \in \mathcal{A}_t \backslash \mathcal{D}_t$ is μ-essentially disjoint with each element in $\mathcal{D}_t$, which contradicts maximality of $\mathcal{D}_t$. Take any measurable $B \subseteq A_t$. By distributivity of supremum we have $B \overset{\mu}{=} B \cap \sup \mathcal{D}_t \overset{\mu}{=} \sup_{D\in\mathcal{D}_t}(B \cap D)$. Hence by total additivity of μ, see Lemma 2.4, we get

$$\nu(B) = \sum_{D\in\mathcal{D}_t} \nu(B \cap D) \leqslant \sum_{D\in\mathcal{D}_t} t\mu(B \cap D) = t\mu(\sup_{D\in\mathcal{D}_t} B \cap D) = t\mu(B)$$

where we used that $B \cap D \in \mathcal{A}_t$, for every $D \in \mathcal{D}_t$. This proves that $A_t \in \mathcal{A}_t$, and so with respect to μ-essential inclusion it is the largest element in $\mathcal{A}_t$.

Now we claim that for any measurable $B \subseteq A'_t = \Omega \backslash A_t$ we have $\nu(B) \geqslant t\mu(B)$. To show this, choose a maximal family $\mathcal{D}_B$ consisting of μ-essentially pairwise disjoint measurable subsets $D \subseteq B$ such that $\nu(D) \geqslant t\mu(D)$. Then the μ-essential supremum $\sup \mathcal{D}_B$ is in fact μ-essentially equal to B. Indeed, we clearly have $\sup \mathcal{D}_B \overset{\mu}{\subseteq} B$. Moreover, for every positive μ-measure subset $A \subseteq B \backslash \sup \mathcal{D}_B$ we have $A \notin \mathcal{D}_B$ and therefore $\nu(A) < t\mu(A)$, by maximality of $\mathcal{D}_B$. This and $\nu \ll \mu$ implies that $B \in \mathcal{A}_t$ and so $B \backslash \sup \mathcal{D}_B \overset{\mu}{\subseteq} A_t$. Since $B \subseteq A'_t$ we conclude that $\mu(B \backslash \sup \mathcal{D}_B) = 0$. Hence $B \overset{\mu}{=} \sup \mathcal{D}_B$. Using, as before, total additivity of measure, Lemma 2.4, we get

$$\nu(B) = \nu(\sup \mathcal{D}_B) = \sum_{D \in \mathcal{D}_B} \nu(D) \geqslant \sum_{D \in \mathcal{D}_B} t\mu(D) = t\mu(\sup \mathcal{D}_B) = t\mu(B).$$

This proves the claim.

It is clear that $s \leqslant t$ implies $A_s \overset{\mu}{\subseteq} A_t$. Therefore for each n the sets $A_{n,k} := A_{\frac{k+1}{2^n}} \backslash A_{\frac{k}{2^n}}$, $k = 0, 1, \ldots,$ are μ-essentially disjoint and thier union is μ-essentially equal to $A_\infty := \bigcup_{n=1}^\infty A_n$. Also functions (2.8) form μ-essentially monotonne sequence, which is convergent μ-everywhere to a measurable function $f : \Omega \to [0, \infty]$. Since $\nu \ll \mu$, the previous statements are true when "μ-essentially" replaced by "ν-essentially".

Now let $A \in \Sigma$. By the above claims for every n and k we have

$$\frac{k}{2^n}\mu(A \cap A_{n,k}) \leqslant \nu(A \cap A_{n,k}) \leqslant \frac{k+1}{2^n}\mu(A \cap A_{n,k}), \qquad n\mu(A \backslash A_n) \leqslant \nu(A \backslash A_n). \tag{2.9}$$

This implies that

$$\begin{aligned}\int_A f_n \mathrm{d}\mu &= \sum_{k=0}^{n2^n-1} \frac{k}{2^n}\mu(A \cap A_{n,k}) + n\mu(A \backslash A_n) \\ &\leqslant \sum_{k=0}^{n2^n-1} \nu(A \cap A_{n,k}) + \nu(A \backslash A_n) = \nu(A).\end{aligned}$$

Thus $\int_A f \mathrm{d}\mu = \lim_{n\to\infty} \int_A f_n \mathrm{d}\mu \leqslant \nu(A)$. In particular, if $\int_A f \mathrm{d}\mu = \infty$, then $\int_A f \mathrm{d}\mu = \nu(A) = \infty$. Hence we may assume that $\int_A f \mathrm{d}\mu < \infty$. Then for each $n \geqslant 1$ we have $n\mu(A \backslash A_\infty) \leqslant n\mu(A \backslash A_n) \leqslant \int_A f \mathrm{d}\mu < \infty$, which implies $\mu(A \cap A'_\infty) = 0$ or equivalently $A \overset{\mu}{\subseteq} A_\infty$. Then also $A \overset{\nu}{\subseteq} A_\infty$ because $\nu \ll \mu$. Hence by continuity of the measure, finite additivity applied to $A \cap A_n = \bigcup_{k=0}^{n2^n-1} A \cap A_{n,k}$, and using (2.9) we get

$$\begin{aligned}\nu(A) = \lim_{n\to\infty} \nu(A \cap A_n) &= \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \nu(A \cap A_{n,k}) \leqslant \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k+1}{2^n}\mu(A \cap A_{n,k}) \\ &\leqslant \lim_{n\to\infty} \left(\sum_{k=0}^{n2^n-1} \frac{k}{2^n}\mu(A \cap A_{n,k}) \right) + \frac{1}{2^n}\mu(A \cap A_n).\end{aligned}$$

If we assume $\mu(A) < \infty$, then $\frac{1}{2^n}\mu(A \cap A_n) \to 0$ and so the above inequalities yield

$$\nu(A) \leqslant \lim_{n\to\infty} \sum_{k=0}^{n2^n-1} \frac{k}{2^n}\mu(A \cap A_{n,k}) \leqslant \lim_{n\to\infty} \int_A f_n \mathrm{d}\mu = \int_A f \mathrm{d}\mu.$$

Accordingly, $\nu(A) = \int_A f\mathrm{d}\mu$. This proves that $f = \frac{d\nu}{d\mu}$ is the weak Radon-Nikodym derivative. For the Radon-Nikodym derivative, see Theorem 2.20. □

2.3. Set morphisms and composition operators

We now pass to a discussion of morphisms between two measure spaces. To this end, we fix two (possibly) different measure spaces $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$. We slightly generalize what is often called a *regular set isomorphism* [**Lam58**, Section 3], [**Grz85**], [**FJ03**, Definition 3.2.3]. Our definition of set morphism is equivalent to what is called *homomorphism* in [**Sou78**, Definition 2.1], *σ-homomorphism* in [**Kan78**, Definition 4.1], or *measurable set transformation* in [**Phi12**, Definition 5.4].

Definition 2.24. A *set morphism* from $(\Omega_\nu, \Sigma_\nu, \nu)$ to $(\Omega_\mu, \Sigma_\mu, \mu)$ is a map $\Phi : \Sigma_\nu \to \Sigma_\mu$ satisfying

(M1) $\nu(A) = 0 \implies \mu(\Phi(A)) = 0$ for any $A \in \Sigma$;
(M2) for any family of essentially pairwise disjoint sets $\{A_i\}_{i\in\mathbb{N}} \subseteq \Sigma_\nu$, sets $\{\Phi(A_i)\}_{i\in\mathbb{N}}$ are essentially pairwise disjoint and $\Phi(\bigcup_{i=1}^\infty A_i) \overset{\mu}{=} \bigcup_{i=1}^\infty \Phi(A_i)$.

If in addition $\Phi(\Omega_\nu) \overset{\mu}{=} \Omega_\mu$ we say that Φ is *unital*. If the implication in (M1) is the equivalence

(M3) $\nu(A) = 0 \iff \mu(\Phi(A)) = 0$ for any $A \in \Sigma_\nu$,

then we say that Φ is a *set monomorphism*. A monomorphism Φ is a *set isomorphism* if

(M4) for any $B \in \Sigma_\mu$ there exists $A \in \Sigma_\nu$ such that $B \overset{\mu}{=} \Phi(A)$.

Remark 2.25. Recall that $[\Sigma_\nu] := \Sigma_\nu / \overset{\nu}{=}$ is a σ-complete Boolean algebra with operations $[A] \vee [B] := [A \cup B]$, $[A] \wedge [B] := [A \cap B]$ and $[A]\backslash[B] := [A\backslash B]$ for $A, B \in \Sigma$. Obviously, $[\varnothing]$ and $[\Omega_\nu]$ are zero and unit in $[\Sigma_\nu]$. Condition (M1) means that a map $\Phi : \Sigma_\nu \to \Sigma_\mu$ descends to a map $[\Phi] : [\Sigma_\nu] \to [\Sigma_\mu]$ where $[\Phi]([A]) := [\Phi(A)]$, $A \in \Sigma_\nu$. If this holds then (M2) is equivalent to $[\Phi]$ being a Boolean ring homomorphism, see Lemma 2.28 below. Then Φ is unital if and only if $[\Phi]$ is a Boolean algebra homomorphism; Φ is a set monomorphism if and only $[\Phi]$ is injective, and Φ is a set isomorphism if and only if $[\Phi]$ is bijective (an isomorphism of Boolean algebras). Note that if $\Phi : \Sigma_\nu \to \Sigma_\mu$ is a set isomorphism then there is a map $\Phi^{-1} : \Sigma_\mu \to \Sigma_\nu$ such that $[\Phi]^{-1} = [\Phi^{-1}]$, so that Φ^{-1} is also a set isomorphism. In [**Lam58**], [**Grz85**] [**FJ03**], maps preserving complements, disjoint unions, and null-sets are called regular set isomorphisms. In the nomenclature of Definition 2.24 these are unital monomorphisms.

Remark 2.26. By (M2), $\mu \circ \Phi : \Sigma_\nu \to [0, \infty]$ is a measure. Hence (M2) is the absolute continuity $\mu \circ \Phi \ll \nu$, and (M3) means that the measures $\mu \circ \Phi$ and ν are equivalent.

Example 2.27. Suppose that $\varphi : \Omega_\mu \to \Omega_\nu$ is a measurable map which is *non-singular* in the sense that $\mu \circ \varphi^{-1} \ll \nu$. Then the preimage

$$\Phi(A) = \varphi^{-1}(A), \qquad A \in \Sigma_\nu,$$

defines a unital set morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$. It is a monomorphism if the measures $\mu \circ \varphi^{-1}$ and ν are equivalent. All unital set morphisms have this form for instance if (Ω_ν, Σ_ν) is a standard Borel space, see [**Roy73**, Proposition 3 on page 397], cf. also [**Fr03**, Theorem 343B].

Lemma 2.28. *Any set morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ essentially preserves unions, intersections and differences of sets. That is, for all $A, B \in \Sigma_\nu$ we have*

$$\Phi(B \cup A) \overset{\mu}{=} \Phi(B) \cup \Phi(A), \quad \Phi(B \cap A) \overset{\mu}{=} \Phi(B) \cap \Phi(A), \quad \Phi(B \backslash A) \overset{\mu}{=} \Phi(B) \backslash \Phi(A).$$

Moreover, if Φ is a set isomorphism then $\Phi(\Omega_\nu) \overset{\mu}{=} \Omega_\mu$ and $\sup_{i \in I} \Phi(A_i) \overset{\mu}{=} \Phi(\sup_{i \in I} A_i)$ for any family of ν-measurable sets $\{A_i\}_{i \in I}$ such that the essential supremum $\sup_{i \in I} A_i$ exists.

Proof. Condition (M2) implies that $\Phi(\varnothing) \overset{\mu}{=} \varnothing$ and therefore also that $\Phi(A \cup B) \overset{\mu}{=} \Phi(A) \sqcup \Phi(B \backslash A)$. In particular, Φ is monotone in the sense that $\Phi(A) \overset{\mu}{\subseteq} \Phi(B)$ if $A \subseteq B$. Hence $\Phi(B) \overset{\mu}{=} \Phi(B \backslash A) \sqcup \Phi(A \cap B)$ implies that $\Phi(B) \backslash \Phi(A) \overset{\mu}{=} \Phi(B \backslash A)$. Thus $\Phi(A \cup B) \overset{\mu}{=} \Phi(A) \sqcup \Phi(B) \backslash \Phi(A) = \Phi(A) \cup \Phi(B)$. Since $A \cap B = B \backslash (B \backslash A)$ we also get $\Phi(A \cap B) \overset{\mu}{=} \Phi(A) \cap \Phi(B)$. This proves the first part of the assertion and confirms our claims in Remark 2.25. In particular, if Φ is a set isomorphism it induces a Boolean algebra isomorphism $[\Phi] : [\Sigma_\nu] \to [\Sigma_\mu]$ and the second part of the assertion follows. ☐

We denote by $L^0(\mu)$ the complex linear space of measurable complex functions $f : \Omega \to \mathbb{C}$, where functions which agree μ-almost everywhere are identified. In fact it is a vector lattice with order defined by the cone of positive functions. In particular, the real linear space $L^0_{\mathbb{R}}(\mu)$ of measurable real valued functions is naturally ordered: $f \leqslant g$ if and only if $f(\omega) \leqslant g(\omega)$ for almost all $\omega \in \Omega$. Then $L^0_{\mathbb{R}}(\mu)$ is a lattice where $f \vee g = \max\{f, g\}$, $f \wedge g = \min\{f, g\}$, for any $f, g \in L^0_{\mathbb{R}}(\mu)$. A version of the following proposition can be found in [**FJ03**, Remark 3.2.4(v)] or in [**Phi12**, Proposition 5.6], where the authors present different constructions of the map T_Φ using pointwise convergence topology. Sourour [**Sou78**, Lemma 2.2] constructs T_Φ using convergence in measure. We give a yet another construction which mimics the construction of an integral, and allows us to define T_Φ in a formally weaker way that exploits the canonical partial order.

Proposition 2.29. *For any set morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ there is a unique linear operator $T_\Phi : L^0(\nu) \to L^0(\mu)$, which preserves limits of increasing sequences and*

$$T_\Phi(\mathbb{1}_A) = \mathbb{1}_\Phi(A), \qquad \textit{for } A \in \Sigma_\nu,$$

This operator has the following properties

1. *if $\{f_n\}$ is a sequence of measurable function convergent to f a.e., then $\{T_\Phi f_n\}$ converges to Tf μ-almost everywhere;*
2. *$T_\Phi(fg) = (T_\Phi f)(T_\Phi g)$ for all measurable functions f, g;*

(3) *T_Φ is a lattice homomorphism, that is $T_\Phi(f \vee g) = T_\Phi(f) \vee T_\Phi(g)$, $T_\Phi(f \wedge g) = T_\Phi(f) \wedge T_\Phi(g)$ and $T_\Phi|f| = |T_\Phi f|$ for any measurable functions $f, g \in L^0_{\mathbb{R}}(\mu)$.*

Proof. Let Φ be a set morphism and let $L^{00}(\nu)$ denote the set of simple functions on Ω_ν. For $f = \sum_{k=1}^N a_k \mathbb{1}_{A_k} \in L^{00}(\mu)$, where $\{A_i\}_{i=1}^N \subseteq \Sigma_\nu$ are pairwise disjoint, we put $T_\Phi f := \sum a_k \mathbb{1}_{\Phi(A_k)}$. One readily sees that this gives a well defined T_Φ linear operator on $L^{00}(\nu)$. Note that the range of the function $T_\Phi f$ is contained in the range of $f \in L^{00}(\nu)$ Now let $f \in L^0(\mu)$ be a positive measurable function. Pick any sequence $\{f_n\}_{n=1}^\infty \subseteq L^{00}(\nu)$ such that $f_n \nearrow f$ pointwise. Then $\lim_{n\to\infty} T_\Phi f_n = \sup_{n\in\mathbb{N}} T_\Phi f_n$ is in $L^0(\mu)$. We claim it is an essential supremum of the family $\{Th : h \leqslant f, h \in L^{00}(\mu)\}$ and hence does not depend on the sequence we picked. Indeed, suppose that $h \in L^{00}(\nu)$ is such that $h \leqslant f$. Take any $\alpha \in (0,1)$. Then for any $x \in \Omega_\nu$ there is $n \in \mathbb{N}$ such that $\alpha f(x) \leqslant f_n(x)$ and hence the sets $B_n = \{x : \alpha f(x) \leqslant f_n(x)\}$, $n \in \mathbb{N}$, form an increasing sequence that covers Ω_ν. Moreover, $\alpha h \mathbb{1}_{B_n} \leqslant \alpha f \mathbb{1}_{B_n} \leqslant f_n \mathbb{1}_{B_n} \leqslant f_n$ and therefore $T_\Phi(h\mathbb{1}_{B_n}) \leqslant T_\Phi(f_n)$. We also have that $T_\Phi(h\mathbb{1}_{B_n}) \nearrow T_\Phi(h)$. Indeed, for $h = \sum_{k=1}^N a_k \mathbb{1}_{A_k}$ we have

$$\begin{aligned} T_\Phi(h\mathbb{1}_{B_n}) &= \sum_{k=1}^N a_k \mathbb{1}_{\Phi(A_k \cap B_n)} = \sum_{k=1}^N a_k \mathbb{1}_{\Phi(A_k)} \mathbb{1}_{\Phi(B_n)} \\ &= T_\Phi(h)\mathbb{1}_{\Phi(B_n)} \nearrow T_\Phi(h)\mathbb{1}_{\Phi(\Omega_\nu)} = T_\Phi(h). \end{aligned}$$

Thus $\alpha T_\Phi(h) \leqslant \lim_{n\to\infty} T_\Phi(f_n)$ for all $\alpha \in (0,1)$. By passing with α to 1 we obtain $T_\Phi(h) \leqslant \lim_{n\to\infty} T_\Phi(f_n) = \sup_{n\in\mathbb{N}}\{T_\Phi(f_n)\}$. Hence this limit is the essential supremum of $\{Th : h \leqslant f, h \in L^{00}(\mu)\}$ as claimed, and we denote it by $T_\Phi(f)$.

Now extending T_Φ to all real and then to all complex measurable functions is carried out in a standard way. For any $f \in L^0_{\mathbb{R}}(\mu)$ we write $f^+ = \max\{f, 0\}$, $f^- = \max\{-f, 0\}$ to get $f = f^+ - f^-$ and then we set $T_\Phi(f) := T_\Phi(f^+) - T_\Phi(f^-)$. For a complex function $f = \mathrm{Re}(f) + i\mathrm{Im}(f)$ we put $T_\Phi f := T_\Phi(\mathrm{Re}(f)) + iT_\Phi(\mathrm{Im}(f))$. Routine arguments show that in this way we get the well defined linear operator $T_\Phi : L^0(\mu) \to L^0(\mu)$.

(1). Let $f_n \to f$ μ-almost everywhere. Assume first that all f_n are real and put $g_n := f \wedge (f_1 \vee ... \vee f_n)$, $n \in \mathbb{N}$. Then g_n is an increasing sequence by construction and it convergence to f μ-almost everywhere, by squeeze theorem. In general, one can apply the above step to the real and imaginary part.

(2). Let $f = \sum_{i=1}^{N_f} f_i \mathbb{1}_i$, $f = \sum_{j=1}^{N_g} g_i \mathbb{1}_j$ are simple functions, where $\Omega = \bigsqcup_{i=1}^{N_f} A_i = \bigsqcup_{j=1}^{N_g} B_j$. Then $T_\Phi(f_n g_n) = \sum_{i=1}\sum_{j=1} f_i g_j \mathbb{1}_{\Phi(A_i)\cap\Phi(B_j)} = T_\Phi(f_n)T_\Phi(g_n)$. Hence the statement holds for simple functions. Let f, g be measurable functions aand sequences (f_n) and (g_n) of simple functions converge to f and g respectively. Then we have $T_\Phi(f)T_\Phi(g) = \lim_n T_\Phi(f_n)T_\Phi(g_n) = \lim_n T_\Phi(f_n g_n) = T_\Phi(fg)$.

(3). Similarly, as in the proof of (2) it can be shown that the statement holds for simple functions. For any measurable functions f, g and sequences (f_n) and (g_n) of simple functions converging to f and g respectively, the sequence $(f_n \vee g_n)$ is convergent to $f \vee g$. Thus $T_\Phi f \vee T_\Phi g = \lim T_\Phi(f_n) \vee T_\Phi(g_n) = \lim T_\Phi(f_n \vee g_n) = T_\Phi(f \vee g)$. The relation $T_\Phi(f \wedge g) = T_\Phi f \wedge T_\Phi g$ follows now from the relation $f \wedge g = -((-f) \vee (-g))$. Since $|f| = f \vee (-f)$, we also have $T_\Phi(|f|) = |T_\Phi(f)|$. □

Example 2.30. If $\Phi : \Sigma_\nu \to \Sigma_\mu$ is a unital set morphism given by $\Phi(A) = \varphi^{-1}(A)$, $A \in \Sigma_\nu$, for a map $\varphi : \Omega_\mu \to \Omega_\nu$, as in Example 2.27, then

$$T_\Phi(f) = f \circ \varphi, \qquad f \in L^0(\nu),$$

is nothing but a standard composition operator.

Definition 2.31. We call the operator $T_\Phi : L^0(\nu) \to L^0(\mu)$ from Proposition 2.29 a *(generalized) composition operator* with the morphism Φ. Let $h : \Omega_\mu \to \mathbb{C}$ be a measurable function. We call the operator hT_Φ given by $[hT_\Phi]\xi(\omega) := h(\omega) \cdot (T_\Phi \xi)(\omega)$, $\xi \in L^0(\mu)$, $\omega \in \Omega_\mu$, a *weighted (generalized) composition operator*.

We finish this subsection with a simple lemma.

Lemma 2.32 (change of variables). *Let $\Phi : \Sigma_\mu \to \Sigma_\mu$ be a set morphism. A measurable function $f : \Omega_\mu \to \overline{\mathbb{R}}$ is $\mu \circ \Phi$-integrable if and only if $T_\Phi f$ is μ-integrable. Then we have*

$$\int_{\Omega_\mu} f d\mu \circ \Phi = \int_{\Omega_\mu} T_\Phi f d\mu.$$

Proof. It follows by the standard method of gradual complication of the function f. Indeed, for characteristic and simple functions the statements are clear. If f is positive, we may find a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that $f_n \nearrow f$. Using that both the integral and T_Φ preserve limits of increasing sequences we get the assertion for f. Writing arbitrary $f = f^+ - f^-$ as a difference of positive functions f^+, f^- we get the assertion by the previous step and linearity of the integral and T_Φ. □

2.4. Weighted composition operators on L^p-spaces

Let (Ω, Σ, μ) be a measure space. For $p \in [1, \infty)$ we write $L^p(\mu) = \{f : \Omega \to \mathbb{C} : \int_\Omega |f(\omega)|^p d\mu < \infty\}$ for the associated Banach space of p-integrable functions on Ω equipped with the p-norm

$$\|f\|_p = \Big(\int_\Omega |f(\omega)|^p d\mu \Big)^{\frac{1}{p}}.$$

For $p = \infty$ we denote by $L^\infty(\mu) = \{f : \Omega \to \mathbb{C} : \operatorname{ess\,sup}_{\omega\in\Omega} |f(\omega)| < \infty\}$ the Banach space of essentially bounded functions, equipped with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}_{\omega\in\Omega} |f(\omega)| = \inf\{ \sup_{\omega\in\Omega\setminus A} |f(\omega)| : A \in \Sigma \text{ has measure zero}\},$$

and we also identify functions equal almost everywhere. Then $L^p(\Omega)$ are Banach spaces for all $p \in [1, \infty]$, and $L^p(\mu)$ is a Hilbert space if and only if $p = 2$. We recall that the Banach space $L^p(\mu)$ can be recovered from $\mu : \Sigma \to [0, \infty]$, or in fact from the measure $[\mu] : [\Sigma] \to [0, \infty]$ defined on the σ-complete Boolean algebra $[\Sigma] = \Sigma/\overset{\mu}{=}$. For instance, the presentations in [**Gar21**], [**GT22**] use this 'point-free' approach to L^p-spaces. Also if $p < \infty$, the space $L^{00}(\mu)$ of simple integrable functions is dense in $L^p(\mu)$. We clearly have $L^{00}(\mu) = L^{00}(\mu_0)$ where μ_0 is the semifinite part (2.5) of μ. Thus the identity on $L^{00}(\mu) = L^{00}(\mu_0)$ extends to the isometric isomorphism $L^p(\mu) \cong L^p(\mu_0)$, and so without loss of generality we may assume that μ is semifinite. In general, if μ is semifinite, then $[\mu]$

extends uniquely to the Dedekind completion of $[\Sigma]$ giving a localizable measure on the completion [**Fr03**, Proposition 322O]. Using the Loomis–Sikorski Theorem, see for instance [**BGL22**, Theorem 7.15] we may represent this abstract measure by a concrete localizable measure space $(\overline{\Omega}, \overline{\Sigma}, \overline{\mu})$, see [**Fr03**, Theorem 322N]) and again $L^p(\mu) \cong L^p(\overline{\mu})$ canonically:

THEOREM 2.33. *For any measure μ there is a canonical localizable measure $\overline{\mu}$ such that $L^{00}(\mu) \cong L^{00}(\overline{\mu})$ by an isomorphism that sends characteristic functions to characteristic functions. Thus for any $p \in [1, \infty)$ it extends to an isometric isomorphism $L^p(\mu) \cong L^p(\overline{\mu})$.*

REMARK 2.34. Thus when considering L^p-spaces with $p < \infty$ we may always assume the measure is localizable, or even decomposable, see [**Lac74**, Corollary on page 136] or [**BGL22**, Theorem 7.17]. For $p = \infty$, the assumption that μ is localizable is also natural. For instance, μ is localizable if and only if $L^\infty(\mu)$ is canonically isomorphic to the dual of $L^1(\mu)$, see [**Fr02**, Theorem 243G]. For this reason, in this dissertation we will almost exclusively consider localizable measures.

For σ-finite measures it is a classical fact, usually attributed to Ridge [**Rid73**] that a norm (and boundendess) of a composition operator can be phrased in terms of the relevant Radon-Nikodym derivative. When appropriately modified, it is also well known to hold for weighted composition operators. In fact in [**Kan78**, Theorem 4.2] it is discussed already in the context of generalized weighted composition operators as in Definition 2.31. Let us give more details and generalize it to localizable measures. Let us then consider two measure spaces $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$, and a set morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$. For any positive measurable map $h : \Omega_\mu \to [0, \infty]$ the formula

$$\mu_{h,\Phi}(A) = \int_{\Phi(A)} h \, d\mu, \qquad A \in \Sigma_\nu, \tag{2.10}$$

defines a measure on Σ_ν. By definition of set morphism we have $\mu_{h,\Phi} \ll \nu$ and thus if ν is localizable, then the weak Radon-Nikodym derivative $\frac{d\mu_{h,\Phi}}{d\nu}$ exists by Theorem 2.20. One could call $\mu_{h,\Phi}$ the *weighted pullback measure* of μ via Φ with weight h. When $\Phi = \varphi^{-1}$ comes from a point map $\varphi : \Omega_\mu \to \Omega_n$, then $\mu_{h,\varphi^{-1}}$ is a weighted pushforward measure of μ via φ. When $h \equiv 1$, then $\mu_{h,\Phi} = \mu \circ \Phi$ is just a composition of μ and Φ.

PROPOSITION 2.35. *Let $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$ be measure spaces where ν is localizable, and let $p \in [1, \infty)$. For any morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ and a measurable map $h : \Omega_\mu \to \mathbb{C}$ the (generalized) weighted composition operator $T_\Phi : L^0(\nu) \to L^0(\mu)$ restricts to the bounded operator $hT_\Phi : L^p(\nu) \to L^p(\mu)$ if and only if the weak Radon-Nikodym derivative $\frac{d\mu_{|h|^p,\Phi}}{d\nu}$ is essentially bounded, and then*

$$\|hT_\Phi\| = \left\| \frac{d\mu_{|h|^p,\Phi}}{d\nu} \right\|_\infty^{\frac{1}{p}}.$$

PROOF. Suppose first that $hT_\Phi : L^p(\nu) \to L^p(\mu)$ is a well defined bounded operator. For any $A \in \mathcal{F}_\nu$ and any measurable $B \subseteq A$ we have

$$\int_B \frac{d\mu_{|h|^p,\Phi}}{d\nu} d\nu = \mu_{|h|^p,\Phi}(B) = \int_{\Omega_\mu} |h|^p \mathbb{1}_{\Phi(B)}\, d\mu = \int_{\Omega_\mu} |(hT_\Phi)\mathbb{1}_B|^p\, d\mu = \|(hT_\Phi)\mathbb{1}_B\|_p^p$$
$$\leqslant \|hT_\Phi\|^p \|\mathbb{1}_B\|_p^p = \|T_\Phi\|^p \nu(B).$$

This implies that $B = \{\omega \in A : \frac{d\mu_{|h|^p,\Phi}}{d\nu}(\omega) > \|T_\Phi\|^p\}$ is a ν-null set, and hence $\frac{d\mu_{|h|^p,\Phi}}{d\nu}|_A \leqslant \|T_\Phi\|^p$ ν-almost everywhere. Therefore

$$\frac{d\mu_{|h|^p,\Phi}}{d\nu} = \sup_{A\in\mathcal{F}_\nu} \frac{d\mu_{|h|^p,\Phi}}{d\nu} \mathbb{1}_A \leqslant \|hT_\Phi\|^p$$

ν-almost everywhere. Equivalently, $\|\frac{d\mu_{|h|^p,\Phi}}{d\nu}\|_\infty \leqslant \|hT_\Phi\|^p$.

Assume now that $\|\frac{d\mu_{|h|^p,\Phi}}{d\nu}\|_\infty < \infty$. Let $f \in L^p(\nu)$ be a simple function $f = \sum_{k=1}^n \lambda_k \mathbb{1}_{A_k}$, where $\{A_k\}_{k=1}^n \subseteq \mathcal{F}_\nu$ are pairwise disjoint and $\{\lambda_k\}_{k=1}^n \subseteq \mathbb{C}$. Then

$$\|T_\Phi f\|_p^p = \int_{\Omega_\mu} \big(\sum_{k=1}^n |\lambda_k|^p |h|^p \mathbb{1}_{\Phi(A_k)}\big) d\mu = \sum_{k=1}^n |\lambda_k|^p \mu_{|h|^p,\Phi}(A_k) = \sum_{k=1}^n |\lambda_k|^p \int_{A_k} \frac{d\mu_{|h|^p,\Phi}}{d\nu}\, d\nu$$
$$\leqslant \sum_{k=1}^n |\lambda_k|^p \nu(A_k) \cdot \left\| \frac{d\mu_{|h|^p,\Phi}}{d\nu} \right\|_\infty = \|f\|_p^p \cdot \left\| \frac{d\mu_{|h|^p,\Phi}}{d\nu} \right\|_\infty .$$

Hence T_Φ restricts to the bounded operator on the space of integrable simple functions $L^{00}(\nu) \subseteq L^p(\nu)$ and its norm does not exceed $\|\frac{d\mu_{|h|^p,\Phi}}{d\nu}\|_\infty^{\frac{1}{p}}$. Since $L^{00}(\nu)$ is dense in $L^p(\nu)$, T_Φ restricts to the bounded operator $T_\Phi : L^p(\nu) \to L^p(\mu)$ with $\|T_\Phi\| \leqslant \|\frac{d\mu_{|h|^p,\Phi}}{d\nu}\|_\infty^{\frac{1}{p}}$. ☐

COROLLARY 2.36 (Ridge). *Let μ, ν be a localizable measures and $p \in [1, \infty)$. For any morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ the composition operator $T_\Phi : L^0(\nu) \to L^0(\mu)$ restricts to the bounded operator $T_\Phi : L^p(\nu) \to L^p(\mu)$ if and only if the weak Radon-Nikodym derivative $\frac{d\mu\circ\Phi}{d\nu}$ is essentially bounded, and then $\|T_\Phi\| = \|\frac{d\mu\circ\Phi}{d\nu}\|_\infty^{\frac{1}{p}}$.*

The following theorem was independently formulated by Kan [**Kan78**, Theorem 4.1] for σ-finite measures and proved by Sourour [**Sou78**, Theorem 3.1] for finite measures. Now we generalize it to localizable measures. For any function $f : \Omega \to \mathbb{C}$ we denote by

$$\operatorname{supp}(f) := \{\omega \in \Omega : f(\omega) \neq 0\}$$

its support. For elements in $L^0(\mu)$ the support is defined only modulo μ-null sets. In particular $f \cdot g = 0$ in $L^0(\mu)$ means that $\mu(\operatorname{supp}(f) \cap \operatorname{supp}(g)) = 0$ in which case we say that f and g are *disjoint* (they are disjoint in the vector lattice $L^0(\mu)$).

THEOREM 2.37 (characterization of weighted composition operators on L^p-spaces). *Let μ, ν be a localizable measures and $p \in [1, \infty)$. A bounded operator $T : L^p(\nu) \to L^p(\mu)$ is a (generalized) weighted composition operator if and only if preserves disjointness in the sense that $f \cdot g = 0$ implies $(Tf) \cdot (Tg) = 0$ for all $f, g \in L^p(\nu)$.*

Moreover, if the above equivalent conditions hold then there is a presentation of $T = hT_\Phi$ which is optimal in the sense that for any other presentation $T = gT_\Psi$ we have $h = \mathbb{1}_{\mathrm{supp}(h)} g$ μ-almost everywhere and $\Phi(A) \overset{\mu}{\subseteq} \Psi(A)$ for every $A \in \Sigma_\nu$.

PROOF. Proposition 2.29(3) implies that every (generalized) weighted composition operator is disjointness preserving. Now let $T : L^p(\nu) \to L^p(\mu)$ be any linear map. Then the formula

$$\Phi_0(A) := \mathrm{supp}(T\mathbb{1}_A), \qquad A \in \mathcal{F}_\nu,$$

gives a well defined map $\Phi_0 : \mathcal{F}_\nu \to \Sigma_\mu$ such that $\mu(\Phi_0(A)) = 0$ whenever $\nu(A) = 0$. Assuming that T preserves disjointness we get that Φ_0 essentially preserves finite disjoint unions: if $A, B \in \mathcal{F}_\nu$ are disjoint, then

$$\Phi_0(A \sqcup B) = \mathrm{supp}(T\mathbb{1}_{A\sqcup B}) = \mathrm{supp}(T\mathbb{1}_A + T\mathbb{1}_B) = \mathrm{supp}(T\mathbb{1}_A) \sqcup \mathrm{supp}(T\mathbb{1}_B) = \Phi_0(A) \sqcup \Phi_0(B).$$

In particular, this implies that $\Phi_0 : \mathcal{F}_\nu \to \Sigma_\mu$ is essentially monotone, cf. Lemma 2.28. Thus using that μ is Dedekind complete we get that the essential supremum

$$\Phi(A) := \sup\{\Phi_0(A_0) : \mathcal{F}_\nu \ni A_0 \subseteq A\}, \qquad A \in \Sigma_\nu,$$

yields a well defined extension $\Phi : \Sigma_\nu \to \Sigma_\mu$ of Φ_0. This map preserves disjointness, as if $A, B \in \Sigma_\nu$ are disjoint, then every $\mathcal{F}_\nu \ni A_0 \subseteq A$ and $\mathcal{F}_\nu \ni B_0 \subseteq B$ the sets $\Phi_0(A_0)$ and $\Phi_0(B_0)$ are disjoint, because Φ_0 preserves disjointness. By arbitrariness of B_0, this implies that $\Phi(A_0)$ and $\Phi(B) = \sup\{\Phi_0(B_0) : \mathcal{F}_\nu \ni A_0 \subseteq A\}$ are disjoint. By arbitrariness of A_0, this in turn implies that $\Phi(A)$ and $\Phi(B)$ are disjoint. To see that Φ preserves countable disjoint unions, we first show it for Φ_0. Let $\{A_n\}_{n\in\mathbb{N}} \subseteq \mathcal{F}_\nu$ be pairwise disjoint and such that $\bigsqcup_{n=1}^\infty A_n \in \mathcal{F}_\nu$. We already know that $\Phi_0(\bigsqcup_{n=1}^N A_n) = \bigsqcup_{n=1}^N \Phi_0(A_n)$ for any finite $N \in \mathbb{N}$. Note that $\mathbb{1}_{\bigsqcup_{n=1}^N A_n} \nearrow \mathbb{1}_{\bigsqcup_{n=1}^\infty A_n}$ pointwise. Hence by monotone convergence theorem we get $\mathbb{1}_{\bigsqcup_{i=1}^N A_n} \xrightarrow[N\to\infty]{L^p(\nu)} \mathbb{1}_{\bigsqcup_{n=1}^\infty A_n}$ and by continuity of T also $T\mathbb{1}_{\bigsqcup_{n=1}^N A_n} \xrightarrow[N\to\infty]{L^p(\nu)} T\mathbb{1}_{\bigsqcup_{n=1}^\infty A_n}$. In particular, there is a subsequence $\{N_k\}_{k=1}^\infty$ such that $\{T\mathbb{1}_{\bigsqcup_{n=1}^{N_k} A_n}\}_{k=1}^\infty$ converges μ-almost everywhere to $T\mathbb{1}_{\bigsqcup_{n=1}^\infty A_n}$, see, for instance, [**Hal50**, Theorem D on page 92]. Hence

$$\begin{aligned}
\Phi_0(\bigsqcup_{n=1}^\infty A_n) &= \{x : T\mathbb{1}_{\bigsqcup_{n=1}^\infty A_n}(x) \neq 0\} = \{x : \lim_{k\to\infty} T\mathbb{1}_{\bigsqcup_{n=1}^{N_k} A_n}(x) \neq 0\} \\
&= \{x : \lim_{k\to\infty} \left(\sum_{n=1}^{N_k} T\mathbb{1}_{A_n}\right)(x) \neq 0\} = \mathrm{supp}\left(\sum_{n=1}^\infty T\mathbb{1}_{A_n}\right) \\
&= \bigsqcup_{n=1}^\infty \mathrm{supp}\,(T\mathbb{1}_{A_n}) = \bigsqcup_{n=1}^\infty \Phi_0(A_n).
\end{aligned}$$

Now let $\{A_n\}_{n=1}^\infty \subseteq \Sigma_\nu$ be pairwise disjoint arbitrary measurable sets. Note that

$$\Phi(\bigsqcup_{n=1}^\infty A_n) = \sup\left\{\bigsqcup_{n=1}^\infty \Phi_0(A_n^0) : \mathcal{F}_\nu \ni A_n^0 \subseteq A_n, \text{ for all } n \in \mathbb{N}\right\}.$$

Indeed, the inclusion $\subseteq$ follows because for $\mathcal{F}_\nu \ni A^0 \subseteq \bigsqcup_{n=1}^\infty A_n$ we may put $A_n^0 := A^0 \cap A_n$ to get a sequence with $\mathcal{F}_\nu \ni A_n^0 \subseteq A_n$ and $\Phi_0(A_0) = \bigsqcup_{n=1}^\infty \Phi_0(A_n^0)$ by the previous step. The reverse inclusion $\supseteq$ follows because countable unions coincide with essential suprema and hence for any sequence $\{A_n^0\}_{n=1}^\infty \subseteq \mathcal{F}_\nu$ with $A_n^0 \subseteq A_n$ we get $\bigsqcup_{n=1}^\infty \Phi_0(A_n^0) = \sup_n \Phi_0(A_n^0) \subseteq \Phi(\bigsqcup_{n=1}^\infty A_n)$. Now using the "associativity" of supremum we get

$$\Phi(\bigsqcup_{n=1}^{\infty} A_n) = \sup\left\{\sup_{n\in\mathbb{N}} \Phi_0(A_n^0) : \mathcal{F}_\nu \ni A_n^0 \subseteq A_n, \text{ for all } n \in \mathbb{N}\right\}$$

$$= \sup_{n\in\mathbb{N}} \sup\left\{\Phi_0(A_n^0) : \mathcal{F}_\nu \ni A_n^0 \subseteq A_n\right\} = \bigsqcup_{n=1}^{\infty} \Phi(A_n).$$

Hence Φ is a set morphism (it fullfils condition (M1), (M2) of Definition 2.24). Now for any $A \in \mathcal{F}_\nu$ we put $h_A := T\mathbb{1}_A$. By construction support of h_A is $\Phi_0(A)$. In particular, as Φ_0 preserves set differences, for any A, B we have

$$h_A = \mathbb{1}_{\Phi_0(A)} h_A + \mathbb{1}_{\Phi_0(A)} h_{B\setminus A} = \mathbb{1}_{\Phi_0(A)} T(\mathbb{1}_A + \mathbb{1}_{B\setminus A}) = \mathbb{1}_{\Phi_0(A)} h_{A\cup B}.$$

Similarly, $h_B = \mathbb{1}_{\Phi_0(B)} h_{A\cup B}$. Therefore $\mathbb{1}_{\Phi_0(A)\cap\Phi_0(A)} h_A = \mathbb{1}_{\Phi_0(A)\cap\Phi_0(A)} h_B$. This implies that putting $h_{\Phi(A)} := h_A$ for $A \in \mathcal{F}_\nu$ we get a well defined quasi-function $\{h_A\}_{A\in\Phi_0(\mathcal{F}_v)}$. By Proposition 2.11 there is a measurable function $h : \Omega_\mu \to \mathbb{C}$ such that $\mathbb{1}_{\Phi(A)} \cdot h = h_{\Phi(A)}$ for every $A \in \mathcal{F}_\nu$. Hence for any $A \in \mathcal{F}_\nu$ we get

$$T\mathbb{1}_A = h_{\Phi(A)} = \mathbb{1}_{\Phi(A)} \cdot h = h(T_\Phi \mathbb{1}_A) = (hT_\Phi)\mathbb{1}_A.$$

By linearity, $T = hT_\Phi$ on the space $L^{00}(\nu)$ of simple integrable functions. Since $L^{00}(\nu)$ is dense in $L^p(\nu)$, assuming that T is bounded, we get that $T = hT_\Phi$ on $L^p(\nu)$.

Now assume that $T = gT_\Psi$ for some other measurable function $g : \Omega_\mu \to \mathbb{C}$ and a morphism $\Psi : \Sigma_\nu \to \Sigma_\mu$. Then for any $A \in \mathcal{F}_\nu$ we get

$$\Phi(A) = \operatorname{supp}(T\mathbb{1}_A) = \operatorname{supp}(gT_\Psi\mathbb{1}_A) = \operatorname{supp}(g) \cap \Psi(A) \subseteq \Psi(A),$$

and also

$$h \cdot \mathbb{1}_\Phi(A) = T(\mathbb{1}_A) = g\mathbb{1}_\Psi(A) = g\mathbb{1}_\Phi(A).$$

Hence by the above construction of h and Φ we get $\Phi(A) \overset{\mu}{\subseteq} \Psi(A)$ for every $A \in \Sigma_\nu$ and $h = \mathbb{1}_{\operatorname{supp}(h)} g$ in $L^0(\mu)$. ☐

Remark 2.38. For the optimal presentation $T = hT_\Phi$ as described above we necessarily have $\operatorname{supp}(h) = \Phi(\Omega_\nu)$. Moreover, for any other presentation $T = gT_\Psi$ with the property $\operatorname{supp}(g) = \Psi(\Omega_\nu)$, it follows from the above proof that $\Phi(A) \overset{\mu}{=} \Psi(A)$ and $h\mathbb{1}_{\Phi(A)} = g\mathbb{1}_{\Psi(A)}$ μ-almost everywhere for every A in the σ-ring $\{\bigcup_{n=1}^\infty A_n : \{A_n\}_{n=1}^\infty \subseteq \mathcal{F}_\nu\}$. In particular, if ν is σ-finite, then being optimal is equivalent to $\operatorname{supp}(h) = \Phi(\Omega_\nu)$.

2.5. Banach-Lamperti theorems

We have the following measure theoretic characterization of isometric weighted composition operators, which uses the weighted transport of a measure given by (2.10).

Proposition 2.39. *Let $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$ be measure spaces where ν is localizable, and let $p \in [1, \infty)$. For any morphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ and a measurable map $h : \Omega_\mu \to \mathbb{C}$ the (generalized) weighted composition operator $hT_\Phi : L^p(\nu) \to L^p(\mu)$ is a well defined isometry if and only if $\nu = \mu_{|h|^p,\Phi}$ on the ring $\mathcal{F}_\nu = \{A \in \Sigma_\nu : \nu(A) < \infty\}$, that is*

$$\nu(A) = \int_{\Phi(A)} |h|^p d\mu, \qquad \textit{for all } A \in \mathcal{F}_\nu. \tag{2.11}$$

Proof. If $hT_\Phi : L^p(\nu) \to L^p(\mu)$ is an isometry, then for any $A \in \mathcal{F}_\nu$ we have

$$\mu_{|h|^p,\Phi}(A) = \int_{\Omega_\mu} |h|^p \mathbb{1}_{\Phi(A)}\, d\mu = \int_{\Omega_\mu} |(hT_\Phi)\mathbb{1}_A|^p\, d\mu = \|(hT_\Phi)\mathbb{1}_A\|_p^p = \|\mathbb{1}_A\|_p^p = \nu(A).$$

Hence (2.11) holds. Conversely, assume (2.11). Let $\xi = \sum_{i=1}^N \xi_i \mathbb{1}_{A_i}$ be a simple function where $\{A_i\}_{i=1}^N \subseteq \Sigma_\nu$ are pairwise disjoint. Since Φ is a morphism, the sets $\{\Phi(A_i)\}_{i=1}^N$ are essentially pairwise disjoint. By (2.11) we have $\nu(A_i) = \mu_{|h|^p,\Phi}(A_i) = \int_{\Phi(A_i)} |h(\omega)|^p d\mu$. Therefore

$$\|hT_\Phi\xi\|_p^p = \int_{\Omega_\mu} \Big|\sum_{i=1}^N \xi_i h(\omega)\mathbb{1}_{\Phi(A_i)}\Big|^p d\mu = \sum_{i=1}^N |\xi_i|^p \int_{\Phi(A_i)} |h(\omega)|^p d\mu = \sum_{i=1}^N |\xi_i|^p \nu(A_i) = \|\xi\|_p^p.$$

Hence hT_Φ acts isometrically on $L^{00}(\nu)$, and since $L^{00}(\nu)$ is dense in $L^p(\nu)$, we conclude that hT_Φ is an isometry on $L^p(\nu)$. □

Remark 2.40. Condition (2.11) is the equality of measures $\nu = \mu_{|h|^p,\Phi}$ on the whole σ-algebra Σ_ν if and only if the measure $\mu_{|h|^p,\Phi}$ is semifinite (as we assume that ν is semifinite). In general, it implies that $\nu \ll \mu_{|h|^p,\Phi} \ll \mu\circ\Phi$. Hence the morphism in the above proposition is necessary a monomorphism and we have $\nu \sim \mu_{|h|^p,\Phi} \sim \mu \circ \Phi$. In particular, the formula $(\nu \circ \Phi^{-1})(\Phi(A)) = \nu(A)$, for $A \in \Sigma_\nu$, determines a measure $\nu \circ \Phi^{-1} \sim \mu|_{\Phi(\Sigma_\nu)}$ on the σ-algebra $\Phi(\Sigma_\nu) := \{B \in \Sigma_\mu : \Phi(\Omega_\nu) \supseteq B \overset{\mu}{=} \Phi(A),\, A \in \Sigma_\nu\}$ of subsets of $\Phi(\Omega_\nu)$. If $\mu|_{\Phi(\Sigma_\nu)}$ is localizable, then the Radon-Nikodym derivative $\frac{d\nu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma_\nu)}}$ exists and (2.11) assumes the form

$$\int_{\Phi(A)} |h|^p d\mu = \int_{\Phi(A)} \frac{d\nu \circ \Phi^{-1}}{d\mu|_{\Phi(\Sigma_\nu)}} d\mu, \qquad \text{for all } A \in \mathcal{F}_\nu. \tag{2.12}$$

All this works best, when the μ is finite, as then also $\mu|_{\Phi(\Sigma_\nu)}$ is finite, and we can write the above conditision as

$$\mathbb{E}\Big(|h|^p \mid \Phi(\Sigma_\nu)\Big) = \frac{d\nu \circ \Phi^{-1}}{d\mu|_{\Phi(\Sigma_\nu)}}, \tag{2.13}$$

where $\mathbb{E}\big(|h|^p \mid \Phi(\Sigma_\nu)\big)$ is a conditional expectation of $|h|^p : \Phi(\Omega_\nu) \to \mathbb{C}$ with respect to σ-subalgebra $\Phi(\Sigma_\nu)$ of $\Sigma_\nu \cap \Phi(\Omega_\nu) := \{A \cap \Phi(\Omega_\nu) : A \in \Sigma_\nu\}$ and the measure $\mu|_{\Phi(\Sigma_\nu)}$. However, in general, even when ν is finite and μ is σ-finite, the Radon-Nikodym derivative $\frac{d\nu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma_\nu)}}$ may not exist and even if a weak Radon-Nikodym derivative exist it may not be unique and thus conditions (2.12), (2.13) make no sense, see example below.

EXAMPLE 2.41. Let μ, ν be Lebesgue measures on the spaces $\Omega_\mu = \mathbb{R}$, $\Omega_\nu = [0,1]$ equipped with σ-algebras of Borel sets. Consider a measurable map $\varphi : \Omega_\mu \to \Omega_\nu$ given by $\varphi(x) = x \pmod 1$. Then for any $A \subseteq \Sigma_\nu$ we have $\varphi^{-1}(A) = \bigsqcup_{k\in\mathbb{Z}}\{t+k : t \in A\}$. Therefore $\mu(\varphi^{-1}(A)) = 0 \iff \nu(A) = 0$ for any $A \in \Sigma_\nu$. Hence the corresponding map $\Phi : \Sigma_\nu \to \Sigma_\mu$ given by $\Phi(A) = \varphi^{-1}(A)$ is a unital set monomorphism, cf. Remark 2.30. The σ-algebra $\Phi(\Sigma_\nu)$ consists of Borel subsets on $\mathbb{R}$ which are invariant under shifts by integers, and

$$\mu(\Phi(A)) = \begin{cases} \infty, \text{ if } \nu(A) > 0 \\ 0, \text{ if } \nu(A) = 0 \end{cases} \qquad \text{for all } A \in \Sigma_\nu.$$

Thus $\mu|_{\Phi(\Sigma_\nu)}$ is not semifinite even though ν is finite and μ is σ-finite. The weak Radon-Nikodym derivative $\frac{d\nu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma_\nu)}}$ does not exist, as for any positive $\Phi(\Sigma_\nu)$-measurable function f the integrals $\int_{\Phi(A)} f d\mu$, for $A \in \Sigma_\nu$, can only attain two values: 0 or ∞. Moreover, any $\Phi(\Sigma_\nu)$-measurable function f is a weak Radon Nikodym derivative of $\nu \circ \Phi^{-1}$ with respect to $\mu|_{\Phi(\Sigma_\nu)}$, as $\mu(\Phi(A)) < \infty$ is equivalent to $\mu(\Phi(A)) = 0$ is equivalent to $\nu(A) = 0$. Nevertheless, for any $p \in [1,\infty)$ taking for instance $h := \sum_{n\in\mathbb{Z}} 2^{-\frac{|n|+2}{p}} \mathbb{1}_{[n,n+1)}$ we get a strictly positive Borel function satisfying (2.11), and hence $hT_\varphi : L^p[0,1] \to L^p(\mathbb{R})$ is an isometry. Allowing h to have zeros, it is even easier to produce isometries. We could take for instance $h = \mathbb{1}_{[0,1]}$, but then the corresponding isometry hT_φ is just an obvious embedding $L^p[0,1] \hookrightarrow L^p(\mathbb{R})$ that can also be realized by $\Phi \equiv id$ and $h \equiv 1$ (this is its optimal presentation in the sense of Theorem 2.37).

We now show that for $p \neq 2$ all isometries on L^p-spaces are weighted composition operators. In view of Theorem 2.37 this is equivalent to showing that for $p \neq 2$, isometries between L^p-spaces are automatically disjoint preserving. This follows from that an L^p-version of the parallelogram law holds if and only if the spanning vectors are disjoint. Namely, the classical parallelogram law says that

$$2(\|\xi\|_2^2 + \|\eta\|_2^2) = \|\xi+\eta\|_2^2 + \|\xi-\eta\|_2^2$$

for any $\xi, \eta \in L^2(\mu)$ (in fact this law characterizes norms coming from an inner product). Replacing 2's with p's, where $p \neq 2$, the above equality characterizes functions $\xi, \eta \in L^p(\mu)$ with disjoint supports. This fact is often attributed to Clarkson, see [**Cla36**, Theorem 2], [**Lam58**, Corollary 2.1], or [**Roy73**, Lemma 22 on page 415]:

LEMMA 2.42 (L^p-parallelogram law). *Let (Ω, Σ, μ) be a measure space. For any $\xi, \eta \in L^p(\mu)$, where $p \in [1,\infty)$, we have*

$$2(\|\xi\|_p^p + \|\eta\|_p^p) \leqslant \|\xi+\eta\|_p^p + \|\xi-\eta\|_p^p, \qquad \text{when } p \in [2,\infty),$$
$$2(\|\xi\|_p^p + \|\eta\|_p^p) \geqslant \|\xi+\eta\|_p^p + \|\xi-\eta\|_p^p \qquad \text{when } p \in [1,2].$$

For $p \neq 2$ the corresponding inequality becomes equality if and only if $\xi \cdot \eta = 0$.

THEOREM 2.43 (Lamperti theorem for localizable measures). *Let $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$ be localizable measure spaces. If $p \in [1,\infty)\backslash\{2\}$, then every linear isometry $U : L^p(\nu) \to L^p(\mu)$ is a weighted composition operator. Hence $U = hT_\Phi$ for a monomorphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ and a measurable function $h : \Omega \to \mathbb{C}$ such that $\nu = \mu_{|h|^p,\Phi}$ on $\mathcal{F}_\nu$.*

PROOF. Let $\xi, \eta \in L^p(\nu)$ have disjoint supports, that is $\xi \cdot \eta = 0$. By Lemma 2.42 we have

$$2(\|\xi\|_p^p + \|\eta\|_p^p) = \|\xi + \eta\|_p^p + \|\xi - \eta\|_p^p.$$

Since U is a linear isometry, it follows that

$$2(\|U\xi\|_p^p + \|U\eta\|_p^p) = \|U\xi + U\eta\|_p^p + \|U\xi - U\eta\|_p^p.$$

Hence $U\xi$ and $U\eta$ have disjoint supports again by Lemma 2.42. Thus by Theorem 2.37, $U = hT_\Phi$ is a weighted composition operator. By Proposition 2.39, see also Remark 2.40, $\Phi : \Sigma_\nu \to \Sigma_\mu$ is a monomorphism and $\nu = \mu_{|h|^p,\Phi}$ on $\mathcal{F}_\nu$. □

Limiting ourselves to positive isometries we can cover also the case $p = 2$, which is not usually considered in this context. The reason is that for positive functions the "L^p-version of the Pythagorean theorem" holds if and only if the functions have disjoint supports. Namely, it is straightforward to see that for any positive functions $\xi, \eta \in L^p(\mu)^+$ and any $p \in [1, \infty)$, we have

$$\xi \cdot \mu = 0 \iff \|\xi + \eta\|_p^p = \|\xi\|_p^p + \|\eta\|_p^p \tag{2.14}$$

(Bohnenblust used equivalence (2.14) to characterize L^p-spaces amongst all Banach lattices, see [**Lac74**, Theorem 3 on page 135]). An operator on $L^p(\mu)$ is *positive* if it maps positive functions to positive functions.

THEOREM 2.44 (Positive isometries on L^p-spaces). *Let $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$ be localizable measure spaces. For every $p \in [1, \infty)$, every a linear operator $U : L^p(\nu) \to L^p(\mu)$ is a positive isometry if and only if is a weighted composition operator $U = hT_\Phi$ for a positive measurable function $h : \Omega \to [0, \infty)$ and a homomorphism $\Phi : \Sigma_\nu \to \Sigma_\mu$ and such that $\nu = \mu_{h^p,\Phi}$ on $\mathcal{F}_\nu$.*

PROOF. Clearly, a weighted composition operator hT_Φ is positive if and only if $h \geqslant 0$ is a positive function. Hence the "if" part follows from Proposition 2.39. Conversely, assume that $U : L^p(\nu) \to L^p(\mu)$ is a positive isometry. Let $\xi, \eta \in L^p(\nu)^+$ have disjoint supports. By (2.14) we have

$$\|\xi + \eta\|_p^p = \|\xi\|_p^p + \|\eta\|_p^p.$$

Since U is a linear isometry, it follows that

$$\|U\xi + U\eta\|_p^p = \|U\xi\|_p^p + \|U\eta\|_p^p.$$

Hence using that U is positive, (2.14) implies that $U\xi$ and $U\eta$ have disjoint supports. Thus by Theorem 2.37 and Proposition 2.39, $U = hT_\Phi$ is a weighted composition operator with the properties as described in the assertion. □

Finally, we pass to the description of invertible isometries. This simplifies greatly because, as we show below, the associated morphism Φ is necessarily an isomorphism of the associated σ-algebras. Example 2.41 shows that the measures $\nu \circ \Phi^{-1}$ and $\mu|_{\Phi(\Sigma_\nu)}$ induced by an arbitrary set morphism are not suitable for the existence of Radon-Nikodym derivative $\frac{d\nu\circ\Phi^{-1}}{d\mu|_{\Phi(\Sigma_\nu)}}$. However, if $\Phi : \Sigma_\nu \to \Sigma_\mu$ is a set isomorphism, then $\mu = \mu|_{\Phi(\Sigma_\nu)}$ (because $\Phi(\Sigma_\nu) = \Sigma_\mu$) and obstacles as in Example 2.41 cannot occur.

LEMMA 2.45. *Let $\Phi : \Sigma_\nu \to \Sigma_\mu$ be a set isomorphism between two localizable measure spaces $(\Omega_\mu, \Sigma_\mu, \mu)$ and $(\Omega_\nu, \Sigma_\nu, \nu)$. Then the Radon-Nikodym derivatives $\frac{d\nu}{d\mu\circ\Phi}$, $\frac{d\nu\circ\Phi^{-1}}{d\mu}$ exist, take values in $(0,\infty)$, and*

$$T_\Phi\left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right) = \frac{d\nu}{d\mu\circ\Phi}.$$

We also have $\left(\frac{d\nu}{d\mu\circ\Phi}\right)^{-1} = \frac{d\mu\circ\Phi}{d\nu}$, $\left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right)^{-1} = \frac{d\mu}{d\nu\circ\Phi^{-1}}$ and $T_\Phi\left(\frac{d\mu\circ\Phi}{d\nu}\right) = \frac{d\mu}{d\nu\circ\Phi^{-1}}$.

PROOF. Since Φ is an isomorphism and μ is localizable, it is immediate that $\mu\circ\Phi$ localizable and equivalent to ν. Hence $\frac{d\nu}{d\mu\circ\Phi}$ and $\frac{d\mu\circ\Phi}{d\nu}$ exist, , take values in $(0,\infty)$, and $\left(\frac{d\nu}{d\mu\circ\Phi}\right)^{-1} = \frac{d\mu\circ\Phi}{d\nu}$ by Corollary 2.22. Similar claims for $\frac{d\nu\circ\Phi^{-1}}{d\mu}$ and $\frac{d\nu\circ\Phi^{-1}}{d\mu}$ follow by the same argument but applied to the inverse isomorphism $\Phi^{-1}:\Sigma_\nu\to\Sigma_\mu$.

Now let A_t, $t\in\mathbb{R}$ and f_n, $n\in\mathbb{N}$, be objects in Theorem 2.23 associated to measures $\nu\circ\Phi^{-1} \ll \mu$. Similarly, let us denote by B_t, $t\in\mathbb{R}$ and g_n, $n\in\mathbb{N}$, the corresponding objects associated to measures $\nu \ll \mu\circ\Phi$. It is clear by the construction that $\Phi(A_t) = B_t$, and therefore $T_\Phi(f_n) = g_n$, $n\in\mathbb{N}$. Therefore, by Theorem 2.23 and construction of T_Φ, we have

$$T_\Phi\left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right) = \lim_{n\to\infty} T_\Phi(f_n) = \lim_{n\to\infty} g_n = \frac{d\nu}{d\mu\circ\Phi}.$$

Similar arguments show that $T_\Phi\left(\frac{d\mu\circ\Phi}{d\nu}\right) = \frac{d\mu}{d\nu\circ\Phi^{-1}}$. □

We now generalize the classical Banach characterization of invertible isometries to localizables measures. For $p\in(1,\infty)\backslash\{2\}$ it is proved in [**GT22**, Theorem 3.7]. We consider $p\in[1,\infty)\backslash\{2\}$ and for positive isometries we also allow $p=2$. For finite measures this is also proved in [**Czy01**, Theorem 5.1].

THEOREM 2.46 (Banach-Lamperti theorem for localizable measures). *Let μ and ν be localizable measures and let $p\in[1,\infty)$. A linear operator $U: L^p(\nu)\to L^p(\mu)$ is a positive invertible isometry if and only if $U = U_\Phi$ where*

$$U_\Phi := \left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right)^{\frac{1}{p}} T_\Phi$$

and $\Phi:\Sigma_\nu\to\Sigma_\mu$ be a set isomorphism from $(\Omega_\nu,\Sigma_\nu,\nu)$ to $(\Omega_\mu,\Sigma_\mu,\mu)$ (then we have $U_\Phi^{-1} = U_{\Phi^{-1}}$). If $p\neq 2$, then U is an invertible isometry (not necessarily positive) if and only if $U=\omega U_\Phi$ where $\omega:\Omega_\mu\to\mathbb{T}$ is measurable function.

PROOF. Let $\Phi:\Sigma_\nu\to\Sigma_\mu$ be a set isomorphism from $(\Omega_\nu,\Sigma_\nu,\nu)$ to $(\Omega_\mu,\Sigma_\mu,\mu)$. Putting $h = \left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right)^{\frac{1}{p}}$ we get $\nu = \mu_{|h|^p,\Phi}$. Hence U_Φ is an isometry by Proposition 2.39. To show that U_Φ is invertible note that for the isomorphism $\Phi^{-1}:\Sigma_\mu\to\Sigma_\nu$ inverse to Φ we have an isometry $U_{\Phi^{-1}} = \left(\frac{d\mu\circ\Phi}{d\nu}\right)^{\frac{1}{p}} T_{\Phi^{-1}}$. By symmetry, it is sufficient to show that $U_\Phi U_{\Phi^{-1}} = 1$. By Lemma 2.45 we get

$$U_\Phi U_{\Phi^{-1}} = \left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right)^{\frac{1}{p}} T_\Phi \left(\frac{d\mu\circ\Phi}{d\nu}\right)^{\frac{1}{p}} T_{\Phi^{-1}} = \left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right)^{\frac{1}{p}}\left(\frac{d\mu}{d\nu\circ\Phi^{-1}}\right)^{\frac{1}{p}} T_\Phi T_{\Phi^{-1}} = 1.$$

Thus U_Φ is surjective and hence invertible, with $U_{\Phi^{-1}}$ being the inverse. Since multiplication by a measurable function $\omega : \Omega_\mu \to \mathbb{T}$ is clearly an invertible isometry the operator ωU_Φ is also an invertible isometry, with the inverse $(\omega U_\Phi)^{-1} = T_{\Phi^{-1}}(\overline{\omega})U_{\Phi^{-1}}$.

Let now U be an arbitrary invertible isometry, and assume that either $p \neq 2$ or U is positive. Then by Theorems 2.43 and 2.44, we necessarily have $U = hT_\Phi$ where $\Phi : \Sigma_\nu \to \Sigma_\mu$ is a set monomorphism and $h : \Omega_\mu \to \mathbb{C}$ are such that $\nu(A) = \int_{\Phi(A)} |h|^p d\mu$ for $A \in \Sigma_\nu$. If we show that Φ is a set isomorphism, then this last relation implies (is equivalent to) the equality $|h|^p = \frac{d\nu\circ\Phi^{-1}}{d\mu}$, see Remark 2.40, and so $U = \omega U_\Phi$ where $\omega = h\left(\frac{d\nu\circ\Phi^{-1}}{d\mu}\right)^{-\frac{1}{p}}$. Thus we only need to show that Φ is surjective, as then it is a set isomorphism. As U is invertible we need to have $\mathrm{supp}(h) = \Omega_\mu$ and so both h and Φ are uniquely determined by U, see Theorem 2.37. In fact, for any $A \in \Sigma_\nu$ we have

$$\Phi(A) = \sup\{\mathrm{supp}(U\mathbb{1}_{A_0}) : \mathcal{F}_\nu \ni A_0 \subseteq A\} = \sup\{\mathrm{supp}(Uf) : \mathrm{supp}(f) \subseteq A, f \in L^p(\nu)\},$$

where the suprema are μ-essential, see the proof of Theorem 2.37. Therefore, a natural candidate for the inverse to Φ is $\Psi : \Sigma_\mu \to \Sigma_\nu$ defined by the ν-essential supremrum

$$\Psi(B) := \sup\{\mathrm{supp}(U^{-1}g) : \mathrm{supp}(g) \subseteq B, g \in L^p(\mu)\}, \quad B \in \mathcal{F}_\mu.$$

For any $B \in \Sigma_\mu$ we have

$$\begin{aligned}\Phi(\Psi(B)) &= \sup\{\mathrm{supp}(Uf) : \mathrm{supp}(f) \subseteq \Psi(B), f \in L^p(\nu)\}\\ &= \sup\{\mathrm{supp}(Uf) : \mathrm{supp}(f) \subseteq \mathrm{supp}(U^{-1}g), \mathrm{supp}(g) \subseteq B, f \in L^p(\nu), g \in L^p(\mu)\}\\ &= \sup\{\mathrm{supp}(UU^{-1}g : \mathrm{supp}(g) \subseteq B, g \in L^p(\mu)\}\\ &= \sup\{\mathrm{supp}(g) : \mathrm{supp}(g) \subseteq B, g \in L^p(\mu)\} = B.\end{aligned}$$

This completes the proof. □

Remark 2.47. Following Phillips [**Phi12**] we call the invertible isometries of the form ωU_Φ *spatial*, then both ω and Φ are essentially determined by the operator ωU_Φ, cf. the last part of Theorem 2.37. The above Banach-Lamperti theorem says that all positive invertible isometries are spatial and if $p \neq 2$, then all invertible isometries are spatial. Assume that $\mu = \nu$. For any $p \in [1, \infty)$ the spatial invertible isometries on $L^p(\mu)$ form a group and we have

$$(\omega U_\Phi)^{-1} = T_{\Phi^{-1}}(\overline{\omega})U_{\Phi^{-1}}, \qquad (\omega U_\Phi) \circ (\upsilon U_\Psi) = \omega T_\Phi(\upsilon)U_{\Phi\circ\Psi}, \tag{2.15}$$

cf. the proof of Theorem 2.46. This group is naturally isomorphic to the semi-direct product

$$UL^\infty(\mu) \rtimes \mathrm{Aut}([\Sigma_\mu])$$

where the group $\mathrm{Aut}([\Sigma_\mu])$ of automorphisms of the Boolean algebra $[\Sigma_\mu]$ acts on the group $UL^\infty(\mu) := \{\omega \in L^\infty(\mu) : |\omega| = 1\}$ of unitaries in the von Neumann algebra $L^\infty(\mu)$ by the formula $[\Phi]\omega = T_\Phi(\omega)$ for $[\Phi] \in \mathrm{Aut}([\Sigma_\mu])$ and $\omega \in UL^\infty(\mu)$. Positive invertible isometries on $L^p(\mu)$ form a subgroup isomorphic to $\mathrm{Aut}([\Sigma_\mu])$, cf. [**GT22**, Theorem 3.7], [**BKM25**, Proposition 2.4].

EXAMPLE 2.48. Consider $\ell^p(\{1,2\}) \cong \mathbb{C}^2$ with norm $\|(x,y)\|_p = \sqrt[p]{|x|^p + |y|^p}$. Every non-zero (equivalently invertible) isometry $U : \ell^p(\{1,2\}) \to \ell^p(\{1,2\})$ for $p \neq 2$ is of the form

$$\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & \lambda_1 \\ \lambda_2 & 0 \end{pmatrix},$$

where $|\lambda_1| = |\lambda_2| = 1$, that is $(\lambda_1, \lambda_2) \in \mathbb{T}^2$. Accordingly, $U\ell^\infty(\{1,2\}) = \mathbb{T}^2$ and the only non-trivial bijection on $\{1,2\}$ is the flip $1 \mapsto 2$, $2 \mapsto 1$. In particular, the group of invertible isometries is isomorphic to the semidirect product $\mathbb{T}^2 \rtimes \mathbb{Z}_2$ where the action of $\mathbb{Z}_2$ on the torus $\mathbb{T}^2$ is given by the flip $(\lambda_1, \lambda_2) \mapsto (\lambda_2, \lambda_1)$. However, for $p = 2$ all non-zero isometries (equivalently unitary matrices) are of the form

$$U = \begin{pmatrix} \lambda_1 \cos\theta & \lambda_1' \sin\theta \\ -\lambda_2' \sin\theta & \lambda_2 \cos\theta \end{pmatrix}, \quad \lambda_1\lambda_2 = \lambda_1'\lambda_2', \qquad \lambda_1, \lambda_2, \lambda_1', \lambda_2' \in \mathbb{T}^2, \theta \in [0, \pi).$$

Thus they are a "mixture" of the previous two types of unitary matrices and a rotation matrix corresponding to an angle $\theta \in [0, \pi)$. Such an isometry is spatial if and only if $\theta = 0$ or $\theta = \pi/2$. Independently of $p \in [1, \infty)$ there are only two positive non-zero isometries on $\ell^p(\{1,2\})$ and they are given by the matrices

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

2.6. Weighted composition operators on C_0-spaces and L^∞-spaces

We now explain that our final Banach-Lamperti theorem can be extended to the case when $p = \infty$, and in a sense this is a special case of Banach-Stone theorem, as $L^\infty(\mu)$ is isometrically isomorphic to the Banach algebra $C(M)$ of continuous functions on some compact Hausdorff space M. In fact since $L^\infty(\mu)$ is a commutative C^*-algebra we have isomorphism of C^*-algebras $L^\infty(\mu) \cong C(M)$ by Gelfand-Naimark theorem. So up to an isometric isomorphism we may treat L^∞-spaces as examples of C-spaces. In fact we find it useful to consider even more general C_0-spaces. Namely, for any locally compact Hausdorff space M we denote by $C_0(M)$ the Banach space of of complex continuous functions on M that vanish at infinity, i.e. $f \in C_0(M)$ if $f : M \to \mathbb{C}$ is continuous and for every $\varepsilon > 0$ the set $\{t \in M : |f(t)| \geqslant \varepsilon\}$ is compact. Then $C_0(M)$ is a Banach space equipped with the supremum norm $\|f\|_\infty := \sup_{t \in M} |f(t)|$. It is a completion of the space $C_c(M)$ of continuous compactly supported functions on M in $\|\cdot\|_\infty$.

Interestingly enough, there is an analogue of Theorem 2.37 that characterizes weighted composition operators on C_0-spaces as operators preserving disjointness. Namely, we denote by $\operatorname{supp}(f) = \{t \in M : f(t) \neq 0\}$ the open suport of $f \in C_0(M)$, an we say that an operator $T : C_0(M) \to C_0(N)$ *preserves disjointness* if

$$\operatorname{supp}(f) \cap \operatorname{supp}(g) \neq \varnothing \implies \operatorname{supp}(Tf) \cap \operatorname{supp}(Tg) = \varnothing.$$

The proof of the following theorem is based on the reasoning from [**Are83**, Example 2.2].

THEOREM 2.49 (characterization of weighted composition operators on C_0-spaces). *Let M, N be locally compact Hausdorff spaces. A bounded operator $T : C_0(M) \to C_0(N)$*

preserves disjointness if and only if T is a weighted composition operator, i.e. there is a continuous map $\varphi : N_0 \to M$ defined on an open subset $N_0 \subseteq N$ and a continuous bounded function $h : N_0 \to \mathbb{C}\backslash\{0\}$ such that

$$Tf(x) = \begin{cases} h(x)f(\varphi(x)), & x \in N_0 \\ 0, & x \notin N_0 \end{cases}, \qquad f \in C_0(M),\ x \in N.$$

PROOF. Assume T preserves disjointness. We use that identifying points with point masses (Dirac measures) we get the embedding $N \ni x \mapsto \delta_x \in C_0(N)'$ of N into the space of complex Radon measures on N. Let $N_0 := \{x \in N : T'\delta_x \neq 0\}$. Since T' is continuous this set is open in $N \cong \{\delta_x : x \in N\}$. We claim that for any $x \in N_0$ the measure $\mu := T'\delta_x \neq 0$ is accumulated on a point. Equivalently the support of the variation $|\mu|$ is a singleton. Indeed, if this is not the case, then there are positive functions $f_1,\ f_2 \in C_0(M)^+$ with disjoint supports such that $|\mu|(f_1) > 0$ and $|\mu|(f_2) > 0$. For any positive $f \in C_0(M)^+$ we have $|\mu|(f) = \sup\{|\mu(g)| : g \in C_0(M),\ |g| \leqslant f\}$. Hence we may find $g_i \in C_0(M)$ such that $|g_i| \leqslant f_i$ and $|\mu(g_i)| > 0$ for $i = 1, 2$. Thus g_1 and g_2 have disjoint supports but

$$|(Tg_i)(x)| = |\delta_x(Tg_i)| = |(T'\delta_x)(g_i)| = |\mu(g_i)| > 0, \quad i = 1, 2,$$

so the Tg_1 and Tg_2 do not have disjoint supports, which contradicts our assumption.

Thus for every $x \in N_0$ there is a unique point $\varphi(x) \in M$ and a non-zero number $h(x) \in \mathbb{C}\backslash\{0\}$ such that $T'\delta_x = h(x)\delta_{\varphi(x)}$ This defines the map $\varphi : N_0 \to M$ and a function $h : N_0 \to \mathbb{C}\backslash\{0\}$ such that for $f \in C_0(M)$ we have

$$(Tf)(x) = (T'\delta_x)(f) = \begin{cases} h(x)f(\varphi(x)), & x \in N_0 \\ 0, & x \notin N_0 \end{cases}.$$

We first show that φ is continuous any at $x_0 \in N_0$. Indeed, pick an open neighborhood V of $\varphi(x_0)$. By Urysohn lemma there is continuous $f : M \to [0, 1]$ such that $f(\varphi(x_0)) = 1$ and $\operatorname{supp}(f) \subseteq V$. Then $Tf(x_0) = h(x_0) \neq 0$ and $Tf(x) = 0$ if $x \notin \varphi^{-1}(V)$. Putting $U := \{x \in N_0 : |Tf(x) - Tf(x_0)| < |h(x_0)|/2\}$ we have an open neighoboorhood of x_0 such that $Tf(x) \neq 0$ for $x \in U$. Hence $U \subseteq \varphi^{-1}(V)$. This proves continuity of φ.

Any $x_0 \in N_0$ has an open neighbourhood V such that $\overline{V} \subseteq N_0$ is compact. By Urysohn lemma there is function $f \in C_0(M)$ which is 1 on the compact set $\varphi(\overline{V}) \subseteq M$. Then $(Tf)(x) = h(x)$ for all $x \in \overline{V}$, and since $Tf \in C_0(N)$ we see that h is continuous on $\overline{V}$. This proves continuity h. Now boundedness of the operator T readily implies that h is bounded. □

In view of the previous subsection it is tempting to prove Banach-Stone theorem by proving first that every invertible isometry $T : C_0(M) \to C_0(N)$ is necessarily disjoint preserving (by finding some C_0-analogue of the L^p-parallelogram law, see Lemma 2.42) However, we are not aware of proof like this. For a history and a general Banach-Stone theorem that we now state we refer to [**AF97**].

THEOREM 2.50 (Banach-Stone theorem). *Let M, N be locally compact Hausdorff spaces. A linear operator $T : C_0(M) \to C_0(N)$ is an invertible isometry if and only if T is a* *weighted composition operator*

$$Tf(x) = h(x)f(\varphi(x)), \qquad f \in C_0(M), x \in N$$

where $\varphi : N \to M$ is a homeomorphism and $h : N \to \mathbb{T}$ is a continuous function.

COROLLARY 2.51 (Banach–Lamperti theorem for $p = \infty$). *Let $(\Omega_\nu, \Sigma_\nu, \nu)$ and $(\Omega_\mu, \Sigma_\mu, \mu)$ be localizable measure spaces. A linear operator $U : L^\infty(\nu) \to L^\infty(\mu)$ is an invertible isometry if and only if*

$$U := \omega T_\Phi$$

where $\Phi : \Sigma_\nu \to \Sigma_\mu$ is a set isomorphism and $\omega : \Omega_\mu \to \mathbb{T}$ is a measurable function. Let be localizable measures. Then every invertible isometry from $L^p(\mu)$ to $L^p(\nu)$ is spatial.

PROOF. It is clear that the operator ωT_Φ as described above is an invertible isometry. Let us now show that any $U : L^\infty(\nu) \to L^\infty(\mu)$ is an invertible isometry is of this form. By Gelfand theorem we have canonical isomorphisms $L^\infty(\mu) \cong C(N)$ and $L^\infty(\nu) \cong C(M)$ where N and M are compact Hausdorff (hyperstonean) spaces. This allows us to transport U to the invertible isometry $T : C(M) \to C(N)$. By Stone-Banach theorem, T is of the form hT_φ where $h \in C(N)$, $|h| = 1$, and $T_\varphi : C(M) \to C(N)$ is a composition operator with a homeomorphism $\varphi : N \to M$ is a homeomorphism. Thus $T_\varphi : C(M) \to C(N)$ is in fact an isometric algebra isomorphism. The Boolean algebras of idempotent elements in $C(M)$ and $C(N)$ are canonically isomorphic to $[\Sigma_\nu]$ and $[\Sigma_\mu]$, respectively. Thus T_φ induces an isomorphism $[\Sigma_\nu] \cong [\Sigma_\mu]$ which we interpret as a set isomorphism $\Phi : \Sigma_\nu \to \Sigma_\mu$. Namely, the action of $T_\Phi : L^\infty(\nu) \to L^\infty(\mu)$ on idempotents in $L^\infty(\nu)$ agrees with the action of T_φ on the corresponding idempotents in $C(M)$. As the idempotents are linearly dense in the considered spaces we conclude that the isometry $U : L^\infty(\nu) \to L^\infty(\mu)$ corresponding to hT_φ is of the form ωT_Φ where $\omega \in L^\infty(\mu)$ is the function corresponding to h under the isomorphism $C(N) \cong L^\infty(\mu)$. $\square$

CHAPTER 3

Representations of Banach algebras and partial isometries

In this chapter, we establish terminology and recall basic facts about Banach algebras and hermitian operators. In addition, based on [**BKM25**, Section 2], we characterize representations of the algebra $C_0(X)$ and Moore-Penrose partial isometries on L^p-spaces associated with localizable measures, which leads to what we call Generalized Banach-Lamperti theorem (Theorem 3.26). We assume that the reader has basic knowledge about Banach algebras and C^*-algebras, as in three first chapters of [**Mur90**].

3.1. Banach algebra fundamentals

A *Banach algebra* is an associative complex algebra A which is a Banach space where the norm is submultiplicative, i.e. $\|ab\| \leqslant \|a\|\|b\|$, for all $a, b \in A$. The set of bounded operators $B(E)$ on a Banach space E form a Banach algebra with the operator norm, pointwise linear structure and composition given by multiplication. Any norm closed subalgebra of $B(E)$ is a Banach algebra, and up to isometric isomorphisms all Banach algebras are of this form.

A C^**-algebra* is a Banach algebra A which is equipped with an anti-linear and anti-multiplicative involution $* : B \to B$ which satisfies the so-called C^*-equality $\|a\|^2 = \|a^*a\|$ for all $a \in A$. The algebra $B(H)$ of bounded operators on a Hilbert space H is a C^*-algebra with involution given by the hermitian adjoint. Any norm closed $*$-subalgebra of $B(H)$ is a C^*-algebra, and up to an isometric $*$-isomorphism all C^*-algebras are of this form.

All these are standard facts. What is slightly less known is that an algebra homomorphism $A \to B$ between two C^*-algebras is $*$-preserving if and only if it is contractive, see [**BM04**, A.5.8]. This suggests us that in the category of Banach algebras, where we do not have involution, natural morphisms are contractive morphisms. In particular, this motivates the following:

Definition 3.1. A *representation* of a Banach algebra A in a Banach algebra B is a contractive homomorphism $\pi : A \to B$. A *representation of A on a Banach space E* is a representation $\pi : A \to B(E)$ in the Banach algebra $B(E)$.

By an *ideal* in a Banach algebra A (unless stated otherwise) we mean a closed two-sided ideal I. Then the quotient A/I with the standard quotient operations and quotient norm is a Banach algebra. In particular, for every representation $\pi : A \to B$ its kernel $\ker(\pi)$ is an ideal in A, and π descends to an injective representation $A/\ker(\pi) \to B$. When A is a C^*-algebra, then I is necessarily a C^*-algebra and A/I is naturally a C^*-algebra.

One of the most important features of a C^*-norm is that it is minimal. This was already noted by Kaplansky [**Kap49**] and an elegant proof can be found in [**Bon54**, Theorem 10]:

Proposition 3.2 (Kaplansky-Bonsall). *Let A be a C^*-algebra. The C^*-norm on A is a minimal norm among all submultiplicative norms on A.*

Corollary 3.3. *Every representation $\pi : A \to B$ of a C^*-algebra A in a Banach algebra B descends to an isometric representation $\pi_0 : A/\ker(\pi) \to B$. In particular,*

1) $\pi : A \to B$ is injective if and only if it is isometric;

2) the range of π is closed (so $\pi(A)$ is a Banach subalgebra of B).

Proof. Assume first that $\pi : A \to B$ is injective. Then $A \ni a \mapsto \|\pi(a)\|$ is a submultiplicative norm on A. Hence $\|a\| \leqslant \|\pi(a)\|$ by Kaplansky-Bonsall theorem. Since $\|\pi(a)\| \leqslant \|a\|$ by definition of representation we conclude that π is isometric. This proves 1). Now let $\pi : A \to B$ be any representation. It descends to an injectvie representation $\pi_0 : A/\ker(\pi) \to B$, and as $A/\ker(\pi)$ is a C^*-algebra, we have by 1) that π_0 is isometric. A range of an isometry on complete space is complete and hence closed. Thus $\pi(A) = \pi_0(A/\ker(\pi))$ is closed in B. Hence 2) holds. ☐

The above two properties in Corollary 3.3, C^*-algebraists take for granted and their use in the theory of C^*-algebras cannot be overestimated. For Banach algebras these two properties usually fail. This constitutes a major difficulty and obstacle in attempts to generalize C*-algebraic results to more general Banach algebras.

Since Hilbert spaces, up isometric isomorphisms, are nothing but L^2-spaces $L^2(\mu)$, the L^p-spaces $L^p(\mu)$ for $p \in [1, \infty)$ can be imagined as “tilted Hilbert spaces”. This seems to be one of the reasons why, e.g. in Harmonic Analysis and in particular in the group representation theory, operator algebras on such spaces are often considered, see for instance [**Her71**], [**Run02**], [**GT15**], [**Gar21**] and references therein. In [**Phi13b**, Definition 1.1] Phillips gave a name to Banach algebras that are isometrically isomorphic to subalgebras of $B(L^p(\mu))$ for some measure μ and $p \in [1, \infty)$. We extend his to definition to cover also L^∞-spaces and C_0-spaces, cf. [**BKM25**, Remark 2.17].

Definition 3.4. Let $p \in [1, \infty]$. We say that a Banach algebra A is an *L^p-operator algebra* if there exists an isometric representation $\pi : A \to B(L^p(\mu))$ for some measure μ. We say that A is a *C_0-operator algebra* if there is an isometric representation $\pi : A \to B(C_0(\Omega))$ for locally compact Hausdorff space Ω.

Remark 3.5. It is a simple but fundamental observation that the commutative C^*-algebra $C_0(X)$ is an L^p-operator algebra for any $p \in [1, \infty]$, as well as a C_0-operator algebra. In fact, by [**GT20**, Theorem 5.3], if $p \in [1, \infty)\backslash\{2\}$, then an L^p-operator algebra A is a C^*-algebra if and only if A is commutative, and therefore $A \cong C_0(X)$. We also note that any L^∞-operator algebra is also naturally C_0-operator algebra. Indeed, any space $L^\infty(\mu)$ is naturally a unital C^*-algebra and therefore it is isometrically isomorphic to the space (in fact algebra) $C_0(\Omega) = C(\Omega)$ where Ω is a compact Hausdorff space.

We will be mostly interested in non-degenerate representations.

Definition 3.6. We say that a representation $\pi : A \to B(E)$ of a Banach algebra A on a Banach space E is *non-degenrate* if $\overline{\pi(A)E} = E$ where $\overline{\pi(A)E}$ denotes the closed linear span of $\pi(A)E = \{\pi(a)\xi : a \in A, \xi \in E\}$.

Remark 3.7. If A has a left approximate unit then by Cohen–Hewitt factorization theorem $\overline{\pi(A)E} = E$ if and only if $\pi(A)E = E$, see [**Kis20**]. Any L^p-operator algebra A, for $p \in (1, \infty)$, which has an approximate unit has an isometric, non-degenerate representation $\pi : A \to B(L^p(\mu))$, see [**GT20**, Theorem 4.3]. In our considerations we will consider only contractive two-sided approximate units. So by an *approximate unit* we mean a net $(e_i)_i$ of elements in the closed unit ball A_1 in A such that $\|ae_i - a\|, \|e_i a - a\| \to 0$ for any $a \in A$. If A has such an approximate unit, we say that A is *approximately unital*, and then the canonical representation given by multiplication yields an isometric inclusion $A \subseteq B(A)$. We use it to define the minimal unitization of A as the Banach algebra $\widetilde{A}$ generated by A and the unit operator in $B(A)$, cf. [**BP19**, Definition 1.8 and Remark 1.9].

A well established notion of a hermitian operator in a C^*-algebra has a natural generalization to (approximately) unital Banach algebras that can be formulated in a number of equivalent ways, see [**BD71**, page 46], here we define it using the exponent function as it is the quickest and most convenient for our purposes. Also following [**BP19**, Definition 2.8] we define this notion for approximately unital Banach algebras using the minimal unitization.

Definition 3.8. Let B be a Banach algebra with an approximate unit. An element $b \in B$ is *hermitian* if $\|e^{itb}\| = 1$ for all $t \in \mathbb{R}$, where the expontent $e^{itb} = \sum_{n=0}^{\infty} \frac{(itb)^n}{n!}$ is calculated in the minimal unitization $\widetilde{B}$.

Remark 3.9. If B is a C^*-algebra then $b \in B$ is hermitian in the sense above if and only if b is self-adjoint (hermitian in the usual sense) that is $b = b^*$. In general, the definition of a hermitian element does not depend on the ambient algebra as long as we keep the approximate unit:

> if A is a Banach subalgebra of B generated by $a \in B$ and an approximate unit in B, then a is hermitian in B if and only if a is hermitian in A,

see [**BP19**, Lemma 2.10]. Spectral properties of hermitian operators in Banach algebras are similar to those in C^*-algebras. For instance, the spectrum of any hermitian $b \in B$ is real (see [**BD71**, Theorem 6 on page 19]), and by the result of Sinclair [**Sin71**] its spectral radius is equal to the norm of b. Thus

$$\sigma_B(b) := \{\lambda \in \mathbb{C} : b - \lambda 1 \text{ not invertible}\} \subseteq [-\|b\|, \|b\|] \quad \text{and} \quad r(b) = \max_{\lambda} |\lambda| = \|b\|.$$

3.2. Representations of $C_0(X)$ on L^p-spaces

For σ-finite measure μ and $p \in [1, \infty)\setminus\{2\}$, it is a well known that heremitian operators in $B(L^p(\mu))$ are mutliplication operators by real-valued functions. It was generalized to decomposable measures in [**BP19**, Proposition 2.12] and to localizable measures [**CGT24**, Proposition 2.7] where the proof relies on the Banach-Lamperti theorem. We now extend this fact to L^∞-spaces and C_0-spaces, using our Banach-Lamperti theorem for $p = \infty$ or Banach-Stone theorem, see [**BKM25**, Proposition 2.11].

PROPOSITION 3.10. *Assume that $E = L^p(\mu)$ for a localizable measure μ and $p \in [1,\infty]\backslash\{2\}$ or $E = C_0(\Omega)$ for a locally compact Hausdorff space Ω. An operator $a \in B(E)$ is hermitian if and only if a is a multiplication operator by a real-valued function from $L^\infty(\mu)$ if $E = L^p(\mu)$ or from $C_b(\Omega)$ if $E = C_0(\Omega)$.*

PROOF. The case $E = L^p(\mu)$, $p \in [1,\infty)\backslash\{2\}$ is proved in [**CGT24**, Proposition 2.7] and the proof is based on the Banach-Lamperti theorem which is formulated there only for $p \in [1,\infty)\backslash\{2\}$. The same argument works for $p = \infty$ by using Corollary 2.51. Also the statement for $E = L^\infty(\mu)$ can be reduced to the case when $E = C(\Omega)$ for Gelfand spectrum Ω of $L^\infty(\mu)$. Therefore we explain only the case when $E = C_0(\Omega)$ for a locally compact Hausdorff space Ω. Mapping functions from $C_b(\Omega)$ to multiplication operators on E we get an isometric unital homorphism into $B(E)$:

$$m : C_b(\Omega) \to B(C_0(\Omega)), \qquad m(h)\xi(x) := h(x)\xi(x),$$

$h \in C_b(\Omega)$, $\xi \in E = C_0(\Omega)$. Since $C_b(\Omega)$ is a unital C^*-algebra, $a \in C_b(\Omega)$ is hermitian if and only if it is a real-valued function, and hence the corresponding multiplication operator is hermitian. Conversely, let now $a \in B(E)$ be any hermitian operator. By definition operators $u_t := e^{ita}$, $t \in \mathbb{R}$, are contractive. Clearly, they form a one-parameter group, that is $u_t u_s = u_{s+t}$, for $s, t \in \mathbb{R}$ and $u_0 = 1$. Hence in fact each u_t is an invertible isometry (u_{-t} is its contractive inverse). By the Banach–Stone theorem (Theorem 2.50) applied to each $u_t : C_0(\Omega) \to C_0(\Omega)$, $t \in \mathbb{R}$, we ge $\omega_t \in C_b(\Omega)$ with $|\omega_t| = 1$ and a homeomorphism $\varphi_t : \Omega \to \Omega$ such that

$$[u_t\xi](x) = \omega_t(x)\xi(\varphi_t(x)).$$

Let us calculate $\|u_t - u_s\|$ for $s, t \in \mathbb{R}$. If $\varphi_t = \varphi_s$ then $\|u_t - u_s\| = \|\omega_t - \omega_s\|_\infty$ because for any $\xi \in C_0(\Omega)$ we have $\|(u_t - u_s)\xi\|_\infty = \sup_{x\in\Omega} |(\omega_t(x) - \omega_s(x))\xi(\varphi_t(x))| \leqslant \|\omega_t - \omega_s\|_\infty \|\xi\|_\infty$, and $\|(u_t - u_s)\xi_i\|_\infty \to_i \|\omega_t - \omega_s\|_\infty$ for an approximate unit $(\xi_i)_i$ in $C_0(\Omega)$. Suppose then that $\varphi_t \neq \varphi_s$. Let $x_0 \in \Omega$ be a point such that $\varphi_t(x_0) \neq \varphi_s(x_0)$ and take $\xi \in C_0(\Omega)$ such that $\xi(\varphi_t(x_0)) = \overline{\omega_t(x_0)}$, $\xi(\varphi_s(x_0)) = -\overline{\omega_s(x_0)}$ and $\|\xi\|_\infty = 1$. Then $\|(u_t - u_s)\xi\|_\infty = 2$. Since, u_t, u_s are contractive, we always have $\|u_t - u_s\| \leqslant 2$, and therefore we have the following general formula

$$\|u_t - u_s\| = \max\left\{\|\omega_t - \omega_s\|_\infty, 2(1 - \delta_{\varphi_t,\varphi_s})\right\} = \begin{cases} \|\omega_t - \omega_s\|_\infty, & \varphi_t \neq \varphi_s \\ 2, & \varphi_t = \varphi_s \end{cases}$$

where δ is Kronecker symbol. Accordingly, $\|u_t - u_s\| < 2$ implies that $\varphi_t \neq \varphi_s$. Since the map $\mathbb{R} \ni t \mapsto u_t \in B(E)$ is norm continuous and $u_0 = 1$ is the identity operator, there exists $\delta > 0$ such that $\|u_t - u_0\| < 2$ for $t \in (-\delta, \delta)$. Thus $\varphi_t = id$ for $t \in (-\delta, \delta)$. Hence u_t is an operator of multiplication by ω_t for each $t \in (-\delta, \delta)$. Take $t \in (-\delta, \delta)\backslash\{0\}$ such that $\|ta\| \leqslant \frac{\pi}{2}$. Then the spectrum of a is contained in $[-\pi/2, \pi/2]$ by Remark 3.9. Hence by the Spectral Mapping Theorem the spectrum of $u_t = e^{ita}$ is contained in the half circle $\{e^{it} : t \in [-\pi/2, \pi/2]\}$. Therefore the holomorphic inverse log of exp is defined on $\sigma_{B(E)}(u_t)$. Applying analytic functional calculus to u_t we get $iat = \log(u_t)$. Using the unital homomorphism m, we obtain $a = -\frac{i}{t}\log(u_t) = -\frac{i}{t}\log(m(\omega_t)) = m(-\frac{i}{t}\log(\omega_t))$.

Thus a is a multiplication operator by $-\frac{i}{t}\log(\omega_t)$. Since a is bounded and hermitian the function $-\frac{i}{t}\log(\omega_t)$ is bounded and real-valued. $\square$

As a consequence we get a characterization of representations of the algebra $C_0(X)$ on L^p-spaces, for $p \neq 2$ (or C_0-spaces) via $*$-homomorphisms $\pi_0 : C_0(X) \to L^\infty(\mu)$ (resp. $\pi_0 : C_0(X) \to C_b(\Omega)$), which is a crucial tool outside of the Hilbert space framework.

THEOREM 3.11. *Let $\pi : C_0(X) \to B(E)$ be a non-degenerate representation where $E = C_0(\Omega)$ for locally compact Hausdorff space Ω, or $E = L^p(\mu)$ for $p \in [1,\infty]\backslash\{2\}$ and localizable measure space (Ω, Σ, μ). Then there is a unique representation $\pi_0 : C_0(X) \to L^\infty(\mu)$ if $E = L^p(\mu)$, or $\pi_0 : C_0(X) \to C_b(\Omega)$ if $E = C_0(\Omega)$, such that*

$$[\pi(a)\xi](x) = \pi_0(a)(x)\xi(x), \qquad a \in C_0(X),\ \xi \in E,\ x \in \Omega. \tag{3.1}$$

In particular, π is positive in the sense that it maps positive functions to positive operators.

PROOF. If X is not compact we may extend $\pi : C_0(X) \to B(E)$ to a unital representation $\pi : C(X^+) \to B(E)$ where X^+ is the one point compactification of X, see [**GT20**, Theorem 4.1]. Then π maps hermitian elements to hermitian ones by [**CGT24**, Lemma 2.4]. Hermitian elements in $C(X^+)$ are real valued functions $C_{\mathbb{R}}(X^+)$. Let us assume that $E = L^p(\mu)$. By Proposition 3.10, for any hermitian $a \in C_{\mathbb{R}}(X^+)$ there is a unique $\pi_0(a) \in L^\infty_{\mathbb{R}}(\mu)$ such that $\pi(a) = m(\pi_0(a))$ where $m : L^\infty(\mu) \to B(E)$ is the standard embedding of $L^\infty(\mu)$ into $B(E)$ as multiplication operators. For any $a \in C(X^+)$ we put $\pi_0(a) := \pi_0(\mathrm{Re}(a)) + i\pi_0(\mathrm{Im}(a))$. Since both π and m are linear we get $\pi(a) = m(\pi_0(a))$ for every $a \in C(X^+)$. Thus $\pi_0 = m^{-1} \circ \pi : C(X^+) \to L^\infty(\mu)$ is a representation as a composition of a representation π with the isometric homomorphism $m^{-1} : m(L^\infty(\mu)) \to L^\infty(\mu)$. Restriction of π_0 to $C_0(X) \subseteq C(X^+)$ is the desired representation, as by construction π_0 satisfies (3.1), and this condition determines π_0. A similar argument works for the case $E = C_b(\Omega)$. The last statement is now clear. $\square$

COROLLARY 3.12. *A Banach algebra $C_0(X)$ possesses an isometric non-degenerate representation on $L^\infty(\mu)$ for localizable μ if and only if X is compact.*

PROOF. If X is not compact and $\pi : C_0(X) \to B(L^\infty(\mu))$ is isometric and non-degenerate then, in the notation of Theorem 3.11, $\pi_0(C_0(X))$ is a non-unital subalgebra of $L^\infty(\mu)$. Hence $\overline{\pi(C_0(X))L^\infty(\mu)} = \overline{\pi_0(C_0(X))L^\infty(\mu)}$ is a non-trivial (non-unital) ideal in $L^\infty(\mu)$, which contradicts the non-degeneracy of π. If X is compact, then the embedding $\pi_0 : C(X) \to \ell^\infty(X)$ gives an isometric unital representation of $C(X)$ on $\ell^\infty(X)$. $\square$

REMARK 3.13. Corollary 3.12 is perhaps one of the reasons why Phillips [**Phi13b**, Page 3] suggests that instead of L^∞-algebras "it may well be more appropriate to consider" what we have called C_0-operator algebras. As obviously, $C_0(X)$ possesses an isometric non-degenerate representation on $C_0(X)$.

For $p = 2$, Theorem 3.11 holds up to a unitary conjugacy.

LEMMA 3.14. *For any representation $\pi : C_0(X) \to B(H)$ on a Hilbert space H, there is a localizable measure μ and a unitary $U : H \to L^2(\mu)$ such that $U\pi(\cdot)U^* : C_0(X) \to$*

$B(L^2(\mu))$ *is given by multiplication operators, i.e. there is a representation* $\pi_0 : C_0(X) \to L^\infty(\mu)$ *such that* $U\pi(a)U^*\xi = \pi_0(a)\cdot\xi$.

PROOF. The algebra $\pi(C_0(X))$ acts on the orthogonal complement of $H_0 := \overline{\pi(C_0(X))H}$ as zero, and we have $H_0 \cong \ell^2(I)$ for some set I (where I is the set of indexes of an orthonormal basis for H_0).

The compression $\pi : C_0(X) \to B(H_0)$ is a non-degenerate representation, and hence it decomposes into a direct sum of cyclic representations $\{\pi_i\}_{i\in I}$, see [**Mur90**, Theorem 5.1.3]. For any cyclic part $\pi_i : C_0(X) \to B(H_i)$ we have a state ϱ_i on $C_0(X)$ given by the formula $\varrho_i(a) := \langle \pi_i(a)\xi_i|\xi_i\rangle_{H_i}$, where ξ_i is a unit cyclic vector for π_i. By Markov–Riesz Theorem ϱ_i is given by integration with respect to some regular (and hence localizable) measure μ_i on X. Applying the GNS construction for ϱ_i we obtain representation $\tilde{\pi}_i : C_0(X) \to B(L^2(\mu_i))$ by operators of multiplication, which is equivalent to π_i. Taking the direct sum of all the arising measure spaces we get a space $L^2(\mu)$ with localizable μ and a unitary U with desired properties. □

3.3. Moore-Penrose partial isometries

Partial isometries on Hilbert spaces are very well known objects that play a fundamental role in the theory of C^*-algebras. A geometrical definition is as follows. Let H be a Hilbert space. An operator $T \in B(H)$ is a *partial isometry* if it is an isometry on the orthogonal complement of its kernel. Thus T restricts to a unitary operator $T : (\ker T)^\perp \to TH$ between the closed subspaces $(\ker T)^\perp$ and TH of H that are called the *initial* and *final space* for T. Partial isometries have numerous algebraic characterizations. For instance, $T \in B(H)$ is a partial isometry if and only if one of the following equivalent conditions hold:

i) T^*T is an orthogonal projection (onto initial subspace),
ii) TT^* is an orthogonal projection (onto the final subspace),
iii) $TT^*T = T$,
iv) $T^*TT^* = T^*$.

see [**Mur90**, Theorem 2.3.3]. Any of these equivalent conditions may serve as definition of a partial isometry in an abstract C^*-algebra. Partial isometries can also be characterized using only the language of Banach algebras, without the use of involution. Mbekhta used it to define partial isometries on Banach spaces:

PROPOSITION 3.15 ([**Mbe04**] Theorem 3.1, Corollary 3.3). *Let H be a Hilbert space. An operator $T \in B(H)$ is a partial isometry if and only if T is a contraction and there exists a contraction $S \in B(H)$ which is a generalized inverse to T, that is $TST = T$ and $STS = S$ (and then we necessarily have $S = T^*$).*

DEFINITION 3.16 ([**Mbe04**, Definition 4.1]). A linear operator T on a Banach space E is a *partial isometry* if it is a contraction and there is a contraction $S \in \mathcal{B}(E)$ such that

$$TST = T, \qquad STS = S.$$

Remark 3.17. By Proposition 3.15, if $E = H$ is a Hilbert space, then T is a partial isometry in the above sense if and only if it is a partial isometry in the usual sense, and then S as in Definition 3.16 is unique and $S = T^*$. Any invertible isometry T on any Banach space E is a partial isometry with the unique $S = T^{-1}$. However, in general there exist non-invertible isometries that are not partial isometries in the above sense. In fact by [**Mbe04**, Corollary 4.3], an isometry is a partial isometry if and only if there is a contractive projection onto its range. Thus it follows from the result of Ando [**And66**, Theorem 4], which uses the original Lamperti theorem, that isometries on a space $L^p(\mu)$ where $p \in [1, \infty)$ and μ is σ-finite are partial isometries, see also [**Czy01**, Twierdzenie 5.13]. Using our general version of Lamperti theorem (Theorem 2.43) one may generalize it to conclude that on arbitrary L^p-spaces for $p \in [1, \infty)$ isometries are always partial isometries. This is also true for L^∞-spaces by [**CFS79**, Theorem 12].

For partial isometries as defined above the operator S in Definition 3.16 is in general not uniquely determined by T, even when they act on L^p-spaces, see [**KL20**, Example 2.5]. A good way to circumvent this is to restrict to the so called Moore-Penrose inverses. These were studied first for matrices by E. Hastings Moore in 1920, [**Moo20**], and popularized by Roger Penrose in 1951, [**Pen55**]. This notion was generalized to unital complex Banach algebras by Rakočevićh as follows

Definition 3.18 ([**Rak88**]). Let B be a unital Banach algebra. An element $a \in B$ is said to have a *generalized inverse* if there is $b \in B$ such that

$$aba = a, \qquad bab = b. \tag{3.2}$$

If in addition the idempotents ab and ba are hermitian in the sense of Definition 3.8, we say that b is the *Moore-Penrose inverse* of a and we denote it by a^*

Remark 3.19. By [**Rak88**, Lemma 2.1] if the Moore-Penrose inverse exists it is unique. In particular, if a^* is a Moore-Penrose inverse of a, then a is a Moore-Penrose inverse of a^*, and so $(a^*)^* = a$.

The above facts motivated Mbekhta to combine his definition of partial isometries with Moore-Penrose inverses, see [**Mbe04**, Definition 4.3], to coin the notion of an MP-partial isometry on a Banach space. We summarize and formulate these definitions in terms Banach algebras as follows

Definition 3.20. Let B be an approximately unital Banach algebra and denote by B_1 the closed unit ball in B. A *partial isometry in* B is an element in B_1 which has a generalized inverse in B_1. An *MP-partial isometry in* B is an element in B_1 which has Moore-Penrose inverse in B_1. If E is a Banach space, then MP-partial isometries in $B(E)$ are called *MP-partial isometries on* E.

Remark 3.21. Let B be a C^*-algebra. It follows from Remark 3.9 and Proposition 3.15, that partial isometries and MP-partial isometries as defined above coincide with the usual partial isometries that can be defined using involution in B. It is also well known that the product uv of two partial isometries $u, v \in B$ is a partial isometry if and only if the projections u^*u and vv^* commute, see [**Erd68**, Theorem 1]. In particular, for any Hilbert space H, with $\dim(H) > 1$, partial isometries in $B(H)$ do not form a semigroup. For instance, the matrix

$$u = \begin{pmatrix} \frac{\sqrt{2}}{2} & 0 \\ \frac{\sqrt{2}}{2} & 0 \end{pmatrix}$$

is a partial isometry on $\ell^2(\{1,2\}) \cong \mathbb{C}^2$ but its square u^2 is not a partial isometry.

3.4. Spatial partial isometries on L^p-spaces

Let us now consider partial isometries on a space $E = L^p(\mu)$ for a localizable measure μ and $p \in [1, \infty]$. Namely, we will discuss the so called spatial partial isometries, originally introduced (for σ-finite measures and finite p) by Phillips [**Phi12**]. We show that they form a unital inverse semigroup whose group of invertible elements are spatial isometries discussed in Remark 2.47, and for $p \neq 2$ they coincide with MP-partial isometries in $B(E)$ as defined above. This shows once again the striking difference with the case $p = 2$ where (MP-)partial isometries do not form a semigroup, see Remark 3.21.

We start by introducing partial automorphisms of measure spaces (or the associated Boolean algebras), cf. Definition 2.24 and Remark 2.47.

Definition 3.22. By a *subspace* of a measure space (Ω, Σ, μ) we mean a measure space (D, Σ_D, μ_D) where $D \in \Sigma$, $\Sigma_D := \{A \cap D : A \in \Sigma\}$ and $\mu_D := \mu|_{\Sigma_D}$. A *partial set automorphism* of (Ω, Σ, μ) is a set isomorphism between two subspaces of (Ω, Σ, μ). So it is a map $\Phi : \Sigma_{D_{\Phi^*}} \to \Sigma_{D_\Phi}$, where $D_\Phi, D_{\Phi^*} \in \Sigma$, that descends to a Boolean isomorphism $[\Phi] : [\Sigma_{D_{\Phi^*}}] \to [\Sigma_{D_\Phi}]$. We denote by

$$\mathrm{PAut}([\Sigma]) := \{[\Phi] : \Phi \text{ is a partial set automorphism}\}$$

the set of partial automorphisms of the Boolean algebra $[\Sigma]$.

By definition $\mathrm{PAut}([\Sigma])$ is the set of isomorphisms between ideals in the Boolean algebra $[\Sigma]$. It forms an inverse semigroup that can be identified with an inverse semigroup of operators on $L^0(\mu)$. Indeed, let $\Phi : \Sigma_{D_{\Phi^*}} \to \Sigma_{D_\Phi}$ be a partial set automorphism. We denote by Φ^* a map $\Phi^* : \Sigma_{D_\Phi} \to \Sigma_{D_{\Phi^*}}$ that is inverse to $\Phi : \Sigma_{D_\Phi^*} \to \Sigma_{D_\Phi}$ in the sense that $[\Phi^*] = [\Phi]^{-1}$, cf. Remark 2.25. Then Φ^* is a partial set automorphism. The associated (generalized) composition operator $T_\Phi : L^0(\mu_{D_{\Phi^*}}) \to L^0(\mu_{D_\Phi})$, see Proposition 2.29, is a linear isomorphism with inverse given by T_{Φ^*}. Identifying $L^0(\mu_{D_\Phi})$ and $L^0(\mu_{D_{\Phi^*}})$ with subspaces of $L^0(\mu)$, in an obvious way, we see that a partial set isomorphism Φ determines a unique linear operator $T_\Phi : L^0(\mu) \to L^0(\mu)$ such that T_Φ preserves monotone limits and

$$T_\Phi(\mathbb{1}_A) = \mathbb{1}_{\Phi(A \cap D_{\Phi^*})} \qquad \text{for all } A \in \Sigma.$$

Thus T_Φ is the composition operator $T_{\overline{\Phi}}$ (in the sense of Definition 2.31) associated to the set morphism $\overline{\Phi} : \Sigma \to \Sigma$ given by $\overline{\Phi}(A) := \Phi(A \cap D_{\Phi^*})$, $A \in \Sigma$. For any two partial automorphisms Φ, Ψ the composite $\Phi \circ \Psi : \Sigma_{\Psi^*(D_{\Phi^*} \cap D_\Psi)} \to \Sigma_{\Phi(D_{\Phi^*} \cap D_\Psi)}$ is a well defined partial set automorphism and $T_\Phi \circ T_\Psi = T_{\Phi \circ \Psi}$. Also T_{Φ^*} is the unique generalized inverse for T_Φ among the composition operators. Thus operators on $L^0(\mu)$ associated to partial set automorphisms form an inverse semigroup which is naturally isomorphic to $\mathrm{PAut}([\Sigma_\mu])$.

Note that for any partial automorphism $\Phi : \Sigma_{D_{\Phi^*}} \to \Sigma_{D_\Phi}$ the map $\mu\circ\Phi^* : \Sigma_{D_\Phi} \to [0,\infty]$ is a measure equivalent to the measure μ_{D_Φ}. If μ is localizable, then so are $\mu\circ\Phi^*$ and μ_{D_Φ}. Therefore the Radon–Nikodym derivative $\frac{\mu\circ\Phi^*}{\mu_{D_\Phi}}$ exists by Theorem 2.20, see Remark 2.21. Extending $\frac{\mu\circ\Phi^*}{\mu_{D_\Phi}}$ by putting zero outside D_Φ we will treat it as a function defined on Ω.

PROPOSITION 3.23. *Let $p \in [1,\infty]$ and (Ω,Σ_μ,μ) be a localizable measure space. For any $[\Phi] \in \mathrm{PAut}([\Sigma_\mu])$ the weighted composition operator*

$$U_\Phi := \left(\frac{d\mu\circ\Phi^*}{d\mu_{D_\Phi}}\right)^{\frac{1}{p}} T_\Phi \tag{3.3}$$

is a well defined partial isometry on $L^p(\mu)$, and the map $\mathrm{PAut}([\Sigma_\mu]) \ni [\Phi] \mapsto U_\Phi \in B(L^p(\mu))$ is a semigroup embedding. The semigroup generated by $U_\Phi \in B(L^p(\mu))$, $[\Phi] \in \mathrm{PAut}([\Sigma_\mu])$, and $UL^\infty(\mu) \subseteq B(L^p(\mu))$ is an inverse semigroup of MP-partial isometries of the form

$$\omega U_\Phi := \omega\left(\frac{d\mu\circ\Phi^*}{d\mu_{D_\Phi}}\right)^{\frac{1}{p}} T_\Phi \tag{3.4}$$

where $\Phi : \Sigma_{D_{\Phi^*}} \to \Sigma_{D_\Phi}$ is a partial set automorphism and $\omega : D_\Phi \to \{z\in\mathbb{C} : |z| = 1\}$ is measurable. Moreover,

$$(\omega U_\Phi)^* = T_{\Phi^*}(\overline{\omega})U_{\Phi^*}, \qquad (\omega U_\Phi)\circ(\upsilon U_\Psi) = \omega T_\Phi(\upsilon)U_{\Phi\circ\Psi}. \tag{3.5}$$

PROOF. By Theorem 2.46 formula (3.3) defines an invertible isometry $U_\Phi : L^p(\mu_{D_{\Phi^*}}) \to L^p(\mu_{D_\Phi})$ with inverse given by U_{Φ^*}. Treating $L^p(\mu)$ as ℓ^p-direct sums $L^p(\mu_{D_{\Phi^*}})\oplus L^p(\mu_{\Omega\setminus D_{\Phi^*}})$ and $L^p(\mu_{D_\Phi})\oplus L^p(\mu_{\Omega\setminus D_\Phi})$ we see that (3.3) defines a partial isometry U_Φ on $L^p(\mu)$ with generalized inverse given by U_{Φ^*}. This implies the first part of the assertion. Now consider an operator ωU_Φ given by (3.4). Clearly, it is contractive. Moreover, the operator $(\omega U_\Phi)^* := T_{\Phi^*}(\overline{\omega})U_{\Phi^*}$ is of the same form. Moreover, using the first relation in (2.15) we see that

$$(\omega U_\Phi)^*\omega U_\Phi = T_{\Phi^*}(\bar{\omega})U_{\Phi^*}\omega U_\Phi = U_{\Phi^*{}_*\circ\Phi} = \mathbb{1}_{D_\Phi}$$

is the operator of multiplication by the characteristic function of the domain D_{Φ^*} of Φ. Similarly $\omega U_\Phi(\omega U_\Phi)^* = \mathbb{1}_{D_{\Phi^*}}$. Since multiplication operators by characteristic functions are hermitian projections, this implies that ωU_Φ is an MP-partial isometry and $(\omega U_\Phi)^*$ is its Moore-Penrose generalized inverse. The second relation in (2.15) extends to operators of the form (3.4) and the composite $(\omega U_\Phi)\circ(\upsilon U_\Psi) = \omega T_\Phi(\upsilon)U_{\Phi\circ\Psi}$ is again of the same form. This gives the second part of the assertion. □

DEFINITION 3.24. For any $p\in[1,\infty]$ and localizable measure μ we denote by $\mathrm{SPIso}(L^p(\mu))$ the inverse semigroup of operators (3.4) and call them *spatial partial isometries* on $L^p(\mu)$, cf. [**Phi12**, Definition 6.4], [**Gar21**, Definition 6.2].

Remark 3.25. The semilattice of idempotents in $\mathrm{SPIso}(L^p(\mu))$ is isomorphic to $[\Sigma_\mu]$, as it consists of multiplication operators of idempotent elements in $L^\infty(\mu)$. The group of invertible elements in $\mathrm{SPIso}(L^p(\mu))$ is isomorphic to $UL^\infty(\mu) \rtimes \mathrm{Aut}([\Sigma_\mu])$, see Remark 2.21.

We may now generalize Banach-Lamperti theorem that describes invertible isometries, Theorem 2.46, to a result that describes MP-partial isometries

Theorem 3.26 (Generalized Banach-Lamperti theorem). *Let $p \in [1, \infty]\backslash\{2\}$ and (Ω, Σ, μ) be a localizable measure space. An operator $T \in B(L^p(\mu))$ is an MP-partial isometry if and only if T is a spatial partial isometry.*

Proof. By Proposition 3.23 it suffices show that an MP-partial isometry $T \in B(L^p(\mu))$ is spatial. By Proposition 3.10 operators T^*T and TT^* are operators of multiplication. Since T^*T, TT^* are projections, they are multiplication operators by characteristic functions, say $\mathbb{1}_{D*}$, $\mathbb{1}_D$ respectively. Then T restricts to an invertible isometry $T : L^p(\mu_{D*}) \to L^p(\mu_D)$. Hence Theorem 2.46 implies that T is the form (3.4).

□

CHAPTER 4

Inverse semigroup Banach algebra crossed products

Let $\mathcal{G}$ be a locally compact Hausdorff étale groupoid. The space of compactly supported continuous functions $C_c(\mathcal{G}) = \{f \in C(\mathcal{G}) : \overline{\operatorname{supp}}(f) \text{ is compact}\}$ with operations

$$(f * g)(\gamma) := \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma), \qquad f^*(\gamma) := \overline{f(\gamma^{-1})}$$

for all $f, g \in C_c(\mathcal{G})$, $\gamma \in \mathcal{G}$, becomes a $*$-algebra. This convolution $*$-algebra admits the largest C^*-norm. The associated completion $C^*(\mathcal{G})$ of $C_c(\mathcal{G})$ is a C^*-algebra which is universal in the sense that representations of $C^*(\mathcal{G})$ on a Hilbert space H are in bijective correspondence with $*$-homomorphisms $C_c(\mathcal{G}) \to B(H)$. Finding appropriate analogues for this construction when Hilbert spaces are replaced by Banach spaces or even L^p-spaces is a non-trivial task. In particular, there is no maximal submultiplicative norm on $C_c(\mathcal{G})$ and one needs to choose some norm restrictions possibly related with the structure of $\mathcal{G}$.

For group actions and their crossed products there are natural Banach algebra norms that yield bijective correspondences between covariant representations of the action and representations of the algebra. Inspired by this, and the fact that étale groupoids can be viewed as inverse semigroup actions (see Chapter 1), in the present chapter we discuss a theory of groupoid Banach algebras that can be related to inverse semigroup actions and their crossed products. The material here is based on [**BKM25**, Section 3] where twisted and not necessarily Hausdorff groupoids are considered. As we stick to the untwisted Hausdorff case, our presentation and some of our proofs are different (simpler and more transparent). In addition, we formulate and prove here the universal description for Banach algebra inverse semigroup crossed products, Theorem 4.31, which is not present in [**BKM25**]. We report on the content of this chapter in [**BK26**].

4.1. Crossed products for group actions

Before turning to representations of inverse semigroup action and associated Banach algebras, as a starter and motivation, it is useful to discuss group actions and their Banach algebra crossed products first. In our setting, an *action of discrete group G on a Banach algebra A* is a homomorphism $\alpha : G \to \operatorname{Aut}(A)$, where $\operatorname{Aut}(A)$ is a group of isometric automorphisms of A. Then there is a natural candidate for the (universal) crossed product. Namely, we consider the Banach space

$$F(\alpha) := \ell^1(G, A) = \{a : G \to A : \|a\|_1 := \sum_{g\in G} \|a(g)\| < \infty\}$$

with the multiplication

$$(a * b)(g) := \sum_{h\in G} a(h)\alpha_h(b(h^{-1}g)), \quad g \in G, \ a, b \in \ell^1(G, A).$$

Then $F(\alpha)$ is a Banach algebra with $\|\cdot\|_1$ norm. If A is a Banach $*$-algebra and α takes values in $*$-preserving automorphisms of A (which is automatic when A is a C^*-algebra), then $F(\alpha)$ is naturally a Banach $*$-algebra with involution given by $a^*(g) := \alpha_g(a(g^{-1})^*)$, $g \in G$, $a \in F(\alpha)$. We will denote by $a\delta_g$ the element of $F(\alpha)$ which takes value $a \in A$ at $g \in G$ and $0 \in A$ otherwise. Then $C_c(G, A) = \operatorname{span}\{a\delta_g : a \in A, \ g \in G\}$ is a dense subalgebra of $F(\alpha) = \overline{\operatorname{span}}\{a\delta_g : a \in A, \ g \in G\}$, and for any $a, b \in A$, $g, h \in G$ we have $a\delta_g b\delta_h = a\alpha_h(b)\delta_{gh}$. In particular the map $A \ni a \mapsto a\delta_1 \in F(\alpha)$, where 1 is the unit in G, allows us to identify A with a closed subalgebra of $F(\alpha)$. When A has an approximate unit, it is also an approximate unit in $F(\alpha)$ and so the inclusion $A \subseteq F(\alpha)$ is non-degenerate in the sense that $AF(\alpha) = \overline{AF(\alpha)} = F(\alpha)$.

It seems that there is a general agreement that $F(\alpha)$ is the right candidate for the *crossed product for* α, see, for instance, [**BK24**] and references given there. One strong evidence is that it is universal for naturally defined covariant representations.

Definition 4.1. Let α be an action of a discrete group G on a Banach algebra A. A *covariant representation* of α on a Banach space E is a pair (π, v) where $\pi : A \to B(E)$ is a representation of A and $v : G \to \operatorname{Iso}(E)$ is a group homomorphism into the group $\operatorname{Iso}(E)$ of invertible isometries of E such that

$$v_g\pi(a) = \pi(\alpha_g(a))v_g, \qquad \text{for all } a \in A, \, g \in G.$$

We say that (π, v) is *non-degenerate* if π is.

Remark 4.2. Since we assume that $v : G \to \operatorname{Iso}(E)$ is a group homomorphism, the displayed commutation relation could be equivalently written as $v_g\pi(a)v_g^{-1} = \pi(\alpha_g(a))$ for $a \in A$, $g \in G$.

The following proposition was proved in [**ZM68**, Proposition 2.7, 2.8] in the C^*-algebraic setting and in [**BK24**, Lemma 2.8] for $A = C_0(X)$, but the proof works for general Banach algebras that possess an approximate unit.

Proposition 4.3. *Let α be an action of a discrete group G on a Banach algebra A. For any covariant representation (π, v) of α the formula*

$$\pi \rtimes v(a) = \sum_{g\in G} \pi(a(g))v_g, \qquad a \in F(\alpha), \tag{4.1}$$

where the series is norm convergent, defines a representation $\pi \rtimes v$ of $F(\alpha)$. If A has an approximate unit, then (4.1) *yields a one-to-one correspondence between non-degenerate representation of $F(\alpha)$ and non-degenerate covariant representations of α.*

Proof. It is straightforward to see for a covariant representation (π, v) of α the formula (4.1) gives a well defined representation $\pi \rtimes v$. Assume now that A has an approximate unit $(e_i)_i$. Since it is also an approximate unit in $F(\alpha)$ we get that $\pi \rtimes v$ is non-degenerate if and only if π is non-degenerate if and only if $(\pi(e_i))_i$ converges strictly to the identity operator. Let now $\psi : F(\alpha) \to B(E)$ be any non-degenerate representation. We put $\pi := \psi|_A$. Then π is a non-degenerate representation of A. For any $g \in G$, the following strong limit exists

$$v_g := \text{s-}\lim \psi(e_i\delta_g).$$

Indeed, by the Cohen-Hewitt factorization theorem, any element in E has the form $\pi(a)\xi$ for some $\xi \in E$, $a \in A$, and then $\lim_i \psi(e_i\delta_g)\pi(a)\xi = \lim_i \psi(e_i\alpha_g(a)\delta_g)\xi = \psi(\alpha_g(a)\delta_g)\xi$. This calculation also implies that $v_g\pi(a) = \pi(\alpha_g(a))v_g$ and shows that v_g does not depend on the choice of an approximate unit. Namely, v_g is uniquely determined by the relations

$$v_g\pi(a)\xi = \psi(\alpha_g(a)\delta_g)\xi \qquad \text{for all } \xi \in E, a \in A.$$

For any $g, h \in G$ we get

$$\begin{aligned} v_g v_h &= \text{s-}\lim_i \text{s-}\lim_j \psi(e_j\delta_g e_i\delta_h) = \text{s-}\lim_i \text{s-}\lim_j \psi(\alpha_g(e_i)e_j\delta_{gh}) \\ &= \text{s-}\lim_i \pi(\alpha_g(e_i))v_{gh} = v_{gh}, \end{aligned}$$

where we used that $(\alpha_g(e_i))_i$ is an approximate unit in A and so $\pi(\alpha_g(e_i))$ converges strongly to the identity operator. In particular, $v_{g^{-1}}v_g = v_1 = 1$, and so the operators v_g are invertible. Clearly, they are contractive, as strong limits of contractive operators. A contractive operator with a contractive inverse has to be an isometry. Thus $\{v_g\}_{g\in G} \subseteq \text{Iso}(E)$ and so (π, v) is a covariant representation. Clearly, $\psi = \pi \rtimes v$ and (π, v) is uniquely determined by this equality. □

The above procedure of going from a covariant representation of α to a representation of $F(\alpha)$ and back, is called *integration* and *disintegration* respectively. From our perspective, assuming existence of approximate units is natural. In particular, virtually all Banach algebras considered in this thesis will have an approximate unit by construction. From this perspective also considering non-degenerate representations on Banach spaces seems to be natural. However, there are situations where one would like to talk about representations in Banach algebras and then when it comes to disintegration things become more subtle. We will discuss these issues in the realm of inverse semigroup actions.

4.2. Covariant representations of inverse semigroup actions

A *partial automorphism* on a Banach algebra A is an isometric algebra isomorphism $\alpha : I \to J$ between two ideals I, J of A. The set of $\text{PAut}(A)$ of partial automorphisms of A together with composition of partial maps, see Example 1.4, forms an inverse semigroup. When $A = C_0(X)$, for a locally compact Hausdorff space X, then ideals in A correspond to open sets, and partial isomorphisms are given by composition with partial homeomorphisms, cf. Theorem 2.50. In particular, we have a natural isomorphism of semigroups $\text{PAut}(C_0(X)) \cong \text{PHomeo}(X)$, cf. Example 1.6. Using this isomorphism we can view the following as an algebraic version of Definition 1.7 as its generalization.

Definition 4.4. An *action of an inverse semigroup S on a Banach algebra A* is a semigroup homomorphism $\alpha : S \to \text{PAut}(A)$ which is "approximately" unital. Namely, $\alpha = \{\alpha_t\}_{t\in S}$ is a family of partial automorphisms $\alpha_t : I_{t^*} \to I_t$ of A such that

(1) $\alpha_s \circ \alpha_t = \alpha_{st}$ (as partial maps) for all $s, t \in S$;
(2) each ideal I_t, $t \in S$, has an approximate unit and the union $\bigcup_{e\in E(S)} I_e$ union is linearly dense in A.

If S is unital, then the second part of (2) is equivalent to assuming that α is unital, i.e. $\alpha_1 = \mathrm{id}_A$. If S has zero, then we will also assume that $A_0 = \{0\}$. We will also denote such inverse semigroup actions by writing $\alpha : S \rightharpoonup A$.

REMARK 4.5. When A is a C^*-algebra the above definition appeared first in the work of Sieben [**Sie97**, Definition 3.1] (every ideal in a C^*-algebra automatically has an approximate unit). For twisted actions on Banach algebras it was introduced in [**BKM25**]. Preservation of the semigroup law required in (1) has a number of consequences. In particular, as in Remark 1.8, for any action $\alpha : S \rightharpoonup A$, for all $s, t \in S$ and $e \in E(S)$ we have

$$\alpha_s(I_{s^*} \cap I_t) = I_{st}, \quad I_t = I_{tt^*}, \quad \alpha_e = \mathrm{id}|_{I_e}, \quad \alpha_{t^*} = \alpha_t^{-1},$$

and $I_s \subseteq I_t$ whenever $s \leqslant t$. We will use these relations, often without warning.

EXAMPLE 4.6. Having an inverse semigroup action θ on a locally compact Hausdorff space X one can translate it to an action on the Banach algebra $A := C_0(X)$ given by partial automorphisms $\alpha_t : C_0(X_{t^*}) \to C_0(X_t)$ where $\alpha_t(a) = a \circ \theta_{t^*}$ for $a \in C_0(X_{t^*})$. Every inverse semigroup action on $A = C_0(X)$ is of this form.

EXAMPLE 4.7. If $S = G$ is a group and A is a Banach algebra with an approximate unit, then inverse semigroup actions of S on A are equivalent to group actions, that is to unital homomorphisms $S \to \mathrm{Aut}(A)$.

The following is a straightforward generalization of a C^*-algebraic definition [**Sie97**, Definition 3.4], as well as Definition 4.1 for non-degenerate covariant representations of group actions on Banach algebras.

DEFINITION 4.8. Let $\alpha : S \rightharpoonup A$ be an inverse semigroup action. A *covariant representation of* α *on a Banach space* E is a pair (π, v) where $\pi : A \to B(E)$ is a representation and $v : S \to B(E)_1$ is a semigroup homomorphism into contractive operators such that

(SCR1) $v_t\pi(a) = \pi(\alpha_t(a))v_t$ for all $a \in I_t$, $t \in S$;
(SCR2) $v_tE = \overline{\pi(I_t)E}$ for $t \in S$.

We say that (π, v) is non-degenerate, injective, etc. if π has that property.

REMARK 4.9. Since $I_t = I_{tt^*}$ for all $t \in S$ and all idempotents in S are of the form tt^*, we see that (SCR2) is equivalent to assuming that the range of the contractive projection v_e corresponding to an idempotent $e \in E(S)$ is $\overline{\pi(I_e)E}$. This condition together with (SCR1) and the fact that $\alpha_e = \mathrm{id}_{I_e}$ for $e \in E(S)$ imply that $\pi(a) = v_e\pi(a) = \pi(a)v_e$ for all $a \in I_e$, $e \in E(S)$.

The above definition implies that operators $\{v_t\}_{t\in S}$ are partial isometries in the sense of Mbekhta, see Definition 3.16, as they are contractive and

$$v_tv_{t^*}v_t = v_t, \qquad v_{t^*}v_tv_{t^*} = v_{t^*}, \qquad t \in S.$$

For actions on C^*-algebras (so for instance on $C_0(X)$) or more generally when ideals that appear in the action have hermitian approximate units, operators $\{v_t\}_{t\in S}$ are necessarily MP-partial isometries:

Proposition 4.10. *Let $\alpha : S \rightharpoonup A$ be an inverse semigroup action, such that each ideal I_e, $e \in E(S)$, has a hermitian approximate unit. For any non-degenerate covariant representation (π, v) of α on a Banach space E, the operators $\{v_t\}_{t\in S}$ are MP-partial isometries. In particular, if $E = L^p(\mu)$ for localizable measure μ and $p \in [1,\infty]\backslash\{2\}$ we have that $\{v_t\}_{t\in S} \subseteq \mathrm{SPIso}(L^p(\mu))$ are necessarily spatial partial isometries.*

Proof. Since π is non-degenerate we may extend it to the unitization of A and so we may assume that A and π are in fact unital. For each $e \in E(S)$ we may choose an approximate unit $\{\mu_i^e\}_i$ in I_e consisting of hermitian operators. Then $\{\pi(\mu_i^e)\}_i$ are hermitian operators by [**CGT24**, Lemma 2.4]. As v_e is a strong limit of $\{\pi(\mu_i^e)\}_i$, this implies that v_e is also hermitian. Indeed, recall that $a \in B(E)$ is hermitian if and only if its numerical range $V(a) = \{f(ax) : x \in E,\ f \in E',\ \|f\| = \|x\| = 1 = f(x)\}$ is real. Hence for any $x \in E,\ f \in E'$ with $\|f\| = \|x\| = 1 = f(x)$ we get $f(v_e x) = f(\lim_i \pi(\mu_i^e)x) \in \mathbb{R}$, and so v_e is hermitian. As $v_t v_{t^*} = v_{tt^*}$ and $v_{t^*}v_t = v_{t^*t}$, where $tt^*, t^*t \in E(S)$, for all $t \in S$, we conclude that $\{v_t\}_{t\in S}$ are MP-partial isometries and $v_t^* = v_{t^*}$. □

Construction of the C^*-crossed product for the inverse semigroup action from [**Sie97**] can be readily adopted to produce a Banach algebra which is universal in the sense that every covariant representation (π, v) of the action integrates to a representation $(\pi \rtimes v)$ of the algebra. But when it comes disintegration the situation becomes much more subtle. Pictorially speaking looking at a representation of the algebra on a Banach algebra, the arguments from the proof of Proposition 4.3, produce for each $t \in S$ an operator $\overline{\pi(I_{t^*})E} \to \overline{\pi(I_t)E}$ between closed subspaces. However, unless the subspaces in question are canonically complemented, there is no canonical way of extending these operators to operators on the whole E. In general they should be viewed as partial operators on E or "local multipliers" of $B(E)$. Following [**BKM25**] we adopt the second point of view and introduce more general covariant representations in Banach algebras.

In general, we will use the *double dual* B'' of a Banach algebra B. The reason is that "local multipliers" live in B'' rather then in B, and also we will need some sort of weak topology and B'' has a weak* topology induced by B', which will refer to as B'-topology. Recall that B'' is again naturally a Banach algebra with either of the Arens products, in which B sits as a B'-weakly dense Banach algebra. Namely, for $a, b \in B''$, the *first Arens product* is defined as

$$a \square b = B'\text{-}\lim_{\alpha} B'\text{-}\lim_{\beta} a_\alpha b_\beta$$

where $\{a_\alpha\}_\alpha$, $\{b_\beta\}_\beta$ are nets B'-convergent to a, b respectively. Similarly, the *second Arens product* is given by

$$a \cdot b = B'\text{-}\lim_{\beta} B'\text{-}\lim_{\alpha} a_\alpha b_\beta.$$

The Banach space B'' with any of the above products is a Banach algebra. In general these products are different. If the two products coincide, then B is called *Arens regular*, see

[**Dal00**] for more details. For the sake of clarity, in this thesis *we will only use the second Arens product*. Our decision is arbitrary, and in any case we will be primarily interested in products $av \in B$ where $a \in B$ and $v \in B''$, in which case the first and second Arens products always agree. Also recall that every C^*-algebra is Arens regular.

For any representation $\pi : A \to B$ the double adjoint $\pi'' : A'' \to B''$ is a B'-weakly continuous representation that extends π, and π'' is isometric whenever π is. In particular, for every Banach subalgebra $A \subseteq B$ we may identify A'' with a Banach subalgebra of B''. Also any approximate unit $\{\mu_i\}$ in A converges A'-weakly to a left identity 1_A in A'' (and a right identity for the first Arens product in A'') which is a unit in the multiplier algebra $\mathcal{M}(A) := \{b \in A'' : ba, ab \in A \text{ for all } a \in A\} \subseteq A''$, see [**Dal00**, Proposition 2.9.16 and Theorem 2.9.49]. In particular, for each ideal I in A, which has an approximate unit, we will adopt the identifications $I \subseteq \mathcal{M}(I) \subseteq I'' \subseteq A''$.

Definition 4.11. Let $\alpha : S \rightharpoonup A$ be an inverse semigroup action. A *covariant representation of* α *in a Banach algebra* B is a pair (π, v) where $\pi : A \to B$ is a representation and $v : S \to (B'')_1$ is a map satisfying

(CR1) $v_t\pi(a) = \pi(\alpha_t(a))v_t \in B$ for all $a \in I_t$;
(CR2) $\pi(a)v_s v_t = \pi(a)v_{st}$ for all $a \in I_{st}$, $s, t \in S$;
(CR3) $\pi(a)v_e = \pi(a)$ for all $a \in I_e$, $e \in E(S)$.

A covariant representation (π, v) is injective, isometric, non-degenerate, etc. if π has that property. We call $B(\pi, v) := \overline{\operatorname{span}}\{\pi(a)v_t : a \in I_t, t \in S\}$ the *range of* (π, v).

Remark 4.12. Condition (CR1) is equivalent to $\pi(a)v_t = v_t\pi(\alpha_{t^*}(a)) \in B$ for all $a \in I_t$, $t \in S$. Note that this condition means in two things. Firstly, this is a commutation relation. Secondly it requires that the considered product is in B, rather than in B''. In particular it implies that $B(\pi, v) \subseteq B$, rather than $B(\pi, v) \subseteq B''$. Conditions (CR1) and (CR3) imply that $v_e\pi(a) = \pi(a)$ for all $a \in I_e$, $e \in E(S)$ (because $\alpha_e = \mathrm{id}_{I_e}$). Since $I_t = I_{tt^*}$, (CR2) and (CR3) imply that $\pi(a)v_t^* v_t = \pi(a)v_{t^*t} = \pi(a)$ for all $a \in I_t$. Employing also (CR1) we get

$$v_t\pi(a)v_{t^*} = \pi\big(\alpha_t(a)\big) \qquad \text{for all } a \in I_{t^*},\ t \in S. \tag{4.2}$$

Lemma 4.13. *Let (π, v) be a covariant representation of α in a Banach algebra B. The range $B(\pi, v)$ is a Banach subalgebra of B. The spaces $A_t = \{\pi(a_t)v_t : a_t \in I_t\}$, $t \in S$, form a grading of $B(\pi, v)$ over the inverse semigroup S in the sense that*

$$B(\pi, v) = \overline{\operatorname{span}}\{A_t : t \in S\}, \qquad A_sA_t \subseteq A_{st}, \qquad s \leqslant t \text{ implies } A_s \subseteq A_t,$$

for all $s, t \in S$. In fact, for all $a_t \in I_t, a_s \in I_s$, $s, t \in S$, we have

1. *$\pi(a_s)v_s \cdot \pi(a_t)v_t = \pi\big(\alpha_s(\alpha_{s^*}(a_s)a_t)\big)v_{st}$ and $\alpha_s\big(\alpha_{s^*}(a_s)a_t\big) \in I_{st}$;*
2. *$s \leqslant t$ implies $I_s \subseteq I_t$ and $\pi(a_s)v_s = \pi(a_s)v_t$.*

Proof. (1). For any $a_t \in I_t, a_s \in I_s$, $s, t \in S$, we have $\alpha_{s^*}(a_s)a_t \in I_{s^*}I_t \subseteq I_{s^*} \cap I_t$ and so $\alpha_s(\alpha_{s^*}(a_s)a_t) \in I_{st}$ because $\alpha_s(I_{s^*} \cap I_t) = I_{st}$, see Remark 4.5. Using (CR1) we get

$$\pi(a_s)v_s \cdot \pi(a_t)v_t = v_s\pi\big(\alpha_{s^*}(a_s)\big) \cdot \pi(a_t)v_t = v_s\pi\big(\alpha_{s^*}(a_s)a_t\big)v_t = \pi\Big(\alpha_s\big(\alpha_s^*(a_s)a_t\big)\Big)v_{st}.$$

(2). If $s \leqslant t$ then $I_s \subseteq I_t$ by Remark 4.5. By (CR3) and (CR2), for $a \in I_s$ we get $\pi(a)v_t = \pi(a)v_{ss^*}v_t = \pi(a)v_{ss^*t} = \pi(a)v_s$. □

Notice that the elements v_t of Definition 4.11 are not required to be partial isometries, and in particular in general they do not form a semigroup. However, this can always be arranged by "normalizing" the covariant representation. Depending on the structure of the target algebra B, this normalization can be done in different ways. In general, knowing nothing about B we can always use the B'-topology of B'' as follows.

Definition 4.14. Let $\alpha : S \rightharpoondown A$ be an inverse semigroup action. We say that a covariant representation (π, v) of α in a Banach algebra B is *B'-normalized*, if

$$v_t = B'\text{-}\lim_i(\pi(\mu_i^t)v_t) \text{ where } \{\mu_i^t\}_i \text{ is an approximate unit in } I_t, \text{ for all } t \in S. \tag{4.3}$$

Remark 4.15. Since the elements $\pi(\mu_i^t)v_t$ appearing in (4.3) are in the unit ball of $B(\pi, v) \subseteq B(\pi, v)'' \subseteq B''$, Banach-Alaoglu Theorem implies that if the limit in (4.3) exists it sits in $B(\pi, v)''$. In particular, (4.3) is equivalent to

$$v_t = B(\pi, v)'\text{-}\lim_i(\pi(\mu_i^t)v_t) \text{ where } \{\mu_i^t\}_i \text{ is an approximate unit in } I_t, \text{ for all } t \in S.$$

Notation 4.16. We denote by 1_t the unit in multiplier algebra $\mathcal{M}(I_t) \subseteq I_t'' \subseteq A''$. Note that $1_t = 1_{tt^*}$ and $1_{t^*} = 1_{t^*t}$ for all $t \in S$, because $I_t = I_{tt^*}$ and $I_{t^*} = I_{t^*t}$.

Proposition 4.17. *A pair (π, v) is a B'-normalized covariant representation of α if and only if $\pi : A \to B$ is a representation and $v : S \to (B'')_1$ is a semigroup homomorphism such that $\pi(a)v_t \in B$, for $a \in I_t$, $t \in S$, and*

1. *$v_t\pi(a)v_{t^*} = \pi(\alpha_t(a))$ for all $a \in I_{t^*}$, $t \in S$;*
2. *$v_e = \pi''(1_e)$ for all $e \in E(S)$.*

Moreover, for any covariant representation (π, v) of α in B putting

$$\tilde{v}_t = B'\text{-}\lim_i(\pi(\mu_i^t)v_t) = v_t\pi''(1_{t^*}), \qquad t \in S,$$

the pair $(\pi, \tilde{v})$ is a unique B'-normalized covariant representation such that $\pi(a)v_t = \pi(a)\tilde{v}_t$ for all $a \in I_t$ and $t \in S$. In particular, we have $B(\pi, v) = B(\pi, \tilde{v})$.

Proof. Assume (π, v) is a pair with the properties described in the assertion. For every $t \in S$ and $a \in I_{t^*}$ we have $\pi(a)\pi''(1_{t^*}) = \pi(a)$, hence using also (2) and (1) we have

$$v_t\pi(a) = v_t\pi(a)\pi''(1_{t^*}) = v_t\pi(a)v_t^*v_t = \pi\big(\alpha_t(a)\big)v_t,$$

which is (CR1). We assume that v is a semigroup homomorphism, which is clearly stronger than (CR2). Using the semigroup law and (2) we get

$$v_t = v_{tt^*t} = v_tv_{t^*}v_t = \pi''(1_t)v_t,$$

which gives (4.3). Applying the semigroup law and (2) to an idempotent $e \in E(S)$ we get

$$v_e = v_{ee} = v_ev_e = \pi''(1_e),$$

which implies (CR3). Hence (π, v) is a B'-normalized covariant representation of α.

For the converse let (π, v) be any B'-normalized covariant representation. For each $t \in S$ choose an approximate unit $\{\mu_i^t\}_i$ in $I_t = I_{tt^*}$. For $a \in I_{t^*}$ we get

$$v_t\pi(a)v_{t^*} = \pi\big(\alpha_t(a)\big)v_tv_{t^*} = \pi(\alpha_t(a))v_{tt^*} = \pi(\alpha_t(a)),$$

which is (1). If $e \in E(S)$ is an idempotent, then by normalization (4.3) and (CR3)

$$v_e = B'\text{-}\lim_i(\pi(\mu_i^e)v_e) = B'\text{-}\lim_i \pi(\mu_i^e) = \pi''(1_e),$$

which is (2). Now take any $s, t \in S$. Recall that $\alpha_s(I_{s^*} \cap I_t) = I_{st}$ and therefore $\{\alpha_s(\alpha_{s^*}(\mu_j^s)\mu_i^t)\}_{i,j}$ is and approximate unit in I_{st}. Using this, normalization (4.3) and Lemma 4.13(1) we get

$$\begin{aligned} v_sv_t &= v_sB'\text{-}\lim_i[\pi(\mu_i^t)v_t] = B'\text{-}\lim_i[v_s\pi(\mu_i^t)v_t] = B'\text{-}\lim_i B'\text{-}\lim_j[\pi(\mu_j^s)v_s\pi(\mu_i^t)v_t] \\ &= B'\text{-}\lim_i B'\text{-}\lim_j[\pi\big(\alpha_s(\alpha_{s^*}(\mu_j^s)\mu_i^t)\big)v_{st}] = v_{st}. \end{aligned}$$

Hence (π, v) satisfies all properties in the assertion.

Now let (π, v) be any covariant representation of α in B. Using (CR1) we get

$$B'\text{-}\lim_i \big(\pi(\mu_i^t)v_t\big) = B'\text{-}\lim_i \big(v_t\pi(\alpha_{t^*}(\mu_i^t))\big) = v_tB'\text{-}\lim_i \pi\big(\alpha_{t^*}(\mu_i^t)\big) = v_t\pi''(1_{t^*}),$$

because $\{\alpha_{t^*}(\mu_i^t)\}_i$ is an approximate unit in I_{t^*} which is B'-convergent to 1_{t^*}. Thus we that the limit $\tilde{v}_t := B'\text{-}\lim_i(\pi(\mu_i^t)v_t) = v_t\pi''(1_{t^*})$ exists and does not depend on the choice of an approximate unit $\{\mu_i^t\}_i$ in I_t, $t \in S$. Clearly if $(\pi, \tilde{v})$ is a B'-normalized covariant representation satisfying $\pi(a)v_t = \pi(a)\tilde{v}_t$ for all $a \in I_t$ and $t \in S$, then each $\tilde{v}_t$ must be given by such a limit. For $a \in I_{t^*}$ we get

$$\tilde{v}_t\pi(a) = v_t\pi''(1_{t^*})\pi(a) = v_t\pi(a) = \pi\big(\alpha_t(a)\big)v_t = \lim_i \pi\big(\alpha_t(a)\big)\pi(\mu_i^t)v_t = \pi\big(\alpha_t(a)\big)\tilde{v}_t.$$

This shows that $(\pi, \tilde{v})$ satisfies (CR1) and $\pi(a)v_t = \pi(a)\tilde{v}_t$ for all $a \in I_t$ and $t \in S$. Using this we get $\pi(a)\tilde{v}_e = \pi(a)v_e = \pi(a)$ for all $a \in I_e$, $e \in E(S)$, so (CR3) holds. Finally, for all $s, t \in S$ and $a \in I_{st}$ the calculation

$$\pi(a)\tilde{v}_s\tilde{v}_t = v_s\pi\big(\alpha_{s^*}(a)\big)\tilde{v}_t = \pi(a)v_sv_t = \pi(a)v_{st} = \pi(a)\tilde{v}_{st}$$

gives (CR2). As $\tilde{v}$ satisfies (4.3) by construction, conclude that $(\pi, \tilde{v})$ is the desired B'-normalized covariant representation of α. ☐

Remark 4.18. The above proposition implies that for any covariant representation (π, v), the normalization condition (4.3) is equivalent to assuming that $v_t = v_t\pi''(1_{t^*})$ for all $t \in S$ and it implies that $v_t = \pi''(1_t)v_t$ for all $t \in S$. When B is Arens regular, the latter implication can be reversed. In general it is not clear, and this asymmetry results from our arbitrary choice to use the second Arens' product in B''.

Corollary 4.19. *Retaining the assumptions and notation from Lemma 4.13. assume in addition that A and B are C^*-algebras. Then*

$$(\pi(a_t)v_t)^* = \pi\big(\alpha_{t^*}(a_t^*)\big)v_{t^*} \qquad \textit{for all } a_t \in I_t,\ t \in S,$$

so $A_t^* = A_{t^*}$, and $\{A_t\}_{t\in S}$ is a saturated grading of $B(\pi, v)$ over the inverse semigroup of S in the sense of [**KM20$_b$**, Definition 6.15]*, see also* [**Exe08**, Definition 7.1].

Proof. If B is a C^*-algebra then B'' is again a C^*-algebra – the enveloping W^*-algebra of B. If in addition A is a C^*-algebra then $\pi : A \to B$ is also necessarily $*$-preserving. By passing to the normalized covariant representation $(\pi, \tilde{v})$ described in Proposition 4.17 we see that for every $t \in S$, the operator $\tilde{v}_t := \pi''(1_t)v_t = v_t\pi''(1_{t^*})$ is a partial isometry whose adjoint in B'' is $\tilde{v}_{t^*} = \pi''(1_{t^*})v_{t^*} = v_{t^*}\pi''(1_t)$. Thus for $a_t \in I_t$, $t \in S$, we get

$$\big(\pi(a_t)v_t\big)^* = \big(\pi(a_t)\tilde{v}_t\big)^* = \tilde{v}_{t^*}\pi(a_t^*) = \pi(\alpha_{t^*}(a_t^*))\tilde{v}_{t^*} = \pi(\alpha_{t^*}(a_t^*))v_{t^*}. \qquad \square$$

Remark 4.20. If each I_t, $t \in S$, is unital, so that $1_t \in A$, $t \in S$, then for any covariant representation (π, v) of α in B its B'-normalization $\tilde{v}_t = \pi(1_t)v_t$, $t \in S$, takes values in B by (CR1). Hence, in this case without changing the range of covariant representations, one can avoid talking about B'' altogether in this case Proposition 4.17 could be formulated without the use of the bidual algebra B'' and the extended representation π''. In particular, in this case a normalized covariant representation of α is a pair (π, v) where $\pi : A \to B$ is a representation and $v : S \to B_1$ is a semigroup homomorphism such that

(1) $v_t\pi(a)v_{t^*} = \pi(\alpha_t(a))$ for all $a \in I_{t^*}$, $t \in S$;
(2) $v_e = \pi(1_e)$ for all $e \in E(S)$.

Let us now explain that we may restrict to maps taking values in B (rather then in B''), as in Remark 4.20, for an arbitrary inverse semigroup action $\alpha : S \curvearrowright A$, whenever B is a dual Banach algebra.

Definition 4.21. Recall that a *dual Banach algebra* is a pair (B, B_*) where B is a Banach algebra and B_* is a predual Banach space of B, and multiplication in B is B_*-weak separately continuous, see [**Run02**], [**Daw10**].

Examples of dual Banach algebras include all W^*-algebras with their unique preduals, and all algebras $B(E)$ of all bounded operators on a reflexive Banach space E, equipped with the canonical predual Banach space $E'\widehat{\otimes}E$. If (B, B_*) is a dual Banach algebra, then we have the canonical embedding $B_* \subseteq B_*'' = B'$, which is strict (if B is not reflexive), and so the B_*-topology on B'' is stronger than B'-topology. In particular, if a bounded net $\{b_i\}_i$ of elements in B is B_*-convergent to $b \in B''$, then we necessarily have that $b \in B$, as by Banach-Alaoglu Theorem the net $\{b_i\}_i$ has to have a subnet B_*-convergent to an element in B. Thus if we could replace B'-convergence (4.3) by B_*-convergence, then the corresponding elements would need to belong to B (rather than B''). We formalize this as follows.

Definition 4.22. Let $\alpha : S \curvearrowright A$ be an inverse semigroup action and let (B, B_*) be a dual Banach algebra. A covariant representation (π, v) of α in B is B_**-normalized*, if

$$v_t = B_*\text{-}\lim_i(\pi(\mu_i^t)v_t), \text{ for all } t \in S, \text{ where } \{\mu_i^t\}_i \text{ is an approximate unit in } I_t. \tag{4.4}$$

Thus if this happens we necessarily have that $\{v_t\}_{t\in S} \subseteq B$.

We will describe B_*-normalized covariant representations using the following result, which is a consequence of [**IS08**, Theorem 5.6].

Proposition 4.23 (Ilie-Stokke). *Let be A a Banach algebra with an approximate unit. Every representation $\pi : A \to B$ into a dual Banach algebra (B, B_*) has a unique extension to a representation $\overline{\pi} : \mathcal{M}(A) \to B$ which is strictly-B_*-continuous; it is given by $\overline{\pi}(m) := B_*\text{-}\lim_i \pi(m\mu_i)$ for an in A and any approximate unit $\{\mu_i\}_i$ in A.*

We get the following analogue of Proposition 4.17 which boils down to replacing B'-topology with the stronger B_*-topology.

Proposition 4.24. *Let (B, B_*) be a dual Banach algebra. A pair (π, v) is a B_*-normalized covariant representation of an action $\alpha : S \curvearrowright A$ if and only if $\pi : A \to B$ is a representation and $v : S \to B_1$ is a semigroup homomorphism such that*

(1) *$v_t\pi(a)v_{t^*} = \pi(\alpha_t(a))$ for all $a \in I_{t^*}$, $t \in S$;*
(2) *$v_e = \overline{\pi}_e(1_e)$ for all $e \in E(S)$ where $\overline{\pi}_e : \mathcal{M}(I_e) \to B$ is the unique strictly-B_*-continuous extension of $\pi|_{I_e}$, given by Proposition 4.23.*

Moreover, for any covariant representation (π, v) of α in B putting

$$\tilde{v}_t = B_*\text{-}\lim_i \pi(\mu_i^t)v_t = \overline{\pi}_{tt^*}(1_{t^*t})v_t = v_t\overline{\pi}_{t^*t}(1_{t^*t}), \qquad t \in S,$$

the pair $(\pi, \tilde{v})$ is a unique B_*-normalized covariant representation such that $\pi(a)v_t = \pi(a)\tilde{v}_t$ for all $a \in I_t$ and $t \in S$. In particular, we have $B(\pi, v) = B(\pi, \tilde{v})$.

Proof. This can be proved exactly as Proposition 4.17. In fact some of the arguments can be simplified as the multiplication in B is B_*-continuous in both variables. We leave the details to the reader. □

Remark 4.25. It might be tempting to think that Proposition 4.17 can be deduced from Proposition 4.24 by applying it to the pair (B'', B'), as any covariant representation in B is also a covariant representation in B''. However, the pair (B'', B') is a dual Banach algebra if and only if B is Arens regular. Thus in general we can not do that. We were not able to find a nice common generalization of these two statements.

Now coming back to covariant representations on a space E, as described in Definition 4.8, we notice that they can be viewed as covariant representations in the algebra $B(E)$ which are 'normalized' using strong topology. But if E is reflexive, then they are in fact equivalent to B_*-normalized representation for the canonical predual B_* of $B(E)$.

Lemma 4.26. *Let (π, v) be a covariant representation of $\alpha : S \curvearrowright A$ on a Banach space E. Then (π, v) is a covariant representation in the Banach algebra $B(E)$ such that each $v_e \in B(E)$, $e \in E(S)$, is a strong limit of $\{\pi(\mu_i^e)\}_i$, where $\{\mu_i^e\}_i$ is a contractive approximate unit in I_e.*

Proof. It is clear that (π, v) is a covariant representation in the Banach algebra $B(E)$, see Remark 4.9. Since each $v_e \in B(E)$, $e \in E(S)$, is a projection onto $\overline{\pi(I_e)E}$ which commutes with elements of $\pi(I_e)$, and thus v_e is a strong limit of $\{\pi(\mu_i^e)\}_i$. □

Recall that if E is a reflexive Banach space, then the projective tensor product Banach space $E' \widehat{\otimes} E$ is naturally a predual of $B(E)$ making it a dual Banach algebra. Here we identify a simple tensor $\xi \otimes f \in E \widehat{\otimes} E'$ with the functional $\widehat{\xi \otimes f} \in B(E)'$ given by $\widehat{\xi \otimes f}(T) := f(T\xi)$, $T \in B(E)$, see [**Rya02**] for more details.

LEMMA 4.27. *Assume that E is a reflexive Banach space, and treat $B(E)$ as dual Banach algebra with the predual $B_* := E' \widehat{\otimes} E$. Then B_*-normalized covariant representations of α in the Banach algebra $B(E)$ coincide with covariant representations of α on the Banach space E.*

PROOF. Assume that (π, v) is B_*-normalized covariant representation in $B(E)$, and so $\{v_t\}_{t \in S} \subseteq B(E)$ are partial isometries with the associated generalized inverses $\{v_t^*\}_{t \in S} \subseteq B(E)$, satisfying the relations described in Proposition 4.24. Since it is a covariant representation it satisfies (SCR1) which in our case is just (CR1). For any $e \in E(S)$ we have $v_e = \pi_e(1_e) = B_*\text{-}\lim \pi(\mu_i^e)$, where $\{\mu_i^e\}_i$ is an approximate unit in I_e. This B_*-convergence written explicitly means that $f(v_e\xi) = \lim_i f(\pi(\mu_i^e)\xi)$ for all $\xi \in E$ and $f \in\in E'$. This implies that $v_e\xi$, for any $\xi \in E$, sits in the weak closure of $\pi(I_e)E$. As the weak and norm closures of convex sets coincide this means that the range of v_e is contained in $\overline{\pi(I_e)E}$. As the converse inclusion is clear, we get $v_e E = \overline{\pi(I_e)E}$ for every $e \in E(S)$ This is equivalent to (SCR2), see Remark 4.9.

Conversely, assume (π, v) is a covariant representation on E in the sense of Definition 4.8. Clearly, it is a covariant representation in $B(E)$ in the sense of Definition 4.11. Thus we only need to show it is B_*-normalized. Let $e \in E(S)$. By Lemma 4.26, $\{\pi(\mu_i^e)\}_i$ converges strongly to v_e, and by construction $\{\pi(\mu_i^e)\}_i$ B_*-converges to $\overline{\pi}_e(1_e)$. Clearly, strong convergence is stronger than B_*-convergence. Hence $v_e = \overline{\pi}_e(1_e)$. Using this for any $t \in S$ we get

$$v_t = v_{tt^*} v_t = \overline{\pi}_{tt^*}(1_{tt^*}) v_t,$$

which is equivalent to (4.4). □

COROLLARY 4.28. *If E is reflexive, for any covariant representation (π, v) of α in $B(E)$ there is a unique covariant representation $(\pi, \widetilde{v})$ on E such that $\pi(a)v_t = \pi(a)\widetilde{v}_t$ for all $a \in I_{t^*}$, $t \in S$.*

PROOF. By Lemma 4.27 and the second part of Proposition 4.24 the desired pair $(\pi, \tilde{v})$ arises as the $E' \widehat{\otimes} E$-normalization of (π, v). □

4.3. Crossed products for inverse semigroup actions

Let us fix an inverse semigroup action $\alpha : S \rightharpoonup A$ on a Banach algebra A. We consider the ℓ^1-direct sum of the ideals $\{I_t\}_{t \in S}$ which is the Banach space

$$\ell^1(\alpha) := \{f \in \ell^1(S, A) : \ f(t) \in I_t, \ t \in S\}.$$

with norm $\|f\|_1 = \sum_{t \in S} \|f(t)\|$. Following [**Sie97**] we define multiplication in $\ell^1(\alpha)$ by

$$(f * g)(r) := \sum_{st=r} \alpha_s(\alpha_{s^*}(f(s))g(t)), \qquad r \in S.$$

It is well defined as for $f, g \in \ell^1(\alpha)$ we have

$$\|f * g\|_1 = \sum_r \| \sum_{st=r} \alpha_s(\alpha_{s^*}(f(s))g(t))\| \leqslant \sum_r \sum_{st=r} \|f(s)\|\|g(t))\| = \sum_{s,t} \|f(s)\|\|g(t))\| = \|f\|_1 \|g\|_1.$$

The proof of [**Sie97**, Proposition 4.1] shows that this multiplication is associative. It exploits our assumption that each of the ideals I_t, $t \in S$, has an approximate unit. In fact without this assumption, associativity of this partial convolution product may fail, see [**DE05**] for more details on that issue. This is one more reason for including existence of approximate units in the definition of the inverse semigroup action, as we did.

Hence $\ell^1(\alpha)$ is a Banach algebra. Let (π, v) be a covariant representation of α in a Banach algebra B. For any $f \in \ell^1(\alpha)$ the series

$$\pi \times v(f) := \sum_{t \in S} \pi(f(t))v_t$$

is norm convergent in B and $\|\pi \times v\| \leqslant \sum_{t\in S} \|\pi(f(t))v_t\| \leqslant \|f\|_1$. Hence $\pi \rtimes v : \ell^1(\alpha) \to B$ is a contractive linear map. Using Lemma 4.13(1) one readily sees that $\pi \rtimes v$ is also multiplicative. Hence $\pi \rtimes v$ is a representation. Clearly, $\overline{\pi \times v(\ell^1(\alpha))} = B(\pi, v)$. Note that typically $\pi \rtimes v$ will not be injective. Indeed, denoting by $a\delta_t$, for any $a \in I_t$, $t \in S$ the element in $\ell^1(\alpha)$ such that $a\delta_t(s) = 0$ for $s \neq t$ and $a\delta_t(t) = a$ we get that $\mathrm{span}\{a\delta_t : a \in I_t, t \in S\}$ is a dense subalgebra in $\ell^1(\alpha)$. Let us denote by

$$\mathcal{N} := \overline{\mathrm{span}}\{f(a\delta_s - a\delta_t)g : f, g \in \ell^1(\alpha), a \in I_s, s, t \in S, s \leqslant t\}$$

the ideal in $\ell^1(\alpha)$ generated by differences $a\delta_s - a\delta_t$ for $a \in I_s$ and $s, t \in S$ with $s \leqslant t$. It follows from Lemma 4.13(2) that for any covariant representation (π, v) we have

$$\mathcal{N} \subseteq \ker(\pi \times v),$$

and thus we $\pi \rtimes v$ factors to a representation of the quotient Banach algebra $\ell^1(\alpha)/\mathcal{N}$.

Definition 4.29. Let $\alpha : S \rightharpoonup A$ be an inverse semigroup action. We denote by $A \rtimes_\alpha S$ the Hausdorff completion of $\ell^1(\alpha)$ with respect to the submultiplive seminorm

$$\|f\|_{\max} := \sup\{\|\pi \times v(f)\| : (\pi, v) \text{ is a covariant representation of } \alpha\}.$$

We call the arising Banach algebra $A \rtimes_\alpha S$ the *(universal) Banach algebra crossed product* of α. More generally, if $\mathcal{R}$ is a class of some covariant representations of α, we define the $\mathcal{R}$*-crossed product* of α, denoted $A \rtimes_{\alpha,\mathcal{R}} S$, as the Hausdorff completion of $\ell^1(\alpha)$ in the seminorm

$$\|f\|_{\mathcal{R}} = \sup\{\|\pi \times v(f)\| : (\pi, v) \in \mathcal{R}\}.$$

In particular, any $(\pi, v) \in \mathcal{R}$ *integrates* to a representation $\pi \rtimes v$ of $A \rtimes_{\alpha,\mathcal{R}} S$, which sends the image of $a\delta_t$ to $\pi(a)v_t$, for $a \in I_t$, $t \in S$.

Remark 4.30. As we noted above, instead of completions of $\ell^1(\alpha)$ one could consider completions of the quotient algebra $\ell^1(\alpha)/\mathcal{N}$. In fact, it is often the case (though we do not know in general) that the seminorm $\|\cdot\|_{\max}$ on $\ell^1(\alpha)$ often becomes a norm on $\ell^1(\alpha)/\mathcal{N}$. It might even be that $A \rtimes_\alpha S = \ell^1(\alpha)/\mathcal{N}$.

We now give a universal description of $A \rtimes_\alpha S$, which is not present in [**BKM25**].

Theorem 4.31. *Let $\alpha : S \rightharpoondown A$ be an inverse semigroup action in a Banach algebra A. Then there is a $(A \rtimes_\alpha S)'$-normalized covariant representation (ι, u) of α in $A \rtimes_\alpha S$ such that $A \rtimes_\alpha S = B(\iota, u)$ and*

(1) *for any covariant representation (π, v) in a Banach algebra B there is a unique representation $\pi \rtimes v : A \rtimes_\alpha S \to B$ such that*
$$\pi(a)v_t = \iota(a)u_t \text{ for all } a \in I_t, t \in S.$$
(2) *For any representation $\Psi : A \rtimes_\alpha S \to B$ there is a unique B'-normalized covariant representation (π, v) of α in B such that $\Psi = \pi \rtimes v$ is the unique representation introduced in* (1).

Proof. By construction, for each $a \in \ell^1(\alpha)$ and $\varepsilon > 0$ there is a covariant representation (π, v) such that $\|a\|_{\max} < \|\pi \times v(a)\| + \varepsilon$. Thus there exist a family $\{(\pi^i, v^i)\}_{i\in I}$ where (π^i, v^i) is a covariant representation of α in a Banach algebra B_i and such that $\|a\|_{\max} = \sup_{i\in I} \|\pi \times v(a)\|$ for every $a \in \ell^1(\alpha)$. Let us now express it using the ℓ^∞-sum of these representations. Namely, let
$$B := \oplus_{i\in I}^{\ell^\infty} B_i'' = \{f : I \to \bigsqcup_i B_i'' : f(i) \in B_i'', \text{for } i \in I, \|f\|_\infty = \sup \|f(i)\| < \infty\}.$$
We define the algebra homomorphism $\iota : A \to B$ by $\iota(a) = \oplus_{i\in I}^{\ell^\infty} \pi_i(a)$, $a \in A$, and the semigroup homomorphism $u : S \to B$ by $u_t := \oplus_{i\in I}^{\ell^\infty} v_t^i$, $t \in S$. Clearly, (ι, u) is a covariant representation of α in B such that $\iota \times u = \oplus_{i\in I}^{\ell^\infty} \pi^i \times v_t^i$ and so $\|\iota \times u(a)\| = \|a\|_{\max}$ for $a \in \ell^1(\alpha)$. Thus it induces an isometric representation $\iota \rtimes u$ of $A \rtimes_\alpha S$ and so we may identify its range with $A \rtimes_\alpha S$. Then (ι, u) is a covariant representation in $A \rtimes_\alpha S$ such that $A \rtimes_\alpha S = B(\iota, u)$. By Proposition 4.17 we may assume that (ι, u) is B'-normalized, and then in fact we may view it as $(A \rtimes_\alpha S)'$-normalized covariant representation in $A \rtimes_\alpha S$ by Remark 4.15.

Property (1) is clear by construction. For (2), let $\Psi : A \rtimes_\alpha S \to B$ be a representation, and consider the representation $\Psi'' : (A \rtimes_\alpha S)'' \to B''$. Then $(\Psi'' \circ \iota, \Psi'' \circ u)$ is a well defined covariant representation of α in B such that $\Psi = (\Psi'' \circ \iota) \rtimes (\Psi'' \circ u)$. □

Remark 4.32. Every Banach algebra which the range of a covariant representation (ι, u) of α having property (1) above, is canonically isometrically isomorphic to $A \rtimes_\alpha S$. In this sense we could have defined $A \rtimes_\alpha S$ using this universal property (though we would still need to prove its existence). Property (2) describes *disintegration* for inverse semigroup crossed products.

Recall if a Banach algebra A has an approximate unit, then a representation $\pi : A \to B$ is called *non-degenerate* or *approximately unital* if it maps an approximate unit of A to an approximate unit of B. This is equivalent to the equality $\overline{\pi(A)B} = B$ and sometimes is called non-degeneracy.

Corollary 4.33. *Let $\alpha : G \rightharpoondown A$ be an action of a group G on a Banach algebra with an approximate unit (so it is also an inverse semigroup action). Then the group crossed product $F(\alpha)$, the algebra $\ell^1(\alpha)$ and the inverse semigrup crossed product $A \rtimes_\alpha G$ coincide.*

Moreover, for any non-degenerate representation $\Psi : A \rtimes_\alpha G \to B$ there is a unique pair (π, v) where $\pi : A \to B$ is a representation, $v : G \to \mathcal{UM}(B)$ is a group homomorphism into the group of invertible isometries in the multiplier algebra $\mathcal{M}(B)$ such that

$$\Psi(a\delta_g) = \pi(a)v_g \qquad \text{for all } a \in A,\ g \in G.$$

Proof. The first part is straightforward. Note that the canonical inclusion $A \ni a \mapsto a\delta_1 \subseteq F(\alpha) = \ell^1(G, A) = A \rtimes_\alpha G$ is non-degenerate. Thus in view of Theorem 4.31 it suffices to show that for any B'-covariant representation (π, v) of α in B where π is non-degenerate we necessarily have that the semigroup homomorphism $v : G \to (B'')_1$ takes values in the group $\mathcal{UM}(B)$. By definition of the covariant representation we have that $v_g\pi(a) = \pi(\alpha_g(a))v_g \in B$ and $\pi(a)v_g \in B$ for $g \in G$ and $a \in A$. Since $\pi(A)B = B = B\pi(A)$ by non-degeneracy of π, this implies that $v_gB \subseteq B$ and $Bv_g \subseteq B$. Hence $\{v_g\}_{g\in G} \subseteq \mathcal{M}(B)$. By B'-normalization v_1 is the unit in by $\mathcal{M}(B)$. Hence $\{v_g\}_{g\in G} \subseteq \mathcal{UM}(B)$ as they are contractive and invertible. □

Finally let us comment on a situation where an inverse semigroup S acts on a C^*-algebra A. Then the Banach algebra $\ell^1(\alpha)$ is naturally a Banach $*$-algebra with involution given by

$$f^*(t) := \alpha_{t^*}(f(t^*)^*), \qquad t \in S.$$

If (π, v) is a covariant representation of α in a C^*-algebra B, then Corollary 4.19 implies that the representation $\pi \times v : \ell^1(\alpha) \to B$ is automatically a $*$-homomorphism. Sieben [**Sie97**, Definition 4.4] defined the C^*-algebraic crossed product by completing $\ell^1(\alpha)$ using $*$-homomorphism $\pi \times v$ coming from covariant representations (π, v) on Hilbert spaces, as we described in Definition 4.8. Buss and Exel [**BE11**, Page 257] defined the C^*-algebraic crossed product (in the setting of twisted actions) similarly, but they used covariant representations (π, v) in C^*-algebras satisfying properties in Proposition 4.17, which characterize what we call B'-normalized covariant representations. Using our normalization procedure we can now explain why these two slightly different constructions coincide.

Corollary 4.34. *Let $\alpha : S \rightharpoonup A$ be an action of an inverse semigroup S where A is a C^*-algebra. Let $\mathcal{R}_{C^*}$ be the class of all covariant representations α in some C^*-algebra and $\mathcal{R}_H$ be the class of all covariant representations of α on some Hilbert space. Then*

$$A \rtimes_{\alpha, \mathcal{R}_{C^*}} S = A \rtimes_{\alpha, \mathcal{R}_H} S.$$

This algebra is a C^*-algebra which coincides with the crossed products introduced in [**Sie97**]*, and* [**BE11**].

Proof. Since $\mathcal{R}_{\mathrm{H}} \subseteq \mathcal{R}_{C^*}$ we get $\|\cdot\|_{\mathcal{R}_{\mathrm{H}}} \leqslant \|\cdot\|_{\mathcal{R}_{C^*}}$. This is equality, because for any covariant representation $(\pi, v) \in \mathcal{R}_{C^*}$ in a C^*-algebra B, we may assume that the enveloping W^*-algebra B'' is faithfully represented on a Hilbert space H. Then, by Corollary 4.28, the renormalized covariant representation $(\pi, \tilde{v})$ is a covariant representation of α on H, so $(\pi, \tilde{v}) \in \mathcal{R}_{\mathrm{H}}$. Moreover, $\pi \times v(f) = \pi \times \tilde{v}(f)$ for all $f \in \ell^1(\alpha)$. Hence $\|\cdot\|_{\mathcal{R}_{\mathrm{H}}} = \|\cdot\|_{\mathcal{R}_{C^*}}$. Clearly, this is a C^*-seminorm on the $*$-algebra $\ell^1(\alpha)$, and it does not change if we consider normalized representation in $\mathcal{R}_{C^*}$. This gives the claim. □

CHAPTER 5

Inverse semigroup graded groupoid Banach algebras

As we mentioned in the introduction to Chapter 4, one of our aims is to complete the convolution algebra of an étale groupoid $\mathcal{G}$ in a way that allows disintegration-integration theorem analogous to Theorem 4.31. In general such a completion will depend on the choice of an inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$. However, one of our main results that we discuss in this chapter is that completions using representations on L^p-spaces do not depend on the choice of S. This property is crucial when identifying algebras given by generators and relations with groupoid algebras. In addition, we will describe groupoid representations on L^p-spaces via weighted spatial partial isometries, and we establish a number of useful relationships between the corresponding norms on the convolution algebra. The material here is based on [**BKM25**, Sections 4,5], but again some proofs that we give are different and formulations are simpler, as we consider untwisted Hausdorff case. In particular, we avoid using the machinery of Borel extensions from [**BKM25**, Subsection 4.5].

5.1. Completions of the convolution algebra

Throughout this chapter we assume that $\mathcal{G}$ is a locally compact Hausdorff étale groupoid with the unit space X. The space of compactly supported continuous functions $C_c(\mathcal{G}) = \{f \in C(\mathcal{G}) : \overline{\mathrm{supp}}(f) \text{ is compact}\}$ on $\mathcal{G}$ with operations

$$(f * g)(\gamma) := \sum_{d(\alpha)=r(\beta)} f(\alpha)g(\beta) = \sum_{r(\eta)=r(\gamma)} f(\eta)g(\eta^{-1}\gamma), \qquad f^*(\gamma) := \overline{f(\gamma^{-1})}$$

for all $f, g \in C_c(\mathcal{G})$, $\gamma \in \mathcal{G}$, becomes a $*$-algebra. We refer to it as the convolution algebra for $\mathcal{G}$, as the convolution product will play the major role in our considerations. For any open subset $U \subseteq \mathcal{G}$ we may view $C_c(U)$ as a subspace of $C_c(\mathcal{G})$ and looking at such spaces coming from bisections helps to understand the structure of the algebra $C_c(\mathcal{G})$. Namely, recall that the family of all bisections $\mathrm{Bis}(\mathcal{G})$ together with natural operations (1.3) is an inverse semigroup. For any $f \in C_c(U)$, $g \in C_c(V)$, with $U, V \in \mathrm{Bis}(\mathcal{G})$ we have $f * g \in C_c(UV)$ and

$$f * g(\alpha\beta) = f(\alpha) \cdot g(\beta) \qquad \alpha \in U, \beta \in V.$$

Also we have $f^* \in C_c(U^*) = C_c(U^{-1})$ where $f^*(\gamma) = \overline{f(\gamma^{-1})}$ for $\gamma \in U^{-1}$. Thus

$$C_c(U) * C_c(V) = C_c(UV), \qquad C_c(U)^* = C_c(U^*) \qquad \text{for } U, V \in \mathrm{Bis}(\mathcal{G}),$$

and as the space $\{C_c(U)\}_{U \in \mathrm{Bis}(\mathcal{G})}$ linearly span $C_c(\mathcal{G})$, they form an "inverse semigroup grading" of the $*$-algebra $C_c(\mathcal{G})$. This grading restricts to any inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ which covers $\mathcal{G}$:

Lemma 5.1. *$C_c(\mathcal{G}) = \operatorname{span}\{f \in C_c(U) : U \in S\}$ for any $S \subseteq \operatorname{Bis}(\mathcal{G})$ that covers $\mathcal{G}$.*

Proof. Let $f \in C_c(\mathcal{G})$. Since $K := \overline{\operatorname{supp}}(f)$ is compact and S covers K, there are $\{U_i\}_{i=1}^n \subseteq S$ such that $K \subseteq \bigcup_{i=1}^n U_i$. Let $\{h_i\}_{i=1}^n$ be a partition of unity on K subordinated to $\{U_i\}_{i=1}^n$ and put $f_i := h_i \cdot f$, $i = 1, ..., n$. Then $f_i \in C_c(U_i)$ and $f = \sum_{i=1}^n f_i$. □

A priori there is no universal Banach algebra completion of $C_c(\mathcal{G})$. Instead, one usually defines a "universal norm" by taking supremum of norms coming from a fixed class of representations of $C_c(\mathcal{G})$. For this procedure to work, one needs to check the following two technical conditions:

Lemma 5.2. *Let $\mathcal{R}$ be a class of some algebra homomorphisms from $C_c(\mathcal{G})$ into some Banach algebras. Then the formula*

$$\|f\|_{\mathcal{R}} = \sup\{\|\psi(f)\| : \psi \in \mathcal{R}\}$$

defines a submultiplicative norm on $C_c(\mathcal{G})$ if and only if $\mathcal{R}$ is bounded and separates points in the sense that

1. *$\sup\{\|\psi(f)\| : \psi \in \mathcal{R}\} < \infty$ and*
2. *for any non-zero $f \in C_c(\mathcal{G})$ there is $\psi \in \mathcal{R}$ such that $\psi(f) \neq 0$.*

Moreover, every submultiplicative norm on $C_c(\mathcal{G})$ is equal to $\|\cdot\|_{\mathcal{R}}$ for some $\mathcal{R}$.

Proof. Even though $\mathcal{R}$ is a class, $\{\|\psi(f)\| : \psi \in \mathcal{R}\} \subseteq [0, \infty]$ is a set and hence its supremum makes sense and it exists (however it may be infinite). It is straightforward that $\|\cdot\|_{\mathcal{R}}$ is a submultiplicative seminorm if and only if it is finite so that (1) holds. If this is the case then this seminorm is a norm if and only if (2) holds. For the last part of the assertion let $\|\cdot\|$ be any submultiplicative norm on $C_c(\mathcal{G})$. Then taking $\mathcal{R} := \{i_{\|\cdot\|}\}$ where $i_{\|\cdot\|} : C_c(\mathcal{G}) \to \overline{C_c(\mathcal{G})}^{\|\cdot\|}$ is the canonical inclusion we get $\|\cdot\| = \|\cdot\|_{\mathcal{R}}$. □

Remark 5.3. In the sequel we will often denote by $\|\cdot\|_{\mathcal{R}}$ norms on $C_c(\mathcal{G})$ even when they do not come from a prescribed class of representations, which is allowed by the last part of the above lemma.

Example 5.4 (Groupoid C^*-algebras). Let $\mathcal{R}$ be the class of all $*$-homomorphisms from $C_c(\mathcal{G})$ into C^*-algebras. Then the conditions in Lemma 5.2 are satisfied. Indeed, let $\psi : C_c(\mathcal{G}) \to B$ be a $*$-homomorphism into a C^*-algebra B. Note first that $C_c(X)$ is a $*$-subalgebra of $C_c(\mathcal{G})$ and by minimality of C^*-norm $*$-homomorphism $\psi : C_c(X) \to B$ is contractive with respect to supremum norm $\|\cdot\|_\infty$ on $C_c(X)$. More specifically, if $f \in C_c(X)$ then $f \in C_0(U)$ for some precompact $U \subseteq X$ and so $\|\psi(f)\| \leqslant \|f\|_\infty$ as $\psi : C_0(U) \to B$ is a $*$-homomorphism between C^*-algebras. Now for any $f \in C_c(U)$ supported on a bisection $U \in \operatorname{Bis}(\mathcal{G})$, using the C^*-equality and that $f * f^* \in C_c(UU^{-1}) = C_c(r(U)) \subseteq C_c(X)$ that we get

$$\|\psi(f)\|^2 = \|\psi(f)\psi(f^*)\| = \|\psi(f * f^*)\| \leqslant \|f * f^*\|_\infty = \sup_{\gamma \in U} |f(\gamma)|^2 = \|f\|_\infty^2,$$

that is $\|\psi(f)\| \leqslant \|f\|_\infty$. By Lemma 5.1, any $f \in C_c(\mathcal{G})$ can be written as $f = \sum_{i=1}^n f_i$ for some $f_i \in C_c(U_i)$ and $U_i \in \operatorname{Bis}(\mathcal{G})$. Hence by the triangle inequality and contractiveness on each bisection we get

$$\|\psi(f)\| \leqslant \sum_{i=1}^{n} \|\psi(f_i)\| \leqslant \sum_{i=1}^{n} \|f_i\|_\infty.$$

As $\sum_{i=1}^{n} \|f_i\|_\infty$ is finite and does not depend on ψ we conclude that the class $\mathcal{R}$ is bounded. Hence $\|\cdot\|_{\mathcal{R}}$ is a C^*-seminorm. To see that $\mathcal{R}$ separates elements of $C_c(\mathcal{G})$ (equivalently $\|\cdot\|_{\mathcal{R}}$ is a norm) we use the *regular representation* $\Lambda : C_c(\mathcal{G}) \to B(\ell^2(\mathcal{G}))$ given by

$$\Lambda(f)\xi(\gamma) := (f * \xi)(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta) \cdot \xi(\eta^{-1}\gamma), \qquad f \in C_c(\mathcal{G}),\ \xi \in \ell^2(\mathcal{G}).$$

This is a well defined $*$-homomorphism and denoting by $\{1_\gamma\}_{\gamma\in\mathcal{G}}$ the standard orhogonal basis for $\ell^2(\mathcal{G})$, for any $f \in C_c(\mathcal{G})$ and any $\gamma \in \mathcal{G}$ we have $\|\Lambda(f)1_\gamma\|_2 = \left(\sum_{d(\eta)=d(\gamma)} |f(\eta\gamma^{-1})|^2\right)^{1/2}$. This implies that Λ is injective, cf. Proposition 5.24 below. Hence we conclude that both $\|\cdot\|_{C^*} = \|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{C^*}^{\mathrm{r}} = \|\cdot\|_{\{\Lambda\}}$ are C^*-norms on $C_c(\mathcal{G})$. The corresponding completions

$$C^*(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{C^*}} \qquad \text{and} \qquad C^*_{\mathrm{r}}(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|^{\mathrm{r}}_{C^*}}$$

are called the *full groupoid C^*-algebra* and the *reduced groupoid C^*-algebra* for $\mathcal{G}$. In particular, $\|\cdot\|_{C^*}$ is the maximal C^*-norm on $C_c(\mathcal{G})$.

Example 5.5 (Hahn's completions). The domain and range maps $d, r : \mathcal{G} \to X$ induce the following three submultiplicative norms on $C_c(\mathcal{G})$:

$$\|f\|_{d*} := \sup_{x\in X} \sum_{d(\gamma)=x} |f(\gamma)|, \qquad \|f\|_{r*} := \sup_{x\in X} \sum_{r(\gamma)=x} |f(\gamma)|, \tag{5.1}$$

$$\|f\|_I := \max\{\|f\|_{d*}, \|f\|_{r*}\}, \tag{5.2}$$

see [**Pat99**, the proof of Theorem 2.2.1] or [**Ren80**, II, 1.4]. The completions of $C_c(\mathcal{G})$ in these norms are denoted by $F_{*d}(\mathcal{G})$, $F_{*r}(\mathcal{G})$ and $F_I(\mathcal{G})$, respectively. The above norms, especially $\|\cdot\|_I$, are common in the literature and $\|\cdot\|_I$ is often called *Hahn's I-norm*, as it was introduced in [**Hah78**]. Thus we call $F_I(\mathcal{G})$ *Hahn's algebra*. Note that for each $f \in C_c(\mathcal{G})$ we have $\|f\|_{*d} = \|f^*\|_{*r}$ and so Hahn's algebra $F_I(\mathcal{G})$ is a Banach $*$-algebra that can be viewed as a symmetrized version of both $F_{*d}(\mathcal{G})$ and $F_{*r}(\mathcal{G})$.

The important feature of the above norms is that when restricted to any subspace $C_c(U) \subseteq C_c(\mathcal{G})$, where $U \in \mathrm{Bis}(\mathcal{G})$, they coincide with the supremum norm $\|\cdot\|_\infty$. Hence each space $C_0(U)$ embeds naturally into the completions of $C_c(\mathcal{G})$ in each of these norms. In particular, $C_0(X)$ is naturally a Banach subalgebra in the above completions. This is an important part of the structure that we want to keep. However, there are some natural completions of $C_c(\mathcal{G})$ equipped with natural inverse semigroup gradings for some proper inverse semigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$. Here transformation groupoids derived from group actions are perhaps the most notable example. Thus we introduce yet another family of norms that are parametrised by subsemigroups of $\mathrm{Bis}(\mathcal{G})$.

LEMMA 5.6. *Let $S \subseteq \mathrm{Bis}(\mathcal{G})$ be a semigroup covering $\mathcal{G}$. The formula*

$$\|f\|^S_{\max} := \inf\left\{ \sum_{k=1}^n \|f_k\|_\infty : f = \sum_{k=1}^n f_k,\ f_k \in C_c(U_k),\ U_k \in S \right\}. \tag{5.3}$$

defines a submultiplicative norm on $C_c(\mathcal{G})$, which is the largest submultiplicative norm on $C_c(\mathcal{G})$ that agrees with $\|\cdot\|_\infty$ on each $C_c(U) \subseteq C_c(\mathcal{G})$, for $U \in S$. In particular, denoting by $F^S(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|^S_{\max}}$ the relevant completion we have:

1. *If $S \subseteq \mathrm{Bis}(\mathcal{G})$ is unital, i.e. $X \in S$, then $C_0(X) \subseteq F^S(\mathcal{G})$ is a Banach subalgebra;*
2. *If S is an inverse semigroup, then $F^S(\mathcal{G})$ is a Banach $*$-algebra with the involution extending that of $C_c(\mathcal{G})$.*

PROOF. By Lemma 5.1 the infimum in (5.3) is over non-empty set, and hence it exists and is finite. Let $f, g \in C_c(\mathcal{G})$. For any $\varepsilon > 0$ there are $f_k \in C_c(U_k)$ and $g_l \in C_c(V_k)$, where $U_k, V_l \in S$, such that $f = \sum_{k=1}^n f_k$, $g = \sum_{l=1}^m g_l$ and $\|f\|^S_{\max} + \varepsilon > \sum_{k=1}^n \|f_k\|_\infty$ and $\|g\|^S_{\max} + \varepsilon > \sum_{l=1}^m \|g_l\|_\infty$. Then $f * g = \sum_{k,l} f_k * g_l$, and $f_k * g_l \in C_c(U_k V_l)$, $U_k V_l \in S$. Therefore

$$\|f * g\|^S_{\max} \leqslant \sum_{k,l} \|f_k * g_l\|_\infty \leqslant \sum_{k,l} \|f_k\|_\infty \|g_l\|_\infty < \big(\|f\|^S_{\max} + \varepsilon\big)\big(\|g\|^S_{\max} + \varepsilon\big).$$

This shows that $\|\cdot\|^S_{\max}$ is submultiplicative. The proof of the triangle inequality is even simpler. If in addition $f \in C_c(U)$ for some $U \in S$, then there is $\gamma_0 \in U$ where $|f|$ attains its maximum, and retaining the above choice of f_k's we get

$$\|f\|_\infty = |f(\gamma_0)| \leqslant \sum_{k=1}^n |f_k(\gamma_0)| \leqslant \sum_{k=1}^n \|f_k\|_\infty < \|f\|^S_{\max} + \varepsilon,$$

which shows that $\|f\|_\infty \leqslant \|f\|^S_{\max}$. The converse inequality $\|f\|^S_{\max} \leqslant \|f\|_\infty$ holds by (5.3). Hence $\|\cdot\|^S_{\max}$ coincides with $\|\cdot\|_\infty$ on $C_c(U)$ for $U \in S$. If $\|\cdot\|$ is any norm on $C_c(\mathcal{G})$ that agrees with $\|\cdot\|_\infty$ on $C_c(U) \subseteq C_c(\mathcal{G})$, for $U \in S$, then for any $f = \sum_{k=1}^n f_k$, $f_k \in C_c(U_k)$, $U_k \in S$, we have $\|f\| \leqslant \sum_{k=1}^n \|f_k\| = \sum_{k=1}^n \|f_k\|_\infty$, which implies $\|f\| \leqslant \|f\|^S_{\max}$. This proves the first part of the assertion. Item (1) is clear. If S is an inverse semigroup then $f^* = \sum_{k=1}^n f_k^*$, $f_k^* \in C_c(U_k^*)$, where $U_k^* \in S$. Thus $\|f^*\|^S_{\max} = \|f\|^S_{\max}$ by (5.3). This shows (2). $\square$

REMARK 5.7. Using the notation from Lemma 5.2, we have $\|\cdot\|^S_{\max} = \|\cdot\|_{\mathcal{R}}$ where $\mathcal{R}$ is the class of all homomorphisms $\psi : C_c(\mathcal{G}) \to B$ into Banach algebras such that $\|\psi(f)\| \leqslant \|f\|_\infty$ for any $f \in C_c(U)$ and any $U \in S$.

DEFINITION 5.8. Let $S \subseteq \mathcal{G}$ be a unital inverse subsemigroup covering $\mathcal{G}$. We call the associated $*$-algebra $F^S(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|^S_{\max}}$ the *universal S-graded Banach algebra* of $\mathcal{G}$. We also denote by $\|\cdot\|_{\max}$ the norm $\|\cdot\|^{\mathrm{Bis}(\mathcal{G})}_{\max}$ and put $F(\mathcal{G}) := F^{\mathrm{Bis}(\mathcal{G})}(\mathcal{G})$.

REMARK 5.9. For any unital inverse subsemigroup $S \subseteq \mathcal{G}$ covering $\mathcal{G}$ we have

$$\|\cdot\|_\infty \leqslant \|\cdot\|_{d_*}, \|\cdot\|_{C^*}, \|\cdot\|_{r_*} \leqslant \|\cdot\|_I \leqslant \|\cdot\|_{\max} \leqslant \|\cdot\|^S_{\max}$$

on $C_c(\mathcal{G})$. In particular, $\|\cdot\|_{\max}$ is minimal amongst all norms $\|\cdot\|^S_{\max}$ on $C_c(\mathcal{G})$, and in general $S_1 \subseteq S_2 \subseteq \mathrm{Bis}(\mathcal{G}) \implies \|\cdot\|^{S_2}_{\max} \leqslant \|\cdot\|^{S_1}_{\max}$. Usually there is no maximal amongst the norms $\|\cdot\|^S_{\max}$. When considering algebras $F^S(\mathcal{G})$ we may always assume that S is a wide inverse semigroup as the 'saturation' $\widetilde{S} := \{U : U \text{ is an open subset of } V \in S\}$ of S is a wide inverse subsemigroup of $\mathrm{Bis}(\mathcal{G})$ and $\|\cdot\|^S_{\max} = \|\cdot\|^{\widetilde{S}}_{\max}$.

EXAMPLE 5.10. Let $\mathcal{G} = G \times X$ be the transformation groupoid for a discrete group action $\theta : G \to \mathrm{Homeo}(X)$, cf. Example 1.17. The natural G-grading of $\mathcal{G}$ is given by $S := \{\{g\} \times X\}_{g \in G} \cong G$. For this (inverse semi)group grading and $f \in C_c(\mathcal{G})$ we have

$$\|f\|^S_{\max} = \sum_{g \in G} \max_{x \in X} |f(g,x)|.$$

In fact, it is easy to see that we have a natural isometric *-isomorphism

$$F^S(\mathcal{G}) \cong \ell^1(G, C_0(X))$$

where $\ell^1(G, C_0(X))$ is the Banach $*$-algebra crossed product discussed in Section 4.1. The associated Hahn norms are given by the formulas

$$\|f\|_{d*} = \max_{x \in X} \sum_{g \in G} |f(g,x)|, \qquad \|f\|_{r*} = \max_{x \in X} \sum_{g \in G} |f(g, \varphi_g(x))|.$$

The algebra $F_{d*}(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{d*}}$ plays crucial role in the study of amenability in [**Mon11**], where Monod denotes it by $C(X, \ell^1 G)$ and calls it the Banach algebra crossed product. Note that the Hahn norm $\|\cdot\|_I$ is in general strictly smaller than $\|\cdot\|_{\ell^1(G,C_0(X))} = \|\cdot\|^S_{\max}$. Obviously, when X is a singleton, so that $\mathcal{G} = G$ is a group, then $S = \mathrm{Bis}(\mathcal{G})$ and all the norms $\|\cdot\|_{d*}, \|\cdot\|_{r*}, \|\cdot\|_I, \|\cdot\|^S_{\max}$ coincide with the ℓ^1-norm in $\ell^1(G)$.

5.2. Even more completions and basic properties

We fix a unital inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ covering $\mathcal{G}$.

DEFINITION 5.11. For a class $\mathcal{R}$ of representations of $F^S(\mathcal{G})$ we denote by $F^S_{\mathcal{R}}(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{\mathcal{R}}}$ the Hausdorff completion in the seminorm $\|f\|_{\mathcal{R}} := \sup\{\|\psi(f)\| : \psi \in \mathcal{R}\}$. For a class $\mathcal{E}$ of Banach spaces we write $F^S_{\mathcal{E}}(\mathcal{G}) := F^S_{\mathcal{R}}(\mathcal{G})$ and $\|\cdot\|_{\mathcal{E}} := \|\cdot\|_{\mathcal{R}}$ for $\mathcal{R}$ consisting of all representations of $F^S(\mathcal{G})$ in $B(E)$ for $E \in \mathcal{E}$.

REMARK 5.12. If $\|\cdot\|_{\mathcal{R}} \leqslant \|\cdot\|_{\max}$, then the algebra $F^S_{\mathcal{R}}(\mathcal{G})$ does not depend on the choice of S, as it can be viewed as a Hausdorff completion of $F(\mathcal{G})$. In this thesis we will be mainly interested in the case when $\|\cdot\|_{\mathcal{R}} \leqslant \|\cdot\|_I$ and hence all the more $\|\cdot\|_{\mathcal{R}} \leqslant \|\cdot\|_{\max}$.

LEMMA 5.13. *Every Banach algebra of the form $F^S_{\mathcal{R}}(\mathcal{G})$ has an approximate unit. Namely, for any approximate unit $\{\mu_i\}_i$ in $C_0(X)$ its image in $F^S_{\mathcal{R}}(\mathcal{G})$ is an approximate unit in $F^S_{\mathcal{R}}(\mathcal{G})$. In particular, if X is compact then $F^S_{\mathcal{R}}(\mathcal{G})$ is unital.*

PROOF. It suffices to prove that for $f \in C_c(U)$, $U \in S$ the net $\{\mu_i * f\}_i$ converges to f in $\|\cdot\|^S_{\max}$. But then $\mu_i * f - f \in C_c(U)$ and therefore $\|\mu_i * f - f\|^S_{\max} = \|\mu_i * f - f\|_\infty = \max_{\gamma \in U} |\mu_i(r(\gamma))f(\gamma) - f(\gamma)|$, which obviously tends to zero. □

We denote by A^{op} the *opposite algebra* to an algebra A, meaning that the linear structure of A^{op} is the same as that of A but the multiplication is performed in the reverse order. When A is a Banach algebra then A^{op} is a Banach algebra with the same norm. An algebra isomorphic to its opposite is called *self-opposite*. The convolution algebra of the groupoid $\mathcal{G}$ is naturally self-opposite with isomorphism given by the contravariant isomorphism $\mathcal{G} \ni \gamma \to \gamma^{-1} \in \mathcal{G}$. Namely, the formula

$$(\check{f})(\gamma) := f(\gamma^{-1}), \qquad f \in C_c(\mathcal{G}),\ \gamma \in \mathcal{G}, \tag{5.4}$$

yields a $*$-isomorphism

$$C_c(\mathcal{G})^{\mathrm{op}} \cong C_c(\mathcal{G}).$$

It follows from the formulas for the norms $\|\cdot\|_{d*}, \|\cdot\|_{r*}, \|\cdot\|_{I}, \|\cdot\|^S_{\max}$ that the above isomorphism extends to isometric isomorphisms

$$F^S(\mathcal{G})^{\mathrm{op}} \cong F^S(\mathcal{G}), \qquad F_I(\mathcal{G})^{\mathrm{op}} \cong F_I(\mathcal{G}), \qquad F_{d*}(\mathcal{G})^{\mathrm{op}} \cong F_{r*}(\mathcal{G}).$$

Let us consider completions of $C_c(\mathcal{G})$ given by representations on a given class of Banach spaces. If $\mathcal{E}$ is a class of Banach spaces we denote by $\mathcal{E}'$ the class of dual spaces to spaces in $\mathcal{E}$. For any other class $\mathcal{F}$ of Banach spaces we write $\mathcal{E} \subseteq_{\mathrm{iso}} \mathcal{F}$ if for every $E \in \mathcal{E}$ there is $F \in \mathcal{F}$ such that $E \cong F$.

PROPOSITION 5.14. *If $\mathcal{E}' \subseteq_{\mathrm{iso}} \mathcal{F}$ then the map $f \mapsto \check{f}$ induces a representation $F^S_{\mathcal{F}}(\mathcal{G})^{\mathrm{op}} \to F^S_{\mathcal{E}}(\mathcal{G})$. If $\mathcal{E}' \subseteq_{\mathrm{iso}} \mathcal{F}$ and $\mathcal{F}' \subseteq_{\mathrm{iso}} \mathcal{E}$ then this representation is an isometric isomorphism*

$$F^S_{\mathcal{F}}(\mathcal{G})^{\mathrm{op}} \cong F^S_{\mathcal{E}}(\mathcal{G}).$$

PROOF. For any representation $\pi : F^S(\mathcal{G})^{\mathrm{op}} \to B(E)$ the formula $\pi'(f) := \pi(\check{f})'$, $f \in C_c(\mathcal{G})$, defines a representation $\pi' : F^S(\mathcal{G}) \to B(E')$. If $\mathcal{E}' \subseteq_{\mathrm{iso}} \mathcal{F}$ and $E \in \mathcal{E}$, then we may replace E' by $F \in \mathcal{F}$ such that $E' \cong F$, an then we get a representation $\pi' : F^S(\mathcal{G}) \to B(F)$. This implies that $\|\check{f}\|^S_{\mathcal{E}} \leqslant \|f\|^S_{\mathcal{F}}$ for $f \in C_c(\mathcal{G})$. Thus the isomorphism $C_c(\mathcal{G})^{\mathrm{op}} \cong C_c(\mathcal{G})$ extends to a representation $F^S_{\mathcal{F}}(\mathcal{G})^{\mathrm{op}} \overset{\text{dense}}{\twoheadrightarrow} F^S_{\mathcal{E}}(\mathcal{G})$. Similarly, $\mathcal{F}' \subseteq_{\mathrm{iso}} \mathcal{E}$ implies $\|\cdot\|^S_{\mathcal{F}} \leqslant \|\cdot\|^S_{\mathcal{E}}$. Hence $F^S_{\mathcal{F}}(\mathcal{G})^{\mathrm{op}} \cong F^S_{\mathcal{E}}(\mathcal{G})$ if $\mathcal{E}' \subseteq_{\mathrm{iso}} \mathcal{F}$ and $\mathcal{F}' \subseteq_{\mathrm{iso}} \mathcal{E}$. ☐

COROLLARY 5.15. *If $\mathcal{E}$ consists of reflexive Banach spaces, then $F^S_{\mathcal{E}}(\mathcal{G})^{\mathrm{op}} \cong F^S_{\mathcal{E}'}(\mathcal{G})$.*

PROOF. Take $\mathcal{F} = \mathcal{E}'$ in Proposition 5.14. ☐

COROLLARY 5.16. *We have $C^*(\mathcal{G})^{\mathrm{op}} \cong C^*(\mathcal{G})$.*

PROOF. Apply Corollary 5.15 to the case where $\mathcal{E}$ consists of Hilbert spaces. ☐

Corollaries 5.15, 5.16 are known, cf. [**GL17**], [**GT15**]. Our formulation of Proposition 5.14 allows one to use it beyond the case of reflexive spaces. Our main interest here are of course L^1 and L^∞-spaces. Recall that a *Lindenstrauss space* or *L^1-predual space* is a Banach space E whose dual E' is an L^1-space.

COROLLARY 5.17. *Let $\mathcal{E}$ be the class of L^1-spaces, $\mathcal{F}_1$ the class of L^∞-spaces, $\mathcal{F}_2$ the class of C_0-spaces and $\mathcal{F}_3$ the class of Lindenstrauss spaces, cf. Remark 3.13. Then*

$$F^S_{\mathcal{F}_1}(\mathcal{G}) \cong F^S_{\mathcal{F}_2}(\mathcal{G}) \cong F^S_{\mathcal{F}_3}(\mathcal{G}) \cong F^S_{\mathcal{E}}(\mathcal{G})^{\mathrm{op}}.$$

PROOF. It is well known that duals to L^1-spaces are isomorphic to L^∞-spaces and that duals to C_0-spaces are isomorphic to L^1-spaces, see for instance [**Lac74**]. By Gelfand-Naimark theorem L^∞-spaces are isomorphic to C_0-spaces. Thus we have $\mathcal{E}' \subseteq_{\text{iso}} \mathcal{F}_1 \subseteq_{\text{iso}} \mathcal{F}_2 \subseteq \mathcal{F}_3$ and $\mathcal{F}_1 \subseteq \mathcal{F}_2' \subseteq \mathcal{F}_3' \subseteq_{\text{iso}} \mathcal{E}$. Hence the assertion follows from the second part of Proposition 5.14. □

5.3. Disintegration-Integration Theorem

Recall that every inverse semigroup action on X gives rise to an étale groupoid $\mathcal{G}$ with unit space X and every étale groupoid arises in this way, see Section 1.3. In this correspondence the inverse semigroup in question is not uniquely determined, for instance we can take any wide inverse subsemigroup of $\operatorname{Bis}(\mathcal{G})$, see Proposition 1.20. Thus an inverse semigroup action $\theta : S \rightharpoondown X$ is formally a richer structure than the associated transformation groupoid $\mathcal{G}$ (the groupoid remembers orbits of the action rather the action itself). Moreover, by Gelfand duality the action $\theta : S \rightharpoondown X$ is equivalent to an inverse semigroup action $\alpha : S \rightharpoondown C_0(X)$, cf. Example 4.6. In this section we show that the associated Banach algebra crossed product $C_0(X) \rtimes_\alpha S$, as constructed in Section 4.3, is the groupoid Banach algebra $F^{\overline{S}}(\mathcal{G})$ as introduced in Definition 5.8 where $\overline{S}$ denotes the canonical image of S in $\operatorname{Bis}(\mathcal{G})$. In accordance, with Theorem 4.31, we need to show that

- every covariant representation of α “integrates” to a representation of $F^{\overline{S}}(\mathcal{G})$,
- every representation of $F^{\overline{S}}(\mathcal{G})$ “disintegrates” to a covariant representation of α.

Let us then fix an action $\alpha : S \to \operatorname{PAut}(C_0(X))$ of an inverse semigroup S on $C_0(X)$. Equivalently, we fix an inverse semigroup action $\theta : S \to \operatorname{PHomeo}(X)$, as in Definition 1.7, and for every $t \in S$, the isomorphism $\alpha_t : C_0(X_{t^*}) \to C_0(X_t)$ is given by composition $\alpha_t(a) = a \circ \theta_{t^*}$ with the homeomorphism $\theta_t : X_{t^*} \to X_t$. Let $\mathcal{G} = S \ltimes_\theta X$ be the associated transformation groupoid. Recall that we have a canonical semigroup homomorphism $S \ni t \mapsto U_t = \{[t, x] : x \in X_{t^*}\} \subseteq \operatorname{Bis}(\mathcal{G})$. Putting

$$\overline{S} := \{U_t : t \in S\} \cup \{X\},$$

we obtain a unital inverse subsemigroup of $\operatorname{Bis}(\mathcal{G})$. For each $t \in S$ we have a linear isomorphism

$$C_c(X_t) \ni a \mapsto a\delta_t \in C_c(U_t), \qquad a\delta_t[t, x] := a(\theta_t(x)), \qquad x \in X_{t^*},$$

In terms of bisections in $\overline{S}$ this means that for every $U \in \overline{S} \subseteq \operatorname{Bis}(\mathcal{G})$ we have

$$C_c(r(U)) \ni a \mapsto a\delta_U \in C_c(U) \quad \text{where} \quad a\delta_U(\gamma) := a(r(\gamma)), \;\; \gamma \in U. \tag{5.5}$$

This isomorphism gives isometry from $(C_c(U), \|\cdot\|_\infty)$ onto a subspace of $(C_c(\mathcal{G}), \|\cdot\|_{\max}^{\overline{S}})$.

LEMMA 5.18. *$C_c(\mathcal{G})$ is spanned by elements $a_t\delta_t$, $a_t \in C_c(X_t)$, $t \in S$, and*

(1) *$a_s\delta_s \cdot a_t\delta_t = a_s(a_t \circ \theta_{s^*})\delta_{st}$ and $(a_t\delta_t)^* = \overline{a_t \circ \theta_t}\delta_{t^*}$;*
(2) *$s \leqslant t$ implies $X_s \subseteq X_t$ and $a\delta_s = a\delta_t$ for any $a \in C_c(X_s)$.*

PROOF. The initial claim follows from Lemma 5.1. Item (2) follows from Remark 1.8 and (1) is straightforward, cf. [**Exe08**, Propositions 3.10, 7.5, 7.6]. □

Proposition 5.19. *Every covariant representation* (π, v) *of* α *in a Banach algebra* B *integrates to a representation* $\pi \rtimes v : F^{\overline{S}}(\mathcal{G}) \to B$ *such that*

$$\pi \rtimes v(a_t\delta_t) = \pi(a_t)v_t, \qquad a_t \in C_c(X_t),\ t \in S.$$

If B *is a* C^**-algebra then* $\pi \rtimes v$ *is* $*$*-preserving. If* $(\pi, \tilde{v})$ *is a* B'*-normalization (or* B_**-normalization) of* (π, v) *as in Proposition 4.17 (or Proposition 4.24) then* $\pi \rtimes v = \pi \rtimes \tilde{v}$.

Proof. We claim that the formula $\pi \rtimes v(\sum_{t\in F} a_t\delta_t) = \sum_{t\in F} \pi(a_t)v_t$, where $F \subseteq S$ is finite, yields a well defined map, or equivalently, that $\sum_{t\in F} a_t\delta_t = 0$ implies $\sum_{t\in F} \pi(a_t)v_t = 0$. This is proved in [**Exe08**, Lemma 8.4] using measure theoretical methods, under the assumption that $\mathcal{G}$ is second countable. We prove this in general using topological tools and induction on the cardinality of F. So suppose that $\sum_{t\in F} a_t\delta_t = 0$. If $|F| = 1$, then $\sum_{t\in F} \pi(a_t)v_t = 0$ (because $a\delta_t = 0$ iff $a_t = 0$). So assume that this happens for all sets with cardinality smaller than that of F. Pick any $t_0 \in F$ and put $F_0 = F\backslash\{t_0\}$. Then $\sum_{t\in F_0} a_t\delta_t = -a_{t_0}\delta_{t_0}$. So the closure of $\{\gamma \in S \times X : (a_{t_0}\delta_{t_0})(\gamma) \neq 0\}$ in U_{t_0}, which is a compact set K in U_{t_0}, is covered by $\{U_t \cap U_{t_0}\}_{t\in F}$. By construction $U_t \cap U_{t_0} = \bigcup_{s\leqslant t,t_0} U_s$. So for each $t \in F_0$ we may find $s_1^t, ..., s_{n_t}^t \leqslant t, t_0$ such that $K \subseteq \bigcup_{t\in F}\bigcup_{i=1}^{n_t} U_{s_i^t}$. Let $\{\varrho_{s_i^t}\}_{i=1...,n_t,t\in F_0}$ be a partition of unity on K subordinated to this open cover. Write $\widetilde{\varrho}_{s_i^t} := \varrho_{s_i^t} \circ r|_{U_{s_i^t}} \in C_c(X_{s_i^t}) \subseteq C_c(X_t \cap X_{t_0})$ for each i and t. Then by Lemma 5.18(2) and Lemma 4.13(2) (applied to each t, n_t-times) we get

$$\sum_{t\in F} a_t\delta_t = \sum_{t\in F_0}(a_t + \sum_{i=1}^{n_t} \widetilde{\varrho}_{s_i^t} a_{t_0})\delta_t \qquad \text{and} \qquad \sum_{t\in F} \pi(a_t)v_t = \sum_{t\in F_0} \pi(a_t + \sum_{i=1}^{n_t} \widetilde{\varrho}_{s_i^t} a_{t_0})v_t.$$

Thus the claim follows by the inductive hypotheses.

Now Lemma 4.13(1) and Lemma 5.18(1) readily imply that $\pi \rtimes v : C_c(\mathcal{G}) \to B$ is an algebra homomorphism, which is $*$-preserving if B is a C^*-algebra. It is $\|\cdot\|^S_{\max}$ contractive because $\|\pi \rtimes v(a_t\delta_t)\| = \|\pi(a_t)v_t\| \leqslant \|a_t\|_\infty = \|a_t\delta_t\|_\infty$ for every $a_t\delta_t \in C_c(U_t)$, $t \in S$. Hence it extends to a representation $\pi \rtimes v : F^{\overline{S}}(\mathcal{G}) \to B$, which is $*$-preserving if B is a C^*-algebra. The last statement is clear. ∎

In order to disintegrate representations of $F^{\overline{S}}(\mathcal{G})$ we will use the following lemma.

Lemma 5.20. *Let* $\psi : A \to E$ *be a contractive linear map from a* C^**-algebra into a Banach space* E *with a predual* E_**. There is a contractive element* $\psi(1_A) \in E$ *such that for every approximate unit* $\{\mu_i\}_i$ *in* A *we have* $\psi(1_A) = E_*\text{-}\lim_i \psi(\mu_i) \in E$.

Proof. We identify E_* with a subspace of E'. Let $f \in E_*$. Then $\psi'(f) = f \circ \psi : A \to \mathbb{C}$ is a bounded functional. Hence it decomposes to $\psi'(f) = \sum_{k=0}^{3} i^k\tau_k$ where $\tau_k : A \to \mathbb{C}$, $k = 0, ..., 3$, are positive functionals. Applying the GNS-construction to each τ_k, we get representations $\pi_k : A \to B(H_k)$ and cyclic vectors $\omega_k \in H_k$ such that $\psi'(f)(a) = \sum_{k=0}^{3} i^k\langle\pi_k(a)\omega_k, \omega_k\rangle$. Since $\{\pi_k(\mu_i)\}_i$ is strongly convergent to the identity on H_k we conclude that $\{f(\psi(\mu_i))\}_i$ is convergent in $\mathbb{C}$ to the number $c_{f,\psi} := \sum_{k=0}^{3} i^k\|\omega_k\|^2 = \sum_{k=0}^{3} i^k\|\tau_k\|$ that depends only on ψ and f (it does not depend on $\{\mu_i\}_i$). Now, since the net $\{\psi(\mu_i)\}_i$ is bounded in E, the Banach–Alaoglu Theorem says that there is a subnet

$\{\psi(\mu_{i_j})\}_j$ with an E_*-limit $\psi(1_A) := E_*\text{-}\lim_j \psi(\mu_{i_j}) \in E$. So in particular $f(\psi(1_A)) = \lim_j f(\psi(\mu_{i_j})) = c_{f,\psi}$ for every $f \in E_*$. This implies that every net $\{\psi(\mu_i)\}_i$, where $\{\mu_i\}_i$ is an approximate unit, is E_*-convergent to $\psi(1_A)$. □

THEOREM 5.21 (Disintegration-Integration Theorem). *Let $\mathcal{G}$ be an étale groupoid and let $\alpha : S \rightharpoonup C_0(X)$ be an inverse semigroup action such that the associated transformation groupoid is isomorphic to $\mathcal{G}$ and let $\overline{S}$ denote the unitization of the image of S in* $\mathrm{Bis}(\mathcal{G})$ *(one can always take a wide unital inverse subsemigroup $S = \overline{S} \subseteq \mathrm{Bis}(\mathcal{G})$). Then we have a canonical isometric isomorphism*

$$F^{\overline{S}}(\mathcal{G}) \cong C_0(X) \rtimes_\alpha S.$$

In fact, the equality $\psi = \pi \rtimes v$ yields a bijective correspondence between representations $\psi : F^{\overline{S}}(\mathcal{G}) \to B$ in a Banach algebra B and

(1) *B'-normalized covariant representations (π, v) of α in B;*
(2) *B_*-normalized covariant representations (π, v) of α in B, if (B, B_*) is a dual Banach algebra;*
(3) *covariant representations (π, v) of α on E, if $B = B(E)$ and E is a reflexive Banach space.*

If each X_t, $t \in S$, is compact then the pairs (π, v) in (1)–(3) coincide with covariant representations (π, v) of α in B such that $v_t = \pi(1_{X_t})v_t \in B$ for all $t \in S$.

PROOF. Let $\psi : F^{\overline{S}}(\mathcal{G}) \to B$ be a representation. Then $\pi := \psi|_{C_0(X)}$ is a representation of $C_0(X)$. For every $t \in S$, the map $C_c(X_t) \ni a_t \mapsto \psi(a_t\delta_t) \in B$ extends to a linear contraction $C_0(X_t) \to B \subseteq B''$. Let $v_t \in B''$ be the element associated to this map in Lemma 5.20, where $E = B''$ and $E_* = B'$. Namely, for any approximate unit $\{\mu_i^t\}_i \subseteq C_0(X_t)$ we have $v_t = B'\text{-}\lim_i \psi(\mu_i^t\delta_t)$. Recall that we consider multiplication in B'' which is B'-continuous in the second variable. For $a \in C_c(X_t) = C_c(X_{tt^*})$ we have $(a\delta_{tt^*}) * (\mu_i^t\delta_t) \to a\delta_t$ in $C_0(U_t)$. Thus

$$\pi(a)v_t = \psi(a\delta_{tt^*})B'\text{-}\lim_i \psi(\mu_i^t\delta_t) = B'\text{-}\lim_i \psi(a\delta_{tt^*} * \mu_i^t\delta_t) = \psi(a\delta_t).$$

For $a \in C_c(X_{t^*})$ we have $(\mu_i^t\delta_t) * (a\delta_{t^*t}) = (\alpha_t(a)\mu_i^t)\delta_t \to \alpha_t(a)\delta_t$ in $C_0(U_{t^*})$. Multiplication in B'' is B'-continuous in the first variable if the second variable is in B, so

$$v_t\pi(a) = B'\text{-}\lim_i \psi(\mu_i^t\delta_t * a\delta_{t^*t}) = \psi\big(\alpha_t(a)\delta_t\big) = \pi(\alpha_t(a))v_t.$$

If in addition $t = e \in E(S)$ then $v_e\pi(a) = \psi(\alpha_e(a)\delta_e) = \psi(a\delta_e) = \pi(a)$. If $a \in C_c(X_{st})$ for $s, t \in S$ then $\alpha_s^{-1}(a) = a \circ \theta_s \in C_c(X_{s^*} \cap X_t)$ and therefore $(a\delta_s) * (\mu_i^t\delta_t) = a(\mu_i^t \circ \theta_{s^*})\delta_{st} = \alpha_s(\alpha_s^{-1}(a)\mu_i^t)u(s,t)\delta_{st} \to au(s,t)\delta_t$ in $C_0(U_{st})$. Thus

$$\pi(a)v_sv_t = B'\text{-}\lim_i \psi(a\delta)\psi(\mu_i^t\delta_t) = \psi\big(a\delta_t\big) = \pi\big(a\big)v_t.$$

By construction $v_t = B'\text{-}\lim_i \varphi(\mu_i^t)v_t)$. Hence we see that (π, v) is a B'-normalized covariant representation of α with $\psi = \pi \rtimes v$.

In view of Proposition 5.19, this gives (1) as a B'-normalized a covariant representation (π, v) of α with $\psi = \pi \rtimes v$ has to be the one we constructed above. By Theorem 4.31

this implies the isometric isomorphism $F^{\overline{S}}(\mathcal{G}) \cong C_0(X) \rtimes_\alpha S$. We get items (2) and (3) from (1) by passing to an appropriate normalization as described in Proposition 4.24 and Corollary 4.28. The last part of the assertion follows from Remark 4.20. ☐

Corollary 5.22. *Retain the notation of Theorem 5.21 and let $\mathcal{E}$ be a class of Banach spaces. Denoting by $\mathcal{E}_{\mathrm{alg}}$ the class of all covariant representations of α in Banach algebras $B(E)$, where $E \in \mathcal{E}$, and by $\mathcal{E}_{\mathrm{spa}}$ the class of all covariant representations of α on Banach spaces $E \in \mathcal{E}$, we have an isometric isomorphism and a representation*

$$F^{\overline{S}}_{\mathcal{E}}(\mathcal{G}) \cong C_0(X) \rtimes_{\alpha,\mathcal{E}_{\mathrm{alg}}} S \to C_0(X) \rtimes_{\alpha,\mathcal{E}_{\mathrm{spa}}} S$$

(recall notation from Definition 4.29). This representation is isometric, for instance, if all spaces in $\mathcal{E}$ are reflexive, or if each X_t, $t \in S$, is compact.

Proof. Apply Theorem 5.21. ☐

Corollary 5.23. *For any unital inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ covering $\mathcal{G}$ and any C^*-algebra B, a homomorphism $F^S(\mathcal{G}) \to B$ is contractive (i.e. is a representation) if and only if it is $*$-preserving, and then it factors through a $*$-homomorphism $C^*(\mathcal{G}) \to B$.*

Proof. By Remark 5.9 we may assume S is wide and so Theorem 5.21 applies. Any representation $\psi : F^S(\mathcal{G}) \to B$ is $*$-preserving because any integrated representation is, by Proposition 5.19. Conversely, if ψ is a $*$-homomorphism it is $\|\cdot\|_{C^*}$-contractive on $C_c(\mathcal{G})$ and thus it defines a representation $\overline{\psi} : C^*(\mathcal{G}) \to B$. Since ψ is the composition of the canonical representation $F^S(\mathcal{G}) \to C^*(\mathcal{G})$ and $\overline{\psi}$, it is a representation itself. ☐

5.4. Representations on L^p-spaces

One of the distinguished features of L^p-spaces is that they allow natural construction of regular representations. We begin with recalling these. For $p, q \in [1,\infty]$ writing $\frac{1}{p} + \frac{1}{q} = 1$ where $p = 1$ we mean that $q = \infty$. Recall the definitions of Hahn norms (5.1), (5.2). They are smaller than the norm $\|\cdot\|_{\mathrm{Bis}(\mathcal{G})}$ coming from the largest grading $\mathrm{Bis}(\mathcal{G})$.

Proposition 5.24. *For any $p \in [1,\infty]$ we have an injective $\|\cdot\|_I$-contractive homomorphism $\Lambda_p : C_c(\mathcal{G}) \to B(\ell^p(\mathcal{G}))$ given by*

$$\Lambda_p(f)\xi(\gamma) := (f * \xi)(\gamma) = \sum_{r(\eta)=r(\gamma)} f(\eta)\xi(\eta^{-1}\gamma), \quad f \in C_c(\mathcal{G}), \xi \in \ell^p(\mathcal{G}).$$

In fact, for any $f \in C_c(\mathcal{G})$ and Hölder's conjugate $p, q \in [1,\infty]$ we have

$$\|\Lambda_p(f)\| \leqslant \|f\|_{*d}^{\frac{1}{p}} \cdot \|f\|_{*r}^{\frac{1}{q}} \leqslant \|f\|_I. \tag{5.6}$$

Moreover, $\|\Lambda_1(f)\| = \|f\|_{*d}$, $\|\Lambda_\infty(f)\| = \|f\|_{*r}$ and $\|\Lambda^p(f)\| = \|\Lambda^q(\check{f})\|$, for all $f \in C_c(\mathcal{G})$.

Proof. For each $\gamma \in \mathcal{G}$, let 1_γ be the characteristic function of $\{\gamma\}$. Then $\{1_\gamma\}_{\gamma\in\mathcal{G}}$ is the standard Schauder basis of $\ell^p(\mathcal{G})$ when $p < \infty$. Note that for every $f \in C_c(\mathcal{G})$ and every $\gamma \in \mathcal{G}$ we have $\Lambda_p(f)1_{d(\gamma)}(\gamma) = f(\gamma)$. Thus Λ_p is injective on $C_c(\mathcal{G})$. Recall that the norm of any linear operator $T : \ell^1(\mathcal{G}) \to \ell^1(\mathcal{G})$ is given by $\|T\| = \sup_{\gamma\in\mathcal{G}} \|T1_\gamma\|$

(because $\|T(\sum_{\gamma\in\mathcal{G}}\xi(\gamma)1_\gamma)\| \leqslant \sum_{\gamma\in\mathcal{G}}|\xi(\gamma)|\|T1_\gamma\| \leqslant \|\xi\|_1\sup_{\gamma\in\mathcal{G}}\|T1_\gamma\|$). In our case, for every $f\in C_c(\mathcal{G})$, we have

$$\|\Lambda_1(f)1_\gamma\|_1 = \sum_{d(\eta)=d(\gamma)}|f(\eta\gamma^{-1})|.$$

Therefore

$$\|\Lambda_1(f)\| = \sup_{\gamma\in\mathcal{G}}\sum_{d(\eta)=d(\gamma)}|f(\eta\gamma^{-1})| = \sup_{\gamma^{-1}\in\mathcal{G}}\sum_{d(\eta)=r(\gamma)}|f(\eta\gamma)| = \sup_{x\in\mathcal{G}^0}\sum_{d(\eta)=x}|f(\eta)| = \|f\|_{*d}.$$

It is immediate that, under that the standard isomorphism $\ell^p(\mathcal{G})'\cong\ell^q(\mathcal{G})$ given by the pairing $\langle\xi,\eta\rangle := \sum_{\gamma\in\mathcal{G}}\xi(\gamma)\eta(\gamma)$, for $\xi\in\ell^p(\mathcal{G})$, $\eta\in\ell^q(\mathcal{G})$, we have $\Lambda^p(f)' = \Lambda^q(\check{f})$. In particular, we get

$$\|\Lambda_\infty(f)\| = \|\Lambda^1(\check{f})'\| = \|\check{f}\|_{*d} = \|f\|_{*r}.$$

This proves the last part of the assertion. Now the inequality (5.6) follows from the Riesz–Thorin interpolation theorem, see [**Roy73**]. □

Definition 5.25. Let $p\in[1,\infty]$ and we call the homomorphism Λ_p defined in Proposition 5.24 a *regular representation* of $\mathcal{G}$. We call the induced completion $F^p_r(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\Lambda_p(\cdot)\|}\cong\overline{\Lambda_p(C_c(\mathcal{G}))}$ the *reduced L^p-operator algebra* of $\mathcal{G}$.

Remark 5.26. The above definition agrees with previous definitions of reduced L^p-operator algebras of groupoids in [**CGT24**], [**HO23**], [**GL17**]. Obviously, $F^2_r(\mathcal{G})\cong C^*_r(\mathcal{G})$ is the standard reduced C^*-algebra of $\mathcal{G}$.

By Proposition 5.24, the homomorphism Λ_p extends to a representation of $F_I(\mathcal{G})$ and hence also of $F(\mathcal{G})$. In particular, the Integration-Disintegration Theorem (Theorem 5.21) applies:

Example 5.27. Let $\tilde{\theta}:\mathrm{Bis}(\mathcal{G})\to\mathrm{PHomeo}(\mathcal{G})$ be the extension of the canonical action described in Remark 1.15. The formulas

$$\pi(a)\xi(\gamma) := a(r(\gamma))\xi(\gamma), \qquad v_U\xi(\gamma) := \begin{cases}\xi(\tilde{\theta}_{U^*}(\gamma)), & \gamma\in r^{-1}(r(U)),\\ 0, & \text{otherwise,}\end{cases}$$

where $a\in C_0(X)$, $\xi\in\ell^p(\mathcal{G})$, $\gamma\in\mathcal{G}$, $U\in\mathrm{Bis}(\mathcal{G})$, define a covariant representation (π,v) of the corresponding inverse semigroup action on $C_0(X)$ on $\ell^p(\mathcal{G})$, and we have $\Lambda_p = \pi\rtimes v : C_c(\mathcal{G})\to B(\ell^p(\mathcal{G}))$.

We now prove that not only the regular representation Λ_p satisfies the inequalities (5.6) but in fact every $\|\cdot\|_S$-contractive homomorphism does, independently of the choice of inverse semigroup $S\subseteq\mathrm{Bis}(\mathcal{G})$. This independence is crucial, as it allows one to define universal L^p-operator algebras given by generators and relations in various ways not worrying by the underlying grading. Namely, we prove the following:

Theorem 5.28. *Let $S\subseteq\mathrm{Bis}(\mathcal{G})$ be any unital inverse subsemigroup covering $\mathcal{G}$, and let $\psi: F^S(\mathcal{G})\to B(L^p(\mu))$ be any representation on any L^p-space $L^p(\mu)$. Then*

$$\|\psi(f)\| \leqslant \|f\|_{d_*}^{1/p}\|f\|_{r_*}^{1/q} \leqslant \|f\|_I, \qquad f\in C_c(\mathcal{G}),$$

where $\frac{1}{p}+\frac{1}{q}=1$. *If* $p=\infty$, *the estimate* $\|\psi(f)\|\leqslant\|f\|_{r*}$ *holds also when* $L^\infty(\mu)$ *is replaced by* $C_0(\Omega)$ *for a locally compact Hausdorff space* Ω.

We first note that when $p<\infty$ we may assume that ψ is non-degenerate.

Proposition 5.29. *Let* $p\in[1,\infty)$. *For any representation* $\psi: F^S(\mathcal{G})\to B(L^p(\mu))$ *there is an isometric isomorphism* $\Phi:\overline{\psi(F^S(\mathcal{G}))L^p(\mu)}\to L^p(\overline{\mu})$, *where* $\overline{\mu}$ *is localizable, and then the formula* $\overline{\psi}(f)=\Phi(\psi(f)|_{\overline{\psi(F^S(\mathcal{G}))L^p(\mu)}})$, $f\in F^S(\mathcal{G})$, *defines a non-degenerate representation* $\overline{\psi}: F^S(\mathcal{G})\to B(L^p(\overline{\mu}))$ *such that* $\|\overline{\psi}(f)\|=\|\psi(f)\|$ *for all* $f\in F^S(\mathcal{G})$.

Proof. By Lemma 5.13, $F^S(\mathcal{G})$ and $C_0(X)$ have a common contractive approximate unit $\{\mu_i\}_i$. In particular, $\overline{\psi(F^S(\mathcal{G}))L^p(\mu)}=\overline{\psi(C_0(X))L^p(\mu)}$ and as $L^p(\mu)$ does not contain an isomorphic copy of c_0 (see [**GT20**, 2.6]) there is a norm one projection $P: L^p(\mu)\to\overline{\psi(C_0(X))L^p(\mu)}$ by [**GT20**, Corollary 3.10]. Thus there is a measure $\overline{\mu}$, which can be chosen to be localizable, and an isometric isomorphism $\Phi:\overline{\psi^S(F(\mathcal{G}))L^p(\mu)}\to L^p(\overline{\mu})$ by [**Tza69**, Theorem 6]. Also for all $f\in F^S(\mathcal{G})$ we have $\|\psi(f)\|=\|\psi(f)P\|$ because $\|\psi(f)\xi\|=\lim_i\|\psi(f)\psi(\mu_i)\xi\|=\lim_i\|\psi(f)P\psi(\mu_i)\xi\|\leqslant\|\psi(f)P\|\|\xi\|$. This implies the assertion. ☐

Remark 5.30. Proposition 5.29 fails when $p=\infty$ and X is not compact, see Corollary 3.12. We do not know whether it extends to C_0 or Lindenstrauss spaces.

Let $f=\sum_{k=1}^n f_k$, where $f_k\in C_c(U_k)$ and $U_k\in\mathrm{Bis}(\mathcal{G})$ for $k=1,\dots,n$. For every $x\in X$ we obviously have

$$\sum_{d(\gamma)=x}|f(\gamma)|\leqslant\sum_{k=1}^n|f_k\circ d|_{U_k}^{-1}(x)|,\qquad\sum_{r(\gamma)=x}|f(\gamma)|\leqslant\sum_{k=1}^n|f_k\circ r|_{U_k}^{-1}(x)|,\tag{5.7}$$

and these inequalities are equalities whenever the strict supports $\mathrm{supp}(f_k)=\{\gamma\in\mathcal{G}: f_k(\gamma)\neq 0\}$ are pairwise disjoint. This can be easily arranged when $\mathcal{G}$ is Hausdorff:

Lemma 5.31. *Let* $f=\sum_{k=1}^n g_k$, *where* $g_k\in C_c(U_k)$ *and* $U_k\subseteq\mathcal{G}$ *open subsets (not necessarily bisections) for* $k=1,\dots,n$. *There are* $f_k\in C_c(U_k)$, $k=1,\dots,n$, *with pairwise disjoint strict supports such that* $f=\sum_{k=1}^n f_k$.

Proof. Consider the case when $n=2$. Denoting by $K_i\subseteq U_i$ the compact support of g_i, we get that $K_1\cap K_2$ is compact (because $\mathcal{G}$ is Hausdorff). Hence there is $h\in C_c(U_1\cap U_2)$ with $0\leqslant h\leqslant 1$ and $h|_K\equiv 1$. Putting $f_1:=g_1+h\cdot g_2$ and $f_2:=(1-h)g_2$ we get functions $f_i\in C_c(U_i)$, $i=1,2$, with disjoint strict supports such that $f=f_1+f_2$. Now proceeding by induction one gets the assertion for every n. ☐

When $\mathcal{G}$ is non-Hausdorff the above lemma is not true even if U_k's are bisections. Therefore the proof of [**BKM25**, Theorem 5.5], which is a version of Theorem 5.28 for not necessarily Hausdorff groupoids, is slightly different than the one present below. The main idea to circumvent the lack of Lemma 5.31 in [**BKM25**] was to use Borel extensions and pass to Borel functions. Here we do not need that and our proof is simpler.

Proof of Theorem 5.28. By Remark 5.9, we may assume that S is a wide inverse subsemigroup of $\mathrm{Bis}(\mathcal{G})$. Indeed, for $\widetilde{S}:=\{U\in\mathrm{Bis}(\mathcal{G}): U\subseteq V\in S\}$, we have a canonical representation $F^{\widetilde{S}}(\mathcal{G}) \to F^S(\mathcal{G})$. Thus composing ψ with this representation, we may replace S with $\widetilde{S}$. Let us assume it. Then by Proposition 1.20 we have $\mathcal{G} = S \ltimes_\theta X$ where $\theta : S \to \mathrm{PHomeo}(X)$ is the canonical inverse semigroup action. Let $\alpha : S \rightharpoonup C_0(X)$ be the associated algebraic action. Then $F^S(\mathcal{G}) \cong C_0(X) \rtimes_\alpha S$ and $\psi = \pi \rtimes v$ for a covariant representation (π, v) of α in $B(L^p(\mu))$ by Theorem 5.21. We consider the following three cases.

(1) Assume that $p < \infty$ and $\psi(C_0(X)) = \pi(C_0(X))$ consists of multiplication operators, i.e. we have a representation $\pi_0 : C_0(X) \to L^\infty(\mu)$ such that $\pi(a)\xi = \pi_0(a) \cdot \xi$ for all $\xi \in L^p(\mu)$, $a \in C_0(X)$. Let $f = \sum_{U\in F} f_U \delta_U \in C(\mathcal{G})$, where $F \subseteq S$ is finite and $f_U \in C_c(r(U))$, see (5.5). By Lemma 5.31 we may assume that f_U's have disjoint supports, and so that the inequalities (5.7) become the following equalities:

$$(5.8) \qquad \sum_{d(\gamma)=x} |f(\gamma)| = \sum_{U\in F} |f_U \circ \theta_U(x)|, \qquad \sum_{r(\gamma)=x} |f(\gamma)| = \sum_{U\in F} |f_U(x)|.$$

Let $\xi \in L^p(\mu)$ and $\eta \in L^q(\mu)$. For any $a \in C_0(r(U))$ we have $\int |\pi(a)v_U\xi|^p d\mu = \int |v_U\pi(a \circ h_t)\xi|^p d\mu \leqslant \int |\pi(a\circ\theta_t)\xi|^p d\mu$ because v_U is contractive. Also for any $a \in C_0(X)$ and any continuous multiplicative real function φ, in particular $\varphi(x) = |x|^\alpha$, we have $\varphi \circ (\pi(a)\xi) = \pi(\varphi \circ a)(\varphi \circ \xi)$ because π can be identified with a representation $C_0(X) \to L^\infty(\mu)$, and the functional calculus applies. In particular, $|\pi(a)\xi \cdot \eta| = |\pi(|a|^{1/p})\xi|\,|\pi(|a|^{1/q})\eta|$. Using this and applying Hölders inequality to the measure $\sum_{U\in F}\mu$, when $p > 1$ (equivalentely $q < \infty$), we get

$$\begin{aligned}
\left|\int \big(\pi \rtimes v(f)\xi\big)\eta \, d\mu\right| &\leqslant \int \sum_{U\in F} \left|\pi(|f_U|^{1/p})v_U\xi\right| \left|\pi(|f_U|^{1/q})\eta\right| d\mu \\
&\leqslant \left(\sum_{U\in F}\int \left|\pi(|f_U|^{1/p})v_U\xi\right|^p d\mu\right)^{1/p} \left(\sum_{U\in F}\int \left|\pi(|f_U|^{1/q})\eta\right|^q d\mu\right)^{1/q} \\
&\leqslant \left(\int \pi\left(\sum_{U\in F}|f_U|\circ\theta_U\right)|\xi|^p \, d\mu\right)^{1/p} \left(\int \pi\left(\sum_{U\in F}|f_U|\right)|\eta|^q \, d\mu\right)^{1/q} \\
&\leqslant \left\|\sum_{U\in F}|f_U\circ\theta_U|\right\|_\infty^{1/p} \|\xi\|_p \left\|\sum_{U\in F}|f_U|\right\|_\infty^{1/q} \|\eta\|_q \\
&\overset{(5.8)}{=} \|f\|_{*d}^{1/p}\,\|f\|_{*r}^{1/q}\,\|\xi\|_p\,\|\eta\|_q.
\end{aligned}$$

This implies that $\|\pi \rtimes v(f)\| \leqslant \|f\|_{*d}^{1/p}\,\|f\|_{*r}^{1/q}$. For $p = 1$ the proof is even simpler, as then

$$\left|\int \pi \rtimes v(f)\xi \, d\mu\right| \leqslant \sum_{U\in F}\int \left|v_U\pi(|f_U\circ\theta_U|)\xi\right| d\mu \leqslant \sum_{U\in F}\int \pi(|f_U\circ\theta_U|)|\xi| \leqslant \left\|\sum_{U\in F}|f_U|\circ\theta_U\right\|_\infty \|\xi\|_1.$$

(2) Assume that $p = 2$. This case is reduced to the previous one by Lemma 3.14.

(3) Assume that $\psi : F^S(\mathcal{G}) \to B(C_0(\Omega))$ and $\psi(C_0(X)) = \pi(C_0(X))$ consists of multiplication operators. The dual space E' of $E := C_0(\Omega)$ is naturally an isomorphic as a

Banach lattice to an L^1-space $L^1(\mu)$. Thus the formula $\tilde{\psi}(f) := \psi(\check{f})'$, $f \in C_c(\mathcal{G})^{\mathrm{op}}$, defines a representation $\tilde{\psi} : F^S(\mathcal{G})^{\mathrm{op}} \to B(E')$ with $\tilde{\psi}|_{C_0(X)}$ positive. This representation reduces to a representation $\tilde{\psi}_{\mathrm{r}}$ on the Banach sublattice $\tilde{\psi}(C_0(X)^+)L^1(\mu)$, which is an abstract L^1-space, and thus it is isomorphic to the Banach lattice $L^1(\tilde{\mu})$ for some localizable $\tilde{\mu}$. Thus $\tilde{\psi}_{\mathrm{r}} : F^S(\mathcal{G})^{\mathrm{op}} \to B(L^1(\tilde{\mu}))$ becomes a non-degenerate representation. Hence $\tilde{\psi}_{\mathrm{r}}|_{C_0(X)}$ acts by multiplication operators, by Theorem 3.11. Also as in the proof of Proposition 5.29 one gets that $\|\tilde{\psi}(f)\| = \|\tilde{\psi}_{\mathrm{r}}(f)\|$ for $f \in C_c(\mathcal{G})^{\mathrm{op}}$. In particular, $\|\psi(f)\| = \|\tilde{\psi}_{\mathrm{r}}(\check{f})\|$ for $f \in C_c(\mathcal{G})$. Therefore applying the case (1) to $\tilde{\psi}_{\mathrm{r}}$ we get $\|\psi(f)\| = \|\tilde{\psi}_{\mathrm{r}}(\check{f})\| \leqslant \|\check{f}\|_{*r} = \|f\|_{*d}$.

(4) By step (2) we may assume that $p \neq 2$. If $p < \infty$, then using Proposition 5.29 we may assume that ψ is non-degenerate, and then by Theorem 3.11, ψ necessarily acts by multiplication operators. Hence the assertion follows from step (1). The estimate $\|\psi(f)\| \leqslant \|f\|_{*r}$ for $p = \infty$ follows from the estimate $\|\psi(f)\| \leqslant \|f\|_{*d}$ for $p = 1$ and Corollary 5.17. □

5.5. Full L^p-groupoid algebras and spatial covariant representations

Theorem 5.28 allows us to define full L^p-operator algebras in a number of equivalent ways. Here we define them in a similar way as in [**BKM25**, Definition 5.12], we only skip one condition required in [**BKM25**] to deal with the real case.

Definition 5.32. Let $p \in [1, \infty]$. The *full L^p-operator algebra* of $\mathcal{G}$ is the completion $F^p(\mathcal{G}) = \overline{C(\mathcal{G})}^{\|\cdot\|_{L^p}}$ in the norm $\|f\|_{L^p} := \sup_{\psi \in \mathcal{R}} \|\psi(f)\|$, where $\mathcal{R}$ consists of homomorphisms $\psi : C_c(\mathcal{G}) \to B(E)$ such that:

(R1) $E = L^p(\mu)$ for some measure μ;
(R2) ψ is $\|\cdot\|^S_{\max}$-contractive for a unital inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ covering $\mathcal{G}$;

Using our previous results we obtain the following general properties and other equivalent definitions of full L^p-operator algebras of $\mathcal{G}$ that are summarised in Figure 1 in the introduction.

Theorem 5.33. *Let $p, q \in [1, \infty]$ with $1/p + 1/q = 1$.*

(1) *We have $F^1(\mathcal{G}) = F^1_{\mathrm{r}}(\mathcal{G}) = F_{d_*}(\mathcal{G})$ and $F^\infty(\mathcal{G}) = F^\infty_{\mathrm{r}}(\mathcal{G}) = F_{r_*}(\mathcal{G})$.*
(2) *$\|f\|_{L_p} \leqslant \|f\|_{L^1}^{1/p} \|f\|_{L^\infty}^{1/q} \leqslant \|f\|_I$ for $f \in C_c(\mathcal{G})$, and so condition (R2) in Definition 5.32 can be replaced by $\|\cdot\|_I$-contractiveness of ψ (which alone is stronger than (R2)).*
(3) *$\|\cdot\|_{L^2} = \|\cdot\|_{C^* \max}$ is the largest C^*-norm on $C_c(\mathcal{G})$, so $F^2(\mathcal{G}) = C^*(\mathcal{G})$.*
(4) *If $p < \infty$, the definition of $\|\cdot\|_{L^p}$ is not affected if one restricts to non-degenerate homomorphisms or if one replaces (R2) with "$\psi(C_c(X))$ consists of multiplication operators".*
(5) *If $p = \infty$ condition (R1) in Definition 5.32 can be replaced by "$E = C_0(\Omega)$ for some locally compact Hausdorff space", or even by "E is a Lindenstrauss space".*
(6) *$F^p(\mathcal{G})^{\mathrm{op}} \cong F^q(\mathcal{G})$ and $F^p(\mathcal{G}) \overset{\mathrm{anti}}{\cong} F^q(\mathcal{G})$ where these isometric maps are induced by the involutions $\check{}$ and $*$ on $C_c(\mathcal{G})$, respectively.*

PROOF. By Theorem 5.28 we have $\|f\|_{L^p} \leqslant \|f\|_{d*}^{1/p} \|f\|_{r*}^{1/q} \leqslant \|f\|_I$ for $f \in C_c(\mathcal{G})$. Applying this to $p = 1$ and $p = \infty$ we get $\|f\|_{L^1} \leqslant \|f\|_{d*}$ and $\|f\|_{L^\infty} \leqslant \|f\|_{d*}$. On the other hand, by Proposition 6.17 we have $\|f\|_{*d} = \|\Lambda_1(f)\| \leqslant \|f\|_{L^1}$ and $\|f\|_{*r} = \|\Lambda_\infty(f)\| \leqslant \|f\|_{L^\infty}$. This shows (1). As a consequence the estimates from Theorem 5.28 translate to (2). For $p = 2$, a homomorphism $\psi : C_c(\mathcal{G}) \to B(L^2(\mu))$ is $\| \cdot \|_{\max}^S$-contractive if and only if it is a $*$-homomorphism, see Corollary 5.23. Also by Lemma 3.14 we may assume that for any such homomorphism $\psi(C_c(X))$ acts by multiplication operators, and so operators in $\psi(C_c(X)^+)$ are automatically positive. This implies (3). Proposition 5.29 implies (4). Now (5) follows from the last part of Theorem 5.28 and from Corollary 5.17. The isomorphisms in (6) for $p = 1, \infty$ and $q = \infty, 1$ follow from (1) and Proposition 6.17. Assuming $p \in (1, \infty)$, $F^p(\mathcal{G})^{\mathrm{op}} \cong F^q(\mathcal{G})$ holds by Corollary 5.15. The isomorphism $F^p(\mathcal{G})^{\mathrm{op}} \cong F^q(\mathcal{G})$ translates to an anti-isomorphism $F^p(\mathcal{G}) \overset{\text{anti}}{\cong} F^q(\mathcal{G})$. □

We will combine our Integration-Disintegration Theorem, generalized Banach-Lamperti theorem, and characterization of non-degenerate representations of $C_0(X)$ in Theorem 3.11, to describe all non-degenerate representations of $F^p(\mathcal{G})$ on L^p-spaces $L^p(\mu)$ in terms of spatial data, that is in terms of the measure space.

To this end, we fix until the end of this section

> an inverse semigroup action $\alpha : S \rightharpoonup C_0(X)$ such that the associated transformation groupoid is isomorphic to $\mathcal{G}$.

Recall that covariant representations on spaces which were introduced in Definition 4.8.

DEFINITION 5.34. A covariant representation (π, v) of α on a space $L^p(\mu)$, for some $p \in [1, \infty]$ and a localizble measure μ, is *spatial* if $\pi : C_0(X) \to B(L^p(\mu))$ acts by multiplication operators on $L^p(\mu)$ and $v : S \to \mathrm{SPIso}(L^p(\mu))$ takes values in the inverse semigroup of spatial partial isometries, see Definition 3.24.

In the remainder whenever we write μ we mean a localizable measure.

PROPOSITION 5.35. *If $p \in (1, \infty)$ and (π, v) is a covariant representation of α on $L^p(\mu)$ where π is given by multiplication operators, then (π, v) is spatial. If $p \in (1, \infty)\backslash\{2\}$ then every non-degenerate covariant representation (π, v) of α on $L^p(\mu)$ is spatial.*

PROOF. Let $\pi_0 : C_0(X) \to L^\infty(\mu)$ be the homomorphism that satisfies $\pi(a)\xi = \pi_0(a) \cdot \xi$ for $\xi \in L^p(\mu)$. By Lemma 4.27, for $e \in E(S)$ we have $v_e = B_*\text{-}\lim \pi(\mu_i^e)$, where $\{\mu_i^e\}_i$ is an approximate unit in $C_0(X_e)$ and $B_* := L^q(\mu) \mathbin{\widehat{\otimes}} L^p(\mu)$ is a predual of $B := B(L^p(\mu))$. This means that for all $(\eta, \xi) \in L^q(\mu) \times L^p(\mu)$ we have

$$\int \eta(v_e\xi)\, d\mu = \lim_i \int \eta\pi(\mu_i^e)\xi\, d\mu = \lim_i \int \pi_0(\mu_i^e)(\eta \cdot \xi)\, d\mu.$$

Note that $\eta \cdot \xi \in L^1(\mu)$ and every element in $L^1(\mu)$ can be written as the product of some elements $\eta \in L^q(\mu)$, $\xi \in L^p(\mu)$. Also $L^1(\mu)$ is a predual of $L^\infty(\mu)$ and so the latter is closed under $L^1(\mu)$ limits. All this plus the displayed equalities imply that $\pi_0(\mu_i^e)$ converges to some $\pi_0(1_e) \in L^\infty(\mu)$ and v_e is a multiplication operator by $\pi_0(1_e)$ (see [**BKM25**, Proposition 5.16], for a different proof of this fact, which exploits properties of L^p-projections).

Concluding, the projections v_e, $e \in E(S)$, are multiplication operators by characteristic functions, and so in particular, they are hermitian operators. Hence by Theorem 3.26, the map v takes values in $\mathrm{SPIso}(L^p(\mu))$, and so (π, v) is spatial. The second part of the assertion now follows from Theorem 3.11. □

Below we use the notation introduced in Subsection 3.4.

Definition 5.36. A *spatial covariant triple* for the action α and a localizable measure μ is a triple (π_0, Φ, ω) where $\pi_0 : C_0(X) \to L^\infty(\mu)$ is a representation, $\Phi = \{[\Phi_s]\}_{s\in S} \subseteq \mathrm{PAut}([\Sigma])$ is an inverse semigroup of partial set automorphisms $\Phi_s : \Sigma_{D_s} \to \Sigma_{D_{s^*}}$, $D_s, D_{s^*} \in \Sigma$, $s \in S$, satisfying $[\Phi_s] \circ [\Phi_t] = [\Phi_{st}]$ for $s, t \in S$, and $\omega = \{\omega_s\}_{s\in S}$ is a family (cocycle) of partial unimodal maps $\omega_s \in UL^\infty(\mu|_{D_{s^*}})$, $s \in S$. These must satisfy:

1. $T_{\Phi_t}(\pi_0(a)) = \pi_0(\alpha_t(a))$ for all $a \in C_0(X_{t^*})$, $t \in S$;
2. $\pi_0(\mu_i^e)1_D \nearrow 1_D$, for every measurable $D \subseteq D_e$ with $\mu(D) < \infty$, positive approximate unit $\{\mu_i^e\}_i \subseteq C_0(X_e)$ and $e \in E(S)$;
3. $\pi_0(a)\omega_s T_{\Phi_s}(\omega_t) = \pi_0(a)\omega_{st}$ for $s, t \in S$, $a \in C_0(X_{st})$.

We say that (π_0, Φ, ω) is *non-degenerate* if $\pi_0(\mu_i)1_D \nearrow 1_D$, for every measurable D with $\mu(D) < \infty$ and a positive contractive approximate unit $\{\mu_i\}_i \subseteq C_0(X)$.

Proposition 5.37. *Let $p \in [1, \infty)$. The assignment*

$$\pi(a)\xi = \pi_0(a) \cdot \xi, \qquad v_t := \omega_t \left(\frac{d\mu \circ \Phi_{t^*}}{\mu|_{D_{t^*}}}\right)^{\frac{1}{p}} T_{\Phi_t}, \qquad a \in C_0(X), \xi \in L^p(\mu), t \in S,$$

gives a one-to-one correspondence between spatial representations (π, v) of α on $L^p(\mu)$ and spatial covariant triples (π_0, Φ, v) for α and μ.

Proof. A representation $\pi : C_0(X) \to B(L^p(\mu))$ acting by multiplication operators is equivalent to a homomorphism $\pi_0 : C_0(X) \to L^\infty(\mu)$. By Proposition 3.23, every map $v : S \to \mathrm{SPIso}(L^p(\mu))$ is given by $v_t = \omega_t(\frac{d\mu\circ\Phi^*}{d\mu|_{D_{\Phi^*}}})^{\frac{1}{p}} T_{\Phi_t}$, $t \in S$, where $\Phi = \{[\Phi_s]\}_{s\in S} \subseteq \mathrm{PAut}([\Sigma])$ and $\omega_t \in UL^\infty(\mu|_{D_{t^*}})$, $t \in S$. The final statement of Proposition 3.23 (and uniqueness of the presentation in the Banach–Lamperti theorem, cf. the last part of Theorem 2.37) tells us that $\pi(a)v_s v_t = \pi(a)v_{st}$ holds for all $s, t \in S$, $a \in C_0(X_{st})$ if and only if (3) in Definition 5.36 holds and $\Phi = \{[\Phi_s]\}_{s\in S} \subseteq \mathrm{PAut}([\Sigma])$ is an inverse semigroup. Condition (2) in Definition 5.36 is equivalent to the equality $\overline{\pi(C_0(X_e))L^p(\mu)} = L^p(\mu|_{D_e})$ for all $e \in E(S)$. Assuming this, condition (1) in Definition 5.36 is equivalent to $v_t\pi(a)v_{t^*} = \pi(\alpha_t(a))$ for all $a \in C_0(X_{t^*})$, $t \in S$. This gives the assertion. □

Theorem 5.38. *Let $\mathcal{G}$ be a groupoid and let α be an inverse semigroup action whose transformation groupoid can be identified with $\mathcal{G}$.*

1. *For each $p \in [1, \infty]$ there is an isometric representation $\pi \rtimes v : F^p(\mathcal{G}) \to B(L^p(\mu))$ where (π, v) is a spatial covariant representation of α on $L^p(\mu)$. If $p < \infty$ or when X is compact, then π can be chosen to be non-degenerate.*
2. *For $p \in (1, \infty)\backslash\{2\}$ every non-degenerate representation of $F^p(\mathcal{G})$ on $L^p(\mu)$, where μ is localizable, is of the form $\pi \rtimes v$ for a spatial covariant representation (π, v)*

of α. This gives a bijective correspondence between such representations of $F^p(\mathcal{G})$ on $L^p(\mu)$ and spatial covariant triples for α and μ.

(3) *For any $p, p' \in (1,\infty)\backslash\{2\}$ we have a bijective correspondence between representations of $F^p(\mathcal{G})$ and $F^{p'}(\mathcal{G})$ on L^p and $L^{p'}$-spaces, respectively. This correspondence matches $\pi_p : F^p(\mathcal{G}) \to B(L^p(\mu))$ and $\pi_{p'} : F^{p'}(\mathcal{G}) \to B(L^{p'}(\mu))$, given by*

$$\pi_p(a_t\delta_t) = \pi_0(a_t)\omega_t \left(\frac{d\mu \circ \Phi_{t*}}{d\mu|_{D_{\Phi_{t*}}}}\right)^{\frac{1}{p}} T_{\Phi_t}, \qquad \pi_{p'}(a_t\delta_t) = \pi_0(a_t)\omega_t \left(\frac{d\mu \circ \Phi_{t*}}{d\mu|_{D_{\Phi_{t*}}}}\right)^{\frac{1}{p'}} T_{\Phi_t},$$

where $a_t \in C_c(X_t), t \in S$, and (π_0, Φ, ω) is a non-degenerate spatial covariant triple for α and a localizable μ.

If the domains of the S-action are compact open, then items (2) and (3) are valid for all $p, p' \in [1,\infty]\backslash\{2\}$.

Proof. (1). For $p \in \{1,\infty\}$, the assertion follows from Theorem 5.33(1). For $p \in [1,\infty)$, Theorem 5.33(4) implies that for each $a \in F^p(\mathcal{G})$ and $k \in \mathbb{N}$ there is a non-degenerate representation $\psi_{a,k} : F^p(\mathcal{G}) \to B(L^p(\mu_{a,k}))$ such that $\psi_{a,k}(C_0(X))$ consists of multiplication operators and $\|a\|_{F^p(\mathcal{G})} \leqslant \|\psi_{a,k}(a)\| + 1/k$. Thus the ℓ^p-direct sum of $\{\psi_{a,k}\}_{a\in F^p(\mathcal{G}),k\in\mathbb{N}}$ is an isometric non-degenerate representation ψ of $F^p(\mathcal{G})$ where $\psi(C_0(X))$ consists of multiplication operators. Disintegrating ψ and applying Proposition 5.35 we get that $\psi = \pi \rtimes v$ where (π, v) is a spatial covariant representation of α.

(2). If $p \in (1,\infty)$, Theorem 5.21(3) implies that every representation of $F^p(\mathcal{G})$ on $L^p(\mu)$ is of the form $\pi \rtimes v$ for a covariant representation (π, v) on the space $L^p(\mu)$ (the same holds also for $p = 1, \infty$ if the domains of the action are compact open). If $p \neq 2$ and the representation is non-degenerate then Theorem 3.11 tells us that π acts by multiplication operators. Then (π, v) is necessarily spatial by Proposition 5.35, and so it is given by a spatial covariant triple by Proposition 5.37. This proves (2). Item (3) now follows readily. □

Remark 5.39. The above natural correspondence between representations might seem a bit striking, as for instance, the example of L^p-Cuntz algebras or more generally every simple purely infinite graph L^p-operator algebra, see [**CMR25**], shows that it may happen that there are no non-zero continuous homomorphisms between $F^p(\mathcal{G})$ and $F^{p'}(\mathcal{G})$ for $p \neq p'$.

CHAPTER 6

Reduced groupoid Banach algebras and topological freeness

In the previous chapter we introduced and studied representations of Banach algebra completions of the groupoid convolution algebra, which were graded by some inverse semigroup of bisections. In this chapter we will say more about the internal structure of the corresponding completions. The main new ingredient that we add to our analysis is the restriction map

$$E_X : C_c(\mathcal{G}) \to C_c(X) \subseteq C_c(\mathcal{G}), \qquad E_X(f) := f|_X, \qquad f \in C_c(\mathcal{G}). \tag{6.1}$$

Here $\mathcal{G}$ is a Hausdorff étale groupoid with the unit space X. Recall that for Hausdorff $\mathcal{G}$ the unit space X is not only open but also closed in $\mathcal{G}$, see Lemma 1.21. Therefore, the restricted map $f|_X$ is not only continuous but also has a compact support, i.e the map E_X is well defined. We will consider completions of $C_c(\mathcal{G})$ for which the map E_X extends to contractive projection. In the C^*-algebra setting such projections are called *conditional expectations*. This will allow us to define reduced groupoid Banach algebras and say something about their ideal structure.

The content of this chapter is based on the results of [**BKM26**]. The presentation here is more accessible because we do not need the whole machinery of functions with meager support and essential quotients developed in [**BKM26**] for non-Hausdorff groupoids (not mentioning twists). We focus here mainly on the role of topological freeness and ideal structure. The reader interested only in Banach algebra crossed products is referred to [**BK24**], which was the main source of inspiration for what we present here in the Hausdorff étale groupoid setting.

In addition, we finish this chapter with completely new applications to algebras associated to Renault-Deaconu groupoids in Section 6.3. In particular, we generalize some of the results of [**BKL24**] from C^*-algebras to L^p-operator algebras. As a corollary we get Cuntz-Krieger uniqueness theorem for simple L^p-operator graph algebras in Section 6.49, which improves upon results of [**CoR19**], [**CMR25**].

6.1. Groupoid Banach algebras

Let us start by defining the class of completions of $C_c(\mathcal{G})$ we want to study.

DEFINITION 6.1. Let $\|\cdot\|_{\mathcal{R}}$ be a submultiplicative norm on $C_c(\mathcal{G})$. We say that $F_{\mathcal{R}}(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{\mathcal{R}}}$ is a *groupoid Banach algebra* of $\mathcal{G}$ if

$$\|f|_X\|_\infty = \|f|_X\|_{\mathcal{R}} \leqslant \|f\|_{\mathcal{R}} \text{ for all } f \in C_c(\mathcal{G}).$$

Equivalently, $C_0(X)$ isometrically embeds into $F_{\mathcal{R}}(\mathcal{G})$ and the projection (6.1) extends to the contraction $E_X^{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_0(X) \subseteq F_{\mathcal{R}}(\mathcal{G})$. If this holds, we say that

(1) $F_{\mathcal{R}}(\mathcal{G})$ is a *reduced groupoid Banach algebra* if in addition $E_X^{\mathcal{R}}$ is *faithful* in the sense that the only (closed two-sided) ideal of $F_{\mathcal{R}}(\mathcal{G})$ contained in $\ker E_X^{\mathcal{R}}$ is $\{0\}$.

(2) $F_{\mathcal{R}}(\mathcal{G})$ is an *S-graded groupoid Banach algebra* if we have $\|a\|_{\mathcal{R}} \leqslant \|a\|_\infty$ for every $a \in C_c(U)$ and $U \in S$, where $S \subseteq \mathrm{Bis}(\mathcal{G})$ is a unital inverse semigroup covering $\mathcal{G}$.

Remark 6.2. For any unital inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ covering $\mathcal{G}$, the universal S-graded Banach algebra $F^S(\mathcal{G})$, see Definition 5.8, is an S-graded groupoid Banach algebra, and it is indeed universal in the sense that for any S-graded groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ we have a canonical representation $F^S(\mathcal{G}) \to F_{\mathcal{R}}(\mathcal{G})$.

Remark 6.3. For any $p \in [1,\infty]$ the L^p-operator algebras $F^p(\mathcal{G})$ and $F^p_{\mathrm{r}}(\mathcal{G})$ are groupoid Banach algebras and $F^p_{\mathrm{r}}(\mathcal{G})$ is reduced, cf. Proposition 6.17 below. By Theorem 5.33(2) (or in fact Theorem 5.28), these algebras are $\mathrm{Bis}(\mathcal{G})$-graded. As we explain below, see Corollary 6.13, there is a unique C^*-norm $\|\cdot\|_{C^*_{\mathrm{r}}}$ on $C_c(\mathcal{G})$ yielding a reduced Banach algebra, and then this algebra coincides with the standard reduced C^*-algebra $C^*_{\mathrm{r}}(\mathcal{G})$ of $\mathcal{G}$. The full C^*-algebra $C^*(\mathcal{G})$, which is a completion of $C_c(\mathcal{G})$ in the maximal C^*-norm $\|\cdot\|_{C^*\max}$, is a groupoid Banach algebra of $\mathcal{G}$. Thus a C^*-norm $\|\cdot\|_{\mathcal{R}}$ on $C_c(\mathcal{G})$ yields a groupoid Banach algebra if and only if it satisfies $\|\cdot\|_{C^*_{\mathrm{r}}} \leqslant \|\cdot\|_{\mathcal{R}} \leqslant \|\cdot\|_{C^*\max}$.

Obviously, not every completion (not even every C^*-completion) of $C_c(\mathcal{G})$ gives a groupoid Banach algebra in the sense of Definition 6.1. Here is the standard example.

Example 6.4 (C^*-norms on the group algebra of $\mathbb{Z}$). Let $\mathcal{G} = \mathbb{Z}$. Then

$$C_c(\mathcal{G}) = C_c(\mathbb{Z}) \cong \mathbb{C}[\mathbb{Z}] = \mathrm{span}\{a_n\delta_n : n \in \mathbb{Z}, a_n \in \mathbb{C}\}$$

and $X = \{0\}$. If $\mathcal{R}$ is the class of all representations of $C_c(\mathbb{Z})$ on a Hilbert space H, then $F_{\mathcal{R}}(\mathbb{Z}) = C^*(\mathbb{Z})$ is the group C^*-algebra of $\mathbb{Z}$ and this is known that $C^*(\mathbb{Z}) \cong C(\mathbb{T})$, see [**BO08**, Example 2.5.1], where isomorphism is given by the Fourier transform. Since $C^*(\mathbb{T})$ is universal for representations of $\mathbb{Z}$, any C^*-completion A of $C_c(\mathbb{Z})$ is a quotient of $C^*(\mathbb{T})$. Hence it is of the form $A = C(Y)$ for a closed subset $Y \subseteq \mathbb{T}$. More explicitly, every C^*-seminorm on $C_c(\mathbb{Z})$ if of the form $\|\cdot\|_Y$ where $Y \subseteq \mathbb{T}$ is a closed set and

$$\|f\|_Y = \sup_{z\in Y} |\sum_{n\in\mathbb{Z}} a_n z^n|, \qquad f \in C_c(\mathbb{Z}).$$

This seminorm is a norm if and only if Y is infinite, and then $\overline{C_c(\mathbb{Z})}^{\|\cdot\|_Y} \cong C(Y)$. The inequality $|f(0)| \leqslant \|f\|_Y$ holds for all $f \in C_c(\mathbb{Z})$ if and only if $Y = \mathbb{T}$. Therefore amongst uncountably many C^*-completions of $C_c(\mathbb{Z})$ there is only one such that the restriction $E_X(f) = f|_X$ extends to a contraction $E_X^{\mathcal{R}} : C^*(\mathbb{Z}) \to C_0(X)$. The uniqueness of such a norm is in fact a consequence of amenability of $\mathbb{Z}$.

We make some preparations to reveal more structure in groupoid Banach algebras. Firstly, for any $U \in \mathrm{Bis}(\mathcal{G})$ we define the restriction

$$E_U(f) := f|_U, \qquad f \in C_c(\mathcal{G}).$$

This gives a linear map $E_U : C_c(\mathcal{G}) \to C_b(U)$ with values in $C_b(U)$ rather than $C_c(U)$ as U need not to be closed. Hence unlike the projection $E_X : C_c(G) \to C_c(X)$, in general we cannot treat E_U as a projection. However, we may consider a smaller domain for E_U.

LEMMA 6.5. *For any $U \in \mathrm{Bis}(\mathcal{G})$ we have $C_c(r(U)) * C_c(\mathcal{G}) = C_c(r^{-1}(r(U)))$ and E_U restricted to this subspace of $C_c(\mathcal{G})$ is a projection onto $C_c(U)$. Moreover, there is a norm one function $b \in C_c(U^*)$ such that $(b * f)|_X = f \circ d|_U^{-1}$.*

PROOF. If $a \in C_c(r(U))$ and $f \in C_c(\mathcal{G})$, then $a * f(\gamma) = a(r(\gamma))f(\gamma)$ and so

$$\overline{\mathrm{supp}}(a * f) = r^{-1}(\overline{\mathrm{supp}}(a)) \cap \overline{\mathrm{supp}}(f) \subseteq r^{-1}(r(U)).$$

Thus $a * f \in C_c(r^{-1}(r(U)))$. Conversely, for every $f \in C_c(r^{-1}(r(U)))$ $r(\overline{\mathrm{supp}}(f))$ is a compact subset of $r(U)$. Thus, there is $a \in C_c(r(U))$ such that $0 \leqslant a \leqslant 1$ and $a|_{r(\overline{\mathrm{supp}}(f))} \equiv 1$. Then $f = a * f \in C_c(r(U)) * C_c(\mathcal{G})$ and

$$\overline{\mathrm{supp}}(f) \cap U \subseteq r^{-1}(\overline{\mathrm{supp}}(a)) \cap U = r^{-1}|_U(\overline{\mathrm{supp}}(a)) \subseteq U,$$

so $E_U(f) = f|_U$ has a compact support. Moreover, putting $b := a \circ d \in C_c(U^*)$ we get

$$(b * f)(x) = \sum_{\gamma\eta = x} b(\gamma)f(\eta) = \sum_{\eta \in d^{-1}(x)} b(\eta^{-1})f(\eta) = \begin{cases} f(d|_U^{-1}(\eta)) & x \in d(U) \\ 0 & x \notin d(U) \end{cases} = f \circ d|_U^{-1}(x).$$

This proves the assertion. □

The above lemma says that $E_U : C_c(r(U)) * C_c(\mathcal{G}) \to C_c(U)$ is a well defined projection onto the subspace $C_c(U) \subseteq C_c(r(U)) * C_c(\mathcal{G})$. Letting $\Phi_U : C_c(U) \xrightarrow{\cong} C_c(r(U))$ be the isomorphim given by the composition with the homeomorphism $r|_U^{-1} : r(U) \to U$ we may consider the map

$$\Phi_U \circ E_U : C_c(r(U)) * C_c(\mathcal{G}) \to C_c(r(U)) \subseteq C_c(X).$$

The second part of Lemma 6.5, says that for any $f \in C_c(r(U)) * C_c(\mathcal{G})$ there is $b \in C_c(U^*)$ such that $E_X(b * f) = \Phi_U(E_U(f))$. Thus we may recover E_U from E_X multiplying by an approximate unit in $C_c(U)$.

EXAMPLE 6.6. Let $\mathcal{G} := G \times X$ be the transformation groupoid for a group action $\theta : G \to \mathrm{Homeo}(X)$, see Examples 1.17, 5.10. Recall that $G \ni g \mapsto U_g := \{g\} \times X \in \mathrm{Bis}(\mathcal{G})$ is an embedding of the group G into the inverse semigroup $\mathrm{Bis}(\mathcal{G})$. Moreover, we may view $C_c(G \times X)$ as $C_c(G, C_c(X))$. Then for any $g \in G$ the evaluation

$$E_g(a) := a(g), \qquad a \in C_c(G, C_c(X)) \cong C(\mathcal{G}),$$

is a map $E_g : C_c(G, C_c(X)) \to C_c(X)$ that coincides with $\Phi_{U_g} \circ E_{U_g}$.

We now show that maps E_U extend to contractive maps on every groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ and in fact we have a natural contractive map $F_{\mathcal{R}}(\mathcal{G}) \to C_0(\mathcal{G})$. Moreover, this map is injective if and only if $F_{\mathcal{R}}(\mathcal{G})$ is reduced. Thus elements in reduced Banach algebra may be naturally treated as functions on $\mathcal{G}$. We will formulate this proposition starting with a seminorm on $C_c(\mathcal{G})$, as this will be usefull in the sequel.

PROPOSITION 6.7. *Let $F_{\mathcal{R}}(\mathcal{G})$ be a Hausdorff completion of $C_c(\mathcal{G})$ in a submultiplicative seminorm $\|\cdot\|_{\mathcal{R}}$. Assume that the canonical homomorphism $i_{\mathcal{R}} : C_c(\mathcal{G}) \to F_{\mathcal{R}}(\mathcal{G})$ is isometric on $C_0(X)$. The following conditions are equivalent*

(1) $\|f|_X\|_\infty \leqslant \|i_{\mathcal{R}}(f)\|_{\mathcal{R}}$ *for all* $f \in C_c(\mathcal{G})$;
(2) $F_{\mathcal{R}}(\mathcal{G})$ *is a groupoid Banach algebra of* $\mathcal{G}$ *(in particular* $i_{\mathcal{R}}$ *is injective);*
(3) $F_{\mathcal{R}}(\mathcal{G})$ *is a completion of* $C_c(\mathcal{G})$ *and for any* $U \in \mathrm{Bis}(\mathcal{G})$ *there is a contractive linear map* $E_U^{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_b(U)$ *that extends* $E_U : C_c(\mathcal{G}) \to C_b(U)$;
(4) $F_{\mathcal{R}}(\mathcal{G})$ *is a completion of* $C_c(\mathcal{G})$ *such that the inclusion* $C_c(\mathcal{G}) \subseteq C_0(\mathcal{G})$ *extends to a contractive map* $j_{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_0(\mathcal{G})$.

Assume the above equivalent conditions. For each $U \in \mathrm{Bis}(\mathcal{G})$, $E_U^{\mathcal{R}}$ *is a* $C_0(X)$*-module map given by* $E_U^{\mathcal{R}}(f) = j_{\mathcal{R}}(f)|_U$, $f \in F_{\mathcal{R}}(\mathcal{G})$, $j_{\mathcal{R}}$ *is homomorphism and for any family* $S \subseteq \mathrm{Bis}(\mathcal{G})$ *that covers* $\mathcal{G}$ *we have*

$$\ker j_{\mathcal{R}} = \bigcap_{U\in S} \ker E_U^{\mathcal{R}} = \{f \in F_{\mathcal{R}}(\mathcal{G}) : E_U^{\mathcal{R}}(af) = 0 \text{ for all } a \in C_c(r(U)),\ U \in S\}.$$

Moreover, $\ker j_{\mathcal{R}}$ *is the largest ideal in* $F_{\mathcal{R}}(\mathcal{G})$ *contained in* $\ker E_X^{\mathcal{R}}$.

PROOF. (4) implies (3) by putting $E_U^{\mathcal{R}}(f) := j_{\mathcal{R}}(f)|_U$ for $f \in F_{\mathcal{R}}(\mathcal{G})$ and $U \in \mathrm{Bis}(\mathcal{G})$. (3) implies (2) because $X \in \mathrm{Bis}(\mathcal{G})$. (2)⇒(1) is trivial. Assume (1). We need to show (4). For any $U \in \mathrm{Bis}(\mathcal{G})$ and $f \in C_c(r(U)\mathcal{G})$, taking b as in Lemma 6.5 we get

$$\|f|_U\|_\infty = \|f \circ s|_U^{-1}\|_\infty = \|(b * f)|_X\|_\infty = \|E_X^{\mathcal{R}}(b * f)\|_\infty \leqslant \|i_{\mathcal{R}}(b * f)\|_{\mathcal{R}} \leqslant \|i_{\mathcal{R}}(f)\|_{\mathcal{R}}.$$

Thus there is a linear contractive map $E_U^{\mathcal{R}} : C_0(r(U))F_{\mathcal{R}}(\mathcal{G}) \to C_0(U)$ with $E_U^{\mathcal{R}}(i_{\mathcal{R}}(f)) = f|_U$ for $f \in C_c(r(U)\mathcal{G})$. Let $f \in C_c(\mathcal{G})$. For any $\varepsilon > 0$ there is $\gamma \in \mathcal{G}$ such that $\|f\|_\infty - \varepsilon < |f(\gamma)|$. Take $U \in \mathrm{Bis}(\mathcal{G})$ with $\gamma \in U$ and norm one $a \in C_c(r(U))$ with $a(r(\gamma)) = 1$. Then

$$|f(\gamma)| = |(a * f)(\gamma)| = |E_U^{\mathcal{R}}(i_{\mathcal{R}}(a * f))(\gamma)| \leqslant \|E_U^{\mathcal{R}}(i_{\mathcal{R}}(a * f))\|_\infty \leqslant \|i_{\mathcal{R}}(a * f)\|_{\mathcal{R}} \leqslant \|i_{\mathcal{R}}(f)\|_{\mathcal{R}}.$$

This implies that $\|f\|_\infty \leqslant \|i_{\mathcal{R}}(f)\|_{\mathcal{R}}$, which in turn implies (4). Thus (1)–(4) are equivalent.

Now assume that (1)–(4) hold. To show that $j_{\mathcal{R}}$ is a $C_c(\mathcal{G})$-bimodule map fix $f \in F_{\mathcal{R}}(\mathcal{G})$ and $g \in C_c(\mathcal{G})$. Pick $\{f_n\}_{n=1}^\infty \subseteq C_c(\mathcal{G})$ converging to f in $\|\cdot\|_{\mathcal{R}}$. Then $\{f_n\}_{n=1}^\infty$ converges to $j_{\mathcal{R}}(f)$ in $\|\cdot\|_\infty$ by continuity of $j_{\mathcal{R}}$. Thus $g * j_{\mathcal{R}}(f) = \lim_{n\to\infty} g * f_n = \lim_{n\to\infty} j_{\mathcal{R}}(gf_n) = j_{\mathcal{R}}(gf)$. Symmetric considerations give $j_{\mathcal{R}}(fg) = j_{\mathcal{R}}(f) * g$. This in particular implies that $\ker j_{\mathcal{R}}$ is an ideal in $F_{\mathcal{R}}(\mathcal{G})$ and that $E_U^{\mathcal{R}}$ is a $C_0(X)$-module map, for any $U \in \mathrm{Bis}(\mathcal{G})$. For any $S \subseteq \mathrm{Bis}(\mathcal{G})$ covering $\mathcal{G}$, the equality $\ker j_{\mathcal{R}} = \bigcap_{U\in S} \ker E_U^{\mathcal{R}}$ is clear as $E_U^{\mathcal{R}}$ is the restriction of $j_{\mathcal{R}}$ to U. Since $E_U^{\mathcal{R}}$ is a continuous $C_0(X)$-module map, we also get $\ker E_U^{\mathcal{R}} = \{f \in F_{\mathcal{R}}(\mathcal{G}) : E_U^{\mathcal{R}}(af) = 0$ for all $a \in C_c(r(U)),\ U \in S\}$. This proves the displayed equalities.

Finally assume that I is an ideal in $F_{\mathcal{R}}(\mathcal{G})$ contained in $\ker E_X^{\mathcal{R}}$, and let $f \in I$. For every $U \in \mathrm{Bis}(\mathcal{G})$ and $a \in C_c(r(U))$, Lemma 6.5 applied to $a * j_{\mathcal{R}}(f) = j_{\mathcal{R}}(af) \in C_c(r(U)) * C_0(\mathcal{G})$ gives $b \in C_c(U^*)$ such that $\|E_X^{\mathcal{R}}(baf)\|_\infty = \|E_U^{\mathcal{R}}(af)\|_\infty$. Since $baf \in I$ we have $E_U^{\mathcal{R}}(af) = 0$ and therefore $f \in \ker(j_{\mathcal{R}})$ by the above description of $\ker(j_{\mathcal{R}})$. Hence $I \subseteq \ker(j_{\mathcal{R}})$. ☐

REMARK 6.8. The contractive map $j_{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_0(\mathcal{G})$, as in Proposition 6.7(4), in the context of C^*-algebras is often called *Renault's j-map*, and it is well known that it is injective on the reduced groupoid C^*-algebra, cf. [**Ren80**, Proposition II.4.2], [**BFPR**, Proposition 2.8], [**KM21**, Proposition 7.10] and [**DWZ22**]. In our more general context, we will refer to it as the *j-map* for $F_{\mathcal{R}}(\mathcal{G})$.

Remark 6.9. In view of Proposition 6.7 we see that a reduced groupoid Banach algebra of $\mathcal{G}$ is a Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ completion of $C_c(\mathcal{G})$ such that the inclusion $C_c(\mathcal{G}) \subseteq C_0(\mathcal{G})$ extends to an *injective contraction* $j_{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_0(\mathcal{G})$ which is isometric on $C_0(X)$. Equivalently, a groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ is reduced if, fixing any $S \subseteq \mathrm{Bis}(\mathcal{G})$ covering $\mathcal{G}$, every $f \in F_{\mathcal{R}}(\mathcal{G})$ is uniquely determined by elements $\{E_U^{\mathcal{R}}(f)\}_{U\in S}$ where $E_U^{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_b(U)$ is a contraction that extends E_U, $U \in S$. In fact, by the displayed equality in Proposition 6.7, elements in $F_{\mathcal{R}}(\mathcal{G})$ are determined by the restricted maps $E_U^{\mathcal{R}} : C_0(r(U))F_{\mathcal{R}}(\mathcal{G}) \to C_0(U)$ that take values in $C_0(U)$ rather than $C_b(U)$, $U \in S$, see Lemma 6.5. Thus apealing to Example 6.6, we see that the maps $E_U^{\mathcal{R}}$ generalize the *Fourier decomposition maps* considered in [**BK24**, Definition 4.4] for Banach algebra crossed products. In particular, our notion of a reduced algebra is consistent with the definition of a *reduced Banach algebra crossed product* from [**BK24**], and a *reduced group algebra* from [**Phi19**].

Corollary 6.10. *Let $F_{\mathcal{R}}(\mathcal{G})$ be a groupoid Banach algebra. For any representation $\psi : F_{\mathcal{R}}(\mathcal{G}) \to B$ such that $\|f|_X\|_\infty \leqslant \|\psi(f)\|$ for $f \in C_c(\mathcal{G})$, we have $\ker\psi \subseteq \ker j_{\mathcal{R}}$, and ψ is necessarily injective on space $C_c(\mathcal{G})$. If $F_{\mathcal{R}}(\mathcal{G})$ is reduced then ψ as above is necessarily injective on $F_{\mathcal{R}}(\mathcal{G})$.*

Proof. Define a seminorm on $C_c(\mathcal{G})$ by the formula $\|f\|_\psi := \|\psi(f)\|$, $f \in C_c(\mathcal{G})$ and put $F_\psi(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|}$. Then for any $f \in C_c(X)$ we have $\|f\|_\infty = \|f|_X\|_\infty \leqslant \|\psi(f)\| \leqslant \|f\|_{\mathcal{R}} = \|f\|_\infty$ and thus $C_0(X)$ embeds isometrically in $F_\psi(\mathcal{G})$. Moreover, the canonical homomorphism $i_\psi : C_c(\mathcal{G}) \to F_\psi(\mathcal{G})$ coincides with $\psi|_{C_c(\mathcal{G})}$ so condition (1) of Proposition 6.7 holds. Hence, $F_\psi(\mathcal{G})$ is a groupoid Banach algebra. Therefore, there exists the contractive homomorphism $j_\psi : F_\psi \to C_0(\mathcal{G})$ such that the following diagram commutes

$$\begin{array}{ccccc} C_c(\mathcal{G}) & \xrightarrow{i_{\mathcal{R}}} & F_{\mathcal{R}}(\mathcal{G}) & \xrightarrow{j_{\mathcal{R}}} & C_0(\mathcal{G}) \\ \big\downarrow{\scriptstyle id} & & \big\downarrow{\scriptstyle \psi} & & \big\downarrow{\scriptstyle id} \\ C_c(\mathcal{G}) & \xrightarrow{i_\psi} & F_\psi(\mathcal{G}) & \xrightarrow{j_\psi} & C_0(\mathcal{G}) \end{array}.$$

Hence, $j_{\mathcal{R}} = j_\psi \circ \psi$ and so $\ker\psi \subseteq \ker j_{\mathcal{R}}$. Moreover, by (2) of Proposition 6.7, $\psi|_{C_c(\mathcal{G})} = i_\psi$ is injective. If $F_{\mathcal{R}}(\mathcal{G})$ is reduced then $\ker j_{\mathcal{R}} = \{0\}$ and so $\ker\psi = \{0\}$, that is ψ is injective on $F_{\mathcal{R}}(\mathcal{G})$. □

Corollary 6.11. *For any groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ the quotient Banach algebra $F_{\mathcal{R},\mathrm{r}}(\mathcal{G}) := F_{\mathcal{R}}(\mathcal{G})/\ker j_{\mathcal{R}}$ is naturally a reduced groupoid Banach algebra of $\mathcal{G}$.*

Proof. The contraction $j_{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_0(\mathcal{G})$ factors to an injective contraction $j_{\mathcal{R}}^{\mathrm{r}} : F_{\mathcal{R},\mathrm{r}}(\mathcal{G},\mathcal{L}) \to C_0(\mathcal{G})$. Since $\ker j_{\mathcal{R}} \cap C_c(\mathcal{G}) = \{0\}$ we may view $F_{\mathcal{R},\mathrm{r}}(\mathcal{G})$ as a completion of $C_c(\mathcal{G})$. Since the quotient map $q_{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to F_{\mathcal{R},\mathrm{r}}(\mathcal{G})$ is injective and contractive on $C_0(X)$, it is isometric on $C_0(X)$, by minimality of the C^*-norm. Therefore, $F_{\mathcal{R},r}(\mathcal{G})$ is a groupoid Banach algebra and $j^r_{\mathcal{R}}$ is a j-map for $F_{\mathcal{R},\mathrm{r}}(\mathcal{G})$. Let $J \lhd F_{\mathcal{R},r}(\mathcal{G})$ with $J \subseteq \ker E^{\mathcal{R},r}_X$. Then $q_{\mathcal{R}}^{-1}(J) \subseteq \ker E^{\mathcal{R}}_X$. Since $\ker J_{\mathcal{R}}$ is the largest ideal in cointained in $\ker E^{\mathcal{R}}_X$ we have $q_{\mathcal{R}}^{-1}(J) \subseteq \ker j_{\mathcal{R}}$ and then $J = \{0\}$. Hence $F_{\mathcal{R},r}(\mathcal{G})$ is a reduced groupoid Banach algebra. □

REMARK 6.12. If $F_{\mathcal{R}}(\mathcal{G})$ is a *groupoid Banach $*$-algebra* meaning that the involution on $C_c(\mathcal{G})$ is isometric in $\|\cdot\|_{\mathcal{R}}$, then the map $j_{\mathcal{R}}$ is $*$-preserving and both the ideal $\ker j_{\mathcal{R}}$ and the quotient reduced algebra $F_{\mathcal{R},\mathrm{r}}(\mathcal{G}) = F_{\mathcal{R}}(\mathcal{G})/\ker j_{\mathcal{R}}$ are Banach $*$-algebras.

COROLLARY 6.13. *A C^*-completion $F_{\mathcal{R}}(\mathcal{G})$ of the $*$-algebra $C_c(\mathcal{G})$ is a groupoid Banach algebra if and only if there is a canonical $*$-homomorphism $F_{\mathcal{R}}(\mathcal{G}) \twoheadrightarrow C^*_{\mathrm{r}}(\mathcal{G})$, and then $F_{\mathcal{R}}(\mathcal{G})$ is reduced if and only if $F_{\mathcal{R}}(\mathcal{G}) = C^*_{\mathrm{r}}(\mathcal{G})$.*

PROOF. Assume first that the C^*-algebra $F_{\mathcal{R}}(\mathcal{G})$ is a reduced groupoid Banach algebra of $\mathcal{G}$. Then we have natural $*$-epimorphisms $\psi : C^*(\mathcal{G}) \twoheadrightarrow F_{\mathcal{R}}(\mathcal{G})$, $\Lambda : C^*(\mathcal{G}) \twoheadrightarrow C^*_{\mathrm{r}}(\mathcal{G})$ and faithful conditional expectations $E^{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \twoheadrightarrow C_0(X)$, $E^{\mathrm{r}} : C^*_{\mathrm{r}}(\mathcal{G}) \twoheadrightarrow C_0(X)$ such that the following diagram commutes

$$\begin{array}{ccc} C^*(\mathcal{G}) & \xrightarrow{\Lambda} & C^*_{\mathrm{r}}(\mathcal{G}) \\ \psi\big\downarrow & & \big\downarrow E^{\mathrm{r}} \\ F_{\mathcal{R}}(\mathcal{G}) & \xrightarrow{E^{\mathcal{R}}} & C_0(X) \end{array} .$$

Since $\psi(\ker\Lambda) \lhd F_{\mathcal{R}}(\mathcal{G})$ and $\Lambda(\ker\psi) \lhd C^*_{\mathrm{r}}(\mathcal{G})$, we conclude that $\ker\psi = \ker\Lambda$. This implies that $F_{\mathcal{R}}(\mathcal{G}) = C^*_{\mathrm{r}}(\mathcal{G})$ as they are both completions in the norm coming from $C^*(\mathcal{G})/\ker\psi = C^*(\mathcal{G})/\ker\Lambda$. The characterization of general groupoid Banach algebras which are C^*-algebras follows now from Corollary 6.11. □

The following lemma generalizes [**BK24**, Lemma 4.10]. It allows us to produce (reduced) groupoid Banach algebras and *-Banach algebras from other (reduced) groupoid Banach algebras.

LEMMA 6.14. *Let $\|\cdot\|_{\mathcal{R}}$ and $\|\cdot\|_{\mathcal{R}_i}$, $i \in I$, be norms on $C_c(\mathcal{G})$ that define groupoid Banach algebras $F_{\mathcal{R}}(\mathcal{G})$ and $F_{\mathcal{R}_i}(\mathcal{G})$, $i \in I$, for $\mathcal{G}$. Then the formulas*

$$(6.2) \qquad \|f\|_{\mathcal{R}*} := \|f^*\|_{\mathcal{R}}, \qquad \|f\|_{\{\mathcal{R}_i\}_i} := \sup_{i\in I} \|f\|_{\mathcal{R}_i}, \qquad f \in C_c(\mathcal{G}),$$

define norms that yield groupoid Banach algebras $F_{\mathcal{R}*}(\mathcal{G})$, $F_{\{\mathcal{R}_i\}_i}(\mathcal{G})$ for $\mathcal{G}$, provided $\|\cdot\|_{\{\mathcal{R}_i\}_i}$ is finite. In particular, the norm $\|f\|_{\mathcal{R},*} := \max\{\|f\|_{\mathcal{R}}, \|f^*\|_{\mathcal{R}}\}$ defines a groupoid algebra $F_{\mathcal{R},*}(\mathcal{G})$ which is a Banach $*$-algebra. Moreover, $F_{\mathcal{R}*}(\mathcal{G})$ and $F_{\mathcal{R},*}(\mathcal{G})$ are reduced when $F_{\mathcal{R}}(\mathcal{G})$ is; and $F_{\{\mathcal{R}_i\}_i}(\mathcal{G})$ is reduced if and only if all $F_{\mathcal{R}_i}(\mathcal{G})$, $i \in I$, are reduced.

PROOF. The involution on $C_c(\mathcal{G})$ extends to an antimultiplicative antilinear isometry from $F_{\mathcal{R}*}(\mathcal{G})$ onto $F_{\mathcal{R}}(\mathcal{G})$. In particular, $\|f|_X\|_\infty = \|f^*|_X\|_\infty \leqslant \|f^*\|_{\mathcal{R}} = \|f\|_{\mathcal{R}*}$ for $f \in C_c(\mathcal{G})$. Hence $F_{\mathcal{R}*}(\mathcal{G})$ is a groupoid Banach algebra. Moreover, $j_{\mathcal{R}*}(f) = j_{\mathcal{R}}(f^*)^*$ for $f \in F_{\mathcal{R}*}(\mathcal{G})$. Thus $j_{\mathcal{R}*}$ is injective if and only if $j_{\mathcal{R}}$ is injective. Equivalently, $F_{\mathcal{R}*}(\mathcal{G})$ is reduced if and only $F_{\mathcal{R}}(\mathcal{G})$ is reduced.

Assume that $\|\cdot\|_{\{\mathcal{R}_i\}_i}$ attains finite values. Clearly, it is a submultiplicative norm on $C_c(\mathcal{G})$, which coincides with $\|\cdot\|_\infty$ on $C_c(X)$ and $\|f\|_\infty \leqslant \|f\|_{\{\mathcal{R}_i\}_i}$ for $f \in C_c(\mathcal{G})$, as this holds for every $\|\cdot\|_{\mathcal{R}_i}$, $i \in I$. Thus $F_{\{\mathcal{R}_i\}_i}(\mathcal{G})$ is a groupoid Banach algebra, and we have a canonical contraction $j_{\{\mathcal{R}_i\}_i} : F_{\{\mathcal{R}_i\}_i}(\mathcal{G}) \to C_0(\mathcal{G})$. Note that $\pi(a) := \prod_{i\in I} a$, for $a \in C_c(\mathcal{G})$, determines an isometric embedding of $F_{\{\mathcal{R}_i\}_i}(\mathcal{G})$ into the direct product $\prod_{i\in I} F_{\mathcal{R}_i}(\mathcal{G})$. Moreover, $\prod_{i\in I} j_{\{\mathcal{R}_i\}_i} = \prod_{i\in I} j_{\mathcal{R}_i} \circ \pi$, as maps from $F_{\{\mathcal{R}_i\}_i}(\mathcal{G})$ to $\prod_{i\in I} C_0(\mathcal{G},)$. Hence $j_{\{\mathcal{R}_i\}_i}$ is injective if all $j_{\mathcal{R}_i}$, $i \in I$, are injective. That is, $F_{\{\mathcal{R}_i\}_i}(\mathcal{G})$ is reduced if all $F_{\mathcal{R}_i}(\mathcal{G})$, $i \in I$, are reduced. □

REMARK 6.15. Let $F_{\mathcal{R}}(\mathcal{G})$ be a groupoid Banach algebra which in addition is S-graded by some unital inverse subsemigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$ which covers $\mathcal{G}$, see Definition 6.1(2). By Proposition 6.7(3), the spaces $C_0(U)$, $U \in S$, forming the grading, embed isometrically into $F_{\mathcal{R}}(\mathcal{G})$. Also then the algebras $F_{\mathcal{R}*}(\mathcal{G})$ and $F_{\mathcal{R},*}(\mathcal{G})$ are S-graded, and if $F_{\mathcal{R}_i}(\mathcal{G})$, $i \in I$, are all S-graded groupoid Banach algebras, then also $F_{\mathcal{R}*}(\mathcal{G})$ is S-graded.

Inspired by [**BK24**, Definition 4.12] we now generalize L^p-groupoid operator algebras to L^P-operator algebras where $P \subseteq [1,\infty]$ is a set of Hölder exponents.

DEFINITION 6.16. For any non-empty $P \subseteq [1,\infty]$ we put

$$F^P(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{L^P}} \quad \text{and} \quad F^P_{\mathrm{r}}(\mathcal{G}) := \overline{C_c(\mathcal{G})}^{\|\cdot\|_{L^P,\mathrm{r}}},$$

where $\|f\|_{L^P} := \sup_{p\in P} \|f\|_{L^p}$ and $\|f\|_{L^P,\mathrm{r}} := \sup_{p\in P} \|f\|_{L^p,\mathrm{r}}$, $f \in C_c(\mathcal{G})$. We denote by $\Lambda_P : F^P(\mathcal{G}) \to F^P_{\mathrm{r}}(\mathcal{G})$ the canonical representation (which is the identity on $C_c(\mathcal{G})$).

PROPOSITION 6.17. *For any non-empty $P \subseteq [1,\infty]$, $F^P_{\mathrm{r}}(\mathcal{G})$ is a reduced groupoid Banach algebra which is* $\mathrm{Bis}(\mathcal{G})$*-graded, and the corresponding j-map $j^{\mathrm{r}}_P : F^P_{\mathrm{r}}(\mathcal{G}) \to C_0(\mathcal{G})$ turns the product in $F^P_{\mathrm{r}}(\mathcal{G})$ into the convolution:*

$$j^{\mathrm{r}}_P(f\cdot g)(\gamma) = \sum_{r(\eta)=r(\gamma)} j^{\mathrm{r}}_P(f)(\eta)\cdot j^{\mathrm{r}}_P(g)(\eta^{-1}\gamma), \qquad f,g \in F^P_{\mathrm{r}}(\mathcal{G}), \tag{6.3}$$

and it preserves the involution in the sense that $j^{\mathrm{r}}_P(f)^* = j^{\mathrm{r}}_{P*}(f^*)$, $f \in F^P_{\mathrm{r}}(\mathcal{G})$, where $P^* := \{q : 1/p + 1/q = 1, p \in P\}$. In particular, $F^P(\mathcal{G})$ is a groupoid Banach algebra with the j-map $j_P := j^{\mathrm{r}}_P \circ \Lambda_P$ and so $\ker j_P = \ker \Lambda_P$.

PROOF. By Lemma 6.14 it suffices to consider the case when $P = \{p\}$ is a singleton (in particular, $j^{\mathrm{r}}_P = \prod_{p\in P} j^{\mathrm{r}}_p \circ \pi$, where $\pi : F^P_{\mathrm{r}}(\mathcal{G}) \to \prod_{p\in P} F^p_{\mathrm{r}}(\mathcal{G})$ is given by $\pi(a) := \prod_{p\in P} a$, for $a \in C_c(\mathcal{G})$). For each $\gamma \in \mathcal{G}$ choose a norm one element $1_\gamma \in L_\gamma$ and treat it as a section of $\mathcal{L}$ which is zero at $\eta \neq \gamma$. Then $\{1_\gamma\}_{\gamma\in\mathcal{G}}$ is a Schauder basis for $\ell^p(\mathcal{G})$ and for any $f \in C_c(\mathcal{G})$ we have $\|f|_X\|_\infty \leqslant \|f\|_\infty = \sup_{\gamma\in\mathcal{G}} |(f1_{d(\gamma)})(\gamma)| \leqslant \sup_{\gamma\in\mathcal{G}} \|f1_{d(\gamma)}\|_p \leqslant \|f\|_{L^p,\mathrm{r}}$. Thus $F^p_r(\mathcal{G})$ is a groupoid Banach algebra and we have a contractive linear map $j^{\mathrm{r}}_p : F^p_{\mathrm{r}}(\mathcal{G}) \to C_0(\mathcal{G})$ extending the inclusion $C_c(\mathcal{G}) \subseteq C_0(\mathcal{G})$. Clearly for every $\gamma \in \mathcal{G}$ and any $f \in C_c(\mathcal{G})$ we have

$$|j^{\mathrm{r}}_p(f)(\gamma)| := |\big(f1_{d(\gamma)}\big)(\gamma)| \qquad \text{and} \qquad \|f1_\gamma\|_p = \left(\sum_{d(\eta)=d(\gamma)} |j^{\mathrm{r}}_p(f)(\eta\gamma^{-1})|^p\right)^{1/p}. \tag{6.4}$$

By continuity these relations hold for any $f \in F^p_{\mathrm{r}}(\mathcal{G})$. The second formula in (6.4) implies that j^{r}_p is injective on $F^p_{\mathrm{r}}(\mathcal{G})$ and hence $F^p_{\mathrm{r}}(\mathcal{G})$ is a reduced groupoid Banach algebra.

Since * yields the isometry $F^p_{\mathrm{r}}(\mathcal{G}) \overset{\text{anti}}{\cong} F^q_{\mathrm{r}}(\mathcal{G})$, where $1/p+1/q = 1$, we get $j^{\mathrm{r}}_p(f)^* = j^{\mathrm{r}}_q(f^*)$, $f \in F^p_r(\mathcal{G})$. Now we prove (6.3), with $P = \{p\}$. Fix $f, g \in F^p_{\mathrm{r}}(\mathcal{G})$ and take nets $(f_i), (g_j)$ in $C_c(\mathcal{G})$ converging to f, g in the norm of $F^p_{\mathrm{r}}(\mathcal{G})$. Using Hölders inequality, (6.4) and the isometry $F^p_{\mathrm{r}}(\mathcal{G}) \overset{\text{anti}}{\cong} F^q_{\mathrm{r}}(\mathcal{G})$, for any $\gamma \in \mathcal{G}$ we get

$$\sum_{r(\eta)=r(\gamma)} \left|j^{\mathrm{r}}_p(f_i - f)(\eta) \cdot j^{\mathrm{r}}_p(g_j)(\eta^{-1}\gamma)\right| \leqslant \|(f_i - f)^* 1_{r(\gamma)}\|_q \|g_j 1_\gamma\|_p \leqslant \|(f_i - f)^*\|_{L^{q},\mathrm{r}} \|g_j\|_{L^p,\mathrm{r}}$$
$$= \|f_i - f\|_{L^p,\mathrm{r}} \|g_j\|_{L^p,\mathrm{r}}.$$

Similarly we get $\sum_{r(\eta)=r(\gamma)} |j^{\mathrm{r}}_p(f)(\eta) \cdot j^{\mathrm{r}}_p(g_j - g)(\eta^{-1}\gamma)| \leqslant \|f\|_{L^p,\mathrm{r}} \|g_j - g\|_{L^p,\mathrm{r}}$. Altogether, since $j^{\mathrm{r}}_p(f_i \cdot g_j) = f_i * g_j$, this implies that

$$\left|j^{\mathrm{r}}_p(f_i \cdot g_j)(\gamma) - \sum_{r(\eta)=r(\gamma)} j_p(f)(\eta) \cdot j_p(g)(\eta^{-1}\gamma)\right| \leqslant \|f_i - f\|_{L^p,\mathrm{r}} \|g_j\|_{L^p,\mathrm{r}} + \|f\|_{L^p,\mathrm{r}} \|g_j - g\|_{L^p,\mathrm{r}}$$

tends to zero, which proves (6.3), as $j^{\mathrm{r}}_p(f_i \cdot g_j)(\gamma)$ tends to $j^{\mathrm{r}}_p(f \cdot g)(\gamma)$. □

Remark 6.18. If $P = P^*$, then $F^P(\mathcal{G})$ and $F^P_{\mathrm{r}}(\mathcal{G})$ are Banach $*$-algebras and Λ_P is a $*$-homomorphism, cf. [**BKM25**, Theorem 5.6(4)]. For $P = \{p, q\}$ with $1/p + 1/q = 1$, the Banach $*$-algebras $F^P_{\mathrm{r}}(\mathcal{G})$ are sometimes called *symmetrized pseudofunction algebras*. They were studied in [**AO22**] and in the group case in [**Elk24**], [**LY17**],[**Phi19**]. For any $P \subseteq [1, \infty]$, by Theorem 5.33(1)(2), we have

$$\{1, \infty\} \subseteq P \implies F^P(\mathcal{G}) = F^P_{\mathrm{r}}(\mathcal{G}) = F_I(\mathcal{G}),$$

In particular, $F_I(\mathcal{G})$ is a reduced groupoid Banach algebra. Also by Theorem 5.33(1), we always have $F^P(\mathcal{G}) = F^P_{\mathrm{r}}(\mathcal{G})$ if $P \subseteq \{1, \infty\}$. Thus [**GL17**, Theorem 6.19] implies that if $\mathcal{G}$ is second countable and amenable, then $F^P(\mathcal{G}) = F^P_{\mathrm{r}}(\mathcal{G})$ for every $P \subseteq [1, \infty]$.

6.2. Topological freeness and intersection properties

A discrete group action $\theta : G \to \mathrm{Homeo}(X)$ on a locally compact Hausdorff space X is called *topologically free* if for every $g \in G\backslash\{1\}$ the set of fixed points $\{x \in X : \theta_g(x) = x\}$ has empty interior. This is a standard and well known condition that appears already in [**ZM68**, Proposition 4.14] where it was used to characterize when $C_0(X)$ is maximal abelian subalgebra of the reduced C^*-algebraic crossed product $C_0(X) \rtimes_{\mathrm{r}} G$. Topological freeness was mostly popularized by the results of Kawamura-Tomiyama [**KT90**] and Archbold-Spielberg [**AS93**] which related it to (generalized) intersection property for the inclusion $C_0(X) \subseteq C_0(X) \rtimes_{\mathrm{r}} G$. The name topological freeness was probably coined by Tomiyama, see [**Tom92**, Definition 2.1(c)]

Topological freeness for étale groupoids was introduced in [**KM21**, Definition 2.20]. Since we assume that $\mathcal{G}$ is Hausdorff, topological freeness coincides with effectiveness, which is a more popular condition, cf., for instance,[**Ren08**], [**BCFS14**], [**CGT24**]. Nevertheless, we will keep on using the name topological freeness, to underline the connection to the aforementioned condition for group actions, and also because that for non-Hausdorff étale groupoids topological freeness is weaker and better condition than effectiveness.

Definition 6.19. For groupoid $\mathcal{G}$ we put $\mathrm{Iso}(\mathcal{G}) := \bigcup_{x\in X} \mathcal{G}(x)$, where $\mathcal{G}(x) := r^{-1}(x) \cap d^{-1}(x)$ denotes the *isotropy group over* x. An étale groupoid $\mathcal{G}$ is *topologically free* if there is no non-empty open set $V \subseteq \mathcal{G}\backslash X$ with $r|_V = d|_V$, equivalently the set $\mathrm{Iso}(\mathcal{G})\backslash X$ has empty interior in $\mathcal{G}$.

Remark 6.20. Recall that any bisection $V \in \mathrm{Bis}(\mathcal{G})$ induces the partial homeomorphism $\theta_V = r \circ d|_V^{-1} : d(V) \to r(V)$ of X. The groupoid $\mathcal{G}$ is topologically free if and only if for every bisection $V \subseteq \mathcal{G}\backslash X$, the set $\{x \in X : \mathcal{G}(x) \cap V \neq \emptyset\} = \{x \in d(V) : \theta_V(x) = x\}$ has empty interior in X. In particular, we see that if $\mathcal{G} = G \times X$ is the transformation groupoid for a group action $\theta : G \to \mathrm{Homeo}(X)$, then $\mathcal{G}$ is topologically free if and only if the group action is topologically free.

Remark 6.21. An étale groupoid $\mathcal{G}$ is *effective* if for any open $V \subseteq \mathcal{G}$ with $r|_V = d|_V$ we have $V \subseteq X$, equivalently interior of $\mathrm{Iso}(\mathcal{G})$ is X. Thus clearly, effectiveness implies topological freeness. Since we assume that $\mathcal{G}$ is Hausdorff, equivalently X is clopen, the converse holds. Hence in this thesis, $\mathcal{G}$ is effective if and only if it is topologically free.

Remark 6.22. An étale groupoid $\mathcal{G}$ is *topologically principle* if the set of point with nontrivial isotoropy $\{x \in X : \mathcal{G}(x) \neq \{x\}\}$ has empty interior. It implies topological freeness. Moreover, if $\mathcal{G}$ is covered by a countable family of bisections (so e.g. when $\mathcal{G}$ is second countable or more generally σ-compact), then using that $\mathcal{G}$ is a Baire space we get that $\mathcal{G}$ is topologically free if and only if $\mathcal{G}$ is topologicall principle.

We say that a subalgebra A of an algebra B is *maximal abelian* if it is maximal element amongst all commutative subalgebras of B partially ordered by inclusion.

Proposition 6.23. *The following conditions are equivalent*

1. $\mathcal{G}$ *is topologically free;*
2. $C_0(X)$ *is maximal abelian in every reduced groupoid Banach algebra* $F_{\mathcal{R}}(\mathcal{G})$*;*
3. $C_0(X)$ *is maximal abelian in some reduced groupoid Banach algebra* $F_{\mathcal{R}}(\mathcal{G})$*.*

Proof. (1)⇒(2). Assume that $\mathcal{G}$ is not topologically free and take non-empty open $V \subseteq \mathcal{G}\backslash X$ with $r|_V = d|_V$. Then every $b \in C_c(V)$ commutes with $C_0(X)$. Hence $C_0(X)$ is not maximal abelian if $F_R(\mathcal{G})$. Implication (2)⇒(3) is trivial.

(3)⇒(1). Let $F_{\mathcal{R}}(\mathcal{G})$ be a reduced groupoid Banach algebra and assume that there is $b \in F_{\mathcal{R}}(\mathcal{G})\backslash C_0(X)$ which commutes with all elements of $C_0(X)$. Since $j_{\mathcal{R}} : F_{\mathcal{R}}(\mathcal{G}) \to C_0(\mathcal{G})$ is injective and $E_X^{\mathcal{R}}(b) = j_{\mathcal{R}}(b)|_X \in C_0(X)$, we get that $c = b - E_X^{\mathcal{R}}(b) \in F_{\mathcal{R}}(\mathcal{G})\backslash C_0(X)$ commutes with all elements in $C_0(X)$ and $\mathrm{supp}(j_{\mathcal{R}}(c))$ has non-empty interior and is contained in $\mathcal{G}\backslash X$. Thus there is a non-empty open bisection $V \subseteq \mathrm{supp}(j_{\mathcal{R}}(c)) \subseteq \mathcal{G}\backslash X$. For each $a \in C_c(V)$ we have $a j_{\mathcal{R}}(c) = j_{\mathcal{R}}(ac) = j_{\mathcal{R}}(ca) = j_{\mathcal{R}}(c)a$. This implies that $r_V = d_V$. Hence $\mathcal{G}$ is not topologically free. □

The following lemma is the technical heart of our simplicity criteria that we discuss in the sequel. It is a groupoid version of [**BK24**, Lemma 5.3] and a simplifed version of [**BKM26**, Lemma 5.8].

LEMMA 6.24 (pinching property). *If $\mathcal{G}$ is topologically free, then for any $f \in C_c(\mathcal{G})$ and $\varepsilon > 0$ there are norm-one functions $a, b \in C_0(X)$ such that $a \cdot f|_X \cdot b \in C_0(X)^+$,*

$$\|f|_X\| \leqslant \|af|_X b\| + \varepsilon \quad \text{and} \quad \|afb - (af|_X b)\|_{\max} \leqslant \varepsilon.$$

PROOF. Let $f \in C_c(\mathcal{G})$. Then $f = \sum_{V \in F} f_V \delta_V$ where $F \subseteq \mathrm{Bis}(\mathcal{G})$ is finite. Let $\varepsilon > 0$ and take an open set $W \subseteq X$ such that $\|f|_X\| < |f(x)| + \varepsilon$ for $x \in W$.

Note that we may assume that $f|_X \neq 0$ and consider only $\varepsilon < \|f|_X\|$, so that $f \neq 0$ on W. Each $V \in F$ decomposes into a disjoint union of two open bisections $V \cap X \subseteq X$ and $V_X := V \backslash X \subseteq \mathcal{G} \backslash X$. By topological freeness and [**KM21**, Proposition 2.24], see also [**ELQ02**, Lemma 2.2], the finite union

$$R := \bigcup_{V \in F} \{x \in d(V_X) : h_{V_X}(x) = x\}$$

has empty interior. Thus there is $x_0 \in W \backslash R$. We claim that for each $V \in F$ there is a function $a_V \in C_0(X)^+$ such that $\|a_V\| = a_V(x_0) = 1$ and

$$|a_V(x) \cdot f_V(x) \cdot a_V(h_V^{-1}(x))| \leqslant \varepsilon/|F| \quad \text{for all } x \in r(V_X). \tag{6.5}$$

Only the following three cases are possible:

1) If $x_0 \in V \cap X$, then any $a_V \in C_0(V \cap X)^+$ with $\|a_V\| = a_V(x_0) = 1$ will do, because $r(V_X) \cap (V \cap X) = \emptyset$ and so $a_V \cdot f_V = 0$ on $r(V_X)$.

2) If $x_0 \in X \backslash r(V)$, then $f_V(x_0) = 0$ so there is a neighobourhood $D \subseteq X$ of x_0 where $|f_V| \leqslant \varepsilon/|F|$, and then it suffices to take any $a_V \in C_0(D)^+$ with $\|a_V\| = a_V(x_0) = 1$.

3) If $x_0 \in r(V) \backslash \overline{V \cap X}$, then $x_0 \in r(V_X)$. Since $x_0 \notin R$, there is a neighobourhood $D \subseteq r(V_X)$ of x_0 such that $h_V^{-1}(D) \cap D = \emptyset$. Then we may take any $a_V \in C_0(D)^+$ with $\|a_V\| = a_V(x_0) = 1$.

Put $a := \prod_{V \in F} a_V$. Then $a \in C_0(X)^+$ satisfies $\|a\| = a(x_0) = 1$ and $af|_X \in C_0(X)$. Let be nd take any norm one $d \in C_0(X)^+$ supported on a precompact neighbourhood $U \subseteq W$ of x_0 such that $d(x_0) = 1$. Then $b := a \cdot d \cdot \frac{\overline{f}}{|f|}$ is a norm one continuous function on X. Moreover, $a \cdot f|_X \cdot b \in C_0(X)^+$ and $\|af|_X b\|_\infty \geqslant |f(x_0)| > \|f|_X\| - \varepsilon$, which is the first desired inequality.

For each $V \in F$ we have $af_V = af_V \mathbb{1}_{V \cap X} + af_V \mathbb{1}_{r(V_X)}$ where $af_V \mathbb{1}_{V \cap X} \in C_0(V \cap X)$ and $af_V \mathbb{1}_{r(V_X)} \in C_0(r(V_X))$ are continuous functions with disjoint supports. Thus

$$af = af|_X + \sum_{V \in F} af_V \mathbb{1}_{r(V_X)} \delta_V.$$

By Lemma 5.18, $\left(af_V \mathbb{1}_{r(V_X)} \delta_V\right) \cdot a = af_V(a \circ h_V^{-1})\delta_V$, and therefore by (6.5)

$$\Big\| \sum_{V \in F} \left(af_V \mathbb{1}_{r(V_X)} \delta_V\right) \cdot b \Big\|_{\max} \leqslant \Big\| \sum_{V \in F} af_V (a \circ h_V^{-1}) \delta_V \Big\|_{\max} \leqslant \varepsilon.$$

Thus $\|a \cdot fb - (af|_X b)\|_{\max} \leqslant \varepsilon$. □

COROLLARY 6.25. *Assume $\mathcal{G}$ is topologically free and let $S \subseteq \mathrm{Bis}(\mathcal{G})$ be a unital inverse subsemigroup covering $\mathcal{G}$. If $\psi : F^S(\mathcal{G}) \to B$ is a representation which is injective on $C_0(X)$, then $\overline{\psi(F^S(\mathcal{G}))}$ is naturally a groupoid Banach algebra.*

Proof. Since $\psi|_{C_0(X)}$ is injective, $\psi|_{C_0(X)}$ is isometric by minimality of the supremum norm. Take $f \in C_c(\mathcal{G})$ and $\epsilon > 0$ and $a, b \in C_0(X)$ as in Lemma 6.24. By the reverse triangle inequality and the contractiveness of ψ, we obtain that

$$\|\psi(abfa)\|_B \leqslant \|\psi(a(bf|_X)a\|_B + \varepsilon = \|\psi(a(bf|_X)a\|_\infty + \varepsilon.$$

Using inequalities in Lemma 6.24 again we get

$$\begin{aligned}\|f|_X\| &\leqslant \|a(b \cdot f|_X)a\|_\infty + \epsilon = \|\psi(a(b \cdot f|_X)a)\|_B + \epsilon \\ &\leqslant \|\psi(a(b \cdot f)a)\|_B + 2\epsilon \leqslant \|\psi(f)\| + 2\epsilon.\end{aligned}$$

Thus we conclude that $\|f|_X\| \leqslant \|\psi(f)\|_B$. It follows that $\overline{\psi(F^S(\mathcal{G}))}$ is a groupoid Banach algebra. □

The last part of Corollary 6.25 can be interpreted as a "generalized intersection property". The following intersections properties were introduced in [**BK24**, Definition 5.6] and [**KM21**, Definitions 5.5, 5.6]. Recall that the ideals we consider are closed and two-sided.

Definition 6.26. Let $A \subseteq B$ be a Banach subalgebra of a Banach algebra B. We say that A *detects ideals in* B, or that $A \subseteq B$ has the *intersection property*, if for every non-zero ideal J in B we have $J \cap A \neq \{0\}$. The inclusion has the *generalized intersection property* if there is a largest ideal $\mathcal{N}$ in B with $\mathcal{N} \cap A = \{0\}$. In this case we call $\mathcal{N}$ the *hidden ideal* and put $B_{\mathrm{r}} := B/\mathcal{N}$. We say $A \subseteq B$ is *minimal* if for every non-zero $a \in A$ we have $\overline{BaB} = B$.

Lemma 6.27. *If $A \subseteq B$ has the generalized intersection property and $\overline{A}$ is the closure of the range of A in the quotient B_{r}, then $\overline{A} \subseteq B_{\mathrm{r}}$ has the intersection property. An inclusion $A \subseteq B$ has the intersection property and is minimal if and only if B is simple.*

Proof. Assume $A \subseteq B$ has the generalized intersection property, let $q : B \to B_r$ be the quotient map and put $\overline{A} := \overline{q(A)}$. If J is a non-zero ideal in B_{r} then $q^{-1}(J)$ is an ideal in B which is strictly larger than $\mathcal{N}$. Thus $A \cap q^{-1}(J) \neq \{0\}$ and because q is injective on A we therefore get $\{0\} \neq q(A \cap q^{-1}(J)) \subseteq \overline{A} \cap J$. Hence $\overline{A} \subseteq B_{\mathrm{r}}$ has the intersection property.

Now let $A \subseteq B$ be any Banach algebra inclusion. If it is minimal and has the intersection property, then for any non-zero ideal J in B we have $I := A \cap J$ is a non-zero ideal in B and therefore $B = \overline{BIB} \subseteq J$, which proves that B is simple. The converse implication is straightforward. □

Remark 6.28. If $A \subseteq B$ has the generalized intersection property and A is a C^*-algebra, so for instance when $A = C_0(X)$, then by minimality of the C^*-norm, we have the isometric embedding $A \subseteq B_{\mathrm{r}}$ (i.e. we have $A = \overline{A}$ in Lemma 6.27).

Definition 6.29. A set $U \subseteq X$ is *$\mathcal{G}$-invariant* if $d(\gamma) \in U$ implies $r(\gamma) \in U$ for all $\gamma \in G$. The groupoid is *minimal* if there are no non-trivial $\mathcal{G}$-invariant open sets in X.

Lemma 6.30. *Let B be a groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$. For any ideal I in $C_0(X)$ the following are equivalent:*

(1) *I is restricted, i.e. $I = J \cap C_0(X)$ for an ideal J in B;*
(2) *$I = C_0(U)$ for an open $\mathcal{G}$-invariant set $U \subseteq X$;*
(3) *I is symmetric, i.e. $\overline{IB} = \overline{BI}$.*

Moreover, the inclusion $C_0(X) \subseteq B$ is minimal if and only if $\mathcal{G}$ is minimal.

Proof. We have $I = C_0(U)$ for an open set $U \subseteq X$. For $V \in \mathrm{Bis}(\mathcal{G})$ we denote $B_V := C_0(V) \subseteq F_{\mathcal{R}}(\mathcal{G})$.

(1)⇒(2). Assume $I = J \cap C_0(X)$ for an ideal J in B. For every $V \in \mathrm{Bis}(\mathcal{G})$ we have $C_0(VUV^*) \subseteq B_V C_0(U) B_{V*} \subseteq J \cap C_0(X) = C_0(U)$. Hence $VUV^* \subseteq U$ for every $V \in \mathrm{Bis}(\mathcal{G})$, which implies U is $\mathcal{G}$-invariant.

(2)⇒(3). If (2) holds then $VU = UV$ for all $V \in \mathrm{Bis}(\mathcal{G})$, which implies that $IB_V = B_V I$ for $V \in \mathrm{Bis}(\mathcal{G})$. Hence $\overline{IB} = \overline{\sum_{V\in \mathrm{Bis}(\mathcal{G})} IB_V} = \overline{\sum_{V\in \mathrm{Bis}(\mathcal{G})} B_V I} = \overline{BI}$.

(3)⇒(1). If (3) holds, then $J = \overline{IB} = \overline{BI}$ is an ideal in B (generated by I) and an approximate unit $\{\mu_i\}_i$ in I is an approximate unit in J. Using this, one gets $I = J \cap C_0(X)$.

Now assume $\mathcal{G}$ is minimal and let $a \subseteq C_0(X)$ be non-zero. By the equivalence (1)⇔(2), $\overline{BaB} \cap C_0(X) = C_0(U)$ where U is an open $\mathcal{G}$-invariant set. Since U contains the open support of a it is non-empty. Hence $U = X$ by $\mathcal{G}$-minimality, and therefore $C_0(X) \subseteq \overline{BaB}$. Since $C_0(X)$ contains an approximate unit for B this implies that $\overline{BaB} = B$. Thus $C_0(X) \subseteq B$ is minimal. For the converse assume $\mathcal{G}$ is not minimal. Let $U \subseteq X$ be a non-trivial open $\mathcal{G}$-invariant set and put $I = C_0(U)$. By the equivalence (2)⇔(3), $\overline{BIB} = \overline{IB}$ and so any approximate unit in I is also an approximate unit in $\overline{BIB}$. This implies that $\overline{BIB} \cap C_0(X) = I \neq C_0(X)$. Hence for any non-zero $a \in I$ we get $\overline{BaB} \neq B$. Thus $C_0(X) \subseteq B$ is not minimal. □

Theorem 6.31. *Let $S \subseteq \mathrm{Bis}(\mathcal{G})$ be a unital inverse subsemigroup covering $\mathcal{G}$. Assume $\mathcal{G}$ is topologically free and consider an S-graded groupoid Banach algebra $F_{\mathcal{R}}(\mathcal{G})$.*

(1) *the inclusion $C_0(X) \subseteq F_{\mathcal{R}}(G)$ has the generalized intersection property with the hidden ideal* $\ker j_{\mathcal{R}}$.
(2) *$F_{\mathcal{R}}(\mathcal{G})$ is a reduced groupoid Banach algebra if and only if $C_0(X) \subseteq F_{\mathcal{R}}(\mathcal{G})$ has the intersection property.*
(3) *The Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ is simple if and only if $F_{\mathcal{R}}(\mathcal{G})$ is a reduced Banach algebra and $\mathcal{G}$ is minimal.*

Proof. For an ideal J in $F_{\mathcal{R}}(\mathcal{G})$ with $J \cap C_0(X) = \{0\}$, Corollaries 6.10 and 6.25 applied to the quotient map $F_{\mathcal{R}}(\mathcal{G}) \to F_{\mathcal{R}}(\mathcal{G})/J$ give that $F_{\mathcal{R}}(\mathcal{G})/J$ is a groupoid Banach algebra and $J \subseteq \ker j_{\mathcal{R}}$. Hence the inclusion $C_0(X) \subseteq F_{\mathcal{R}}(\mathcal{G})$ has the generalized intersection property with the hidden ideal $\ker j_{\mathcal{R}}$. Thus it has the intersection property if and only if $\ker j_{\mathcal{R}} = \{0\}$, which by definition means that $F_{\mathcal{R}}(\mathcal{G})$ is a reduced groupoid Banach algebra. Combining this with Lemmas 6.27, 6.30 one gets that $F_{\mathcal{R}}(\mathcal{G})$ is simple if and only if $F_{\mathcal{R}}(\mathcal{G})$ is reduced and $\mathcal{G}$ is minimal. □

We now show that the generalized intersection property in Theorem 6.31(1) when applied to groupoid L^P-operator algebras is in fact equivalent to topological freeness. To this end, we use the following representation, which in the ℓ^2-context is called the *orbit* *representation* [**KM21**], *augmentation representation* [**BCFS14**], or *trivial representation* [**Ren97**]. As in [**BK24**] we prefer to call it a $C_0(X)$*-trivial representation*.

Lemma 6.32. *For any $p \in [1,\infty]$, we have a representation $\Lambda_p^{\mathrm{tr}} : F_p(\mathcal{G}) \to B(\ell^p(X))$ where*

$$\Lambda_p^{\mathrm{tr}}(f)\xi(x) := \sum_{d(\gamma)=x} f(\gamma)\xi(r(\gamma)), \qquad f \in C_c(\mathcal{G}),\ \xi \in \ell_p(X).$$

Proof. Consider the canonical action $\theta : S \to \mathrm{PAut}(X)$ of any wide inverse semigroup $S \subseteq \mathrm{Bis}(\mathcal{G})$, see Definition 1.14. One readily sees that

$$\pi^{\mathrm{tr}}(a)\xi(x) := a(x)\xi(x), \qquad v_U^{\mathrm{tr}}\xi(x) := \begin{cases} \xi(\theta_{U^*}(x)), & x \in r(U), \\ 0, & x \notin r(U), \end{cases}$$

for $a \in C_0(X)$, $\xi \in \ell^p(X)$, $x \in X$, $U \in S$, defines a covariant representation $(\pi^{\mathrm{tr}}, v^{\mathrm{tr}})$ of the action θ on the space $\ell^p(X)$ in the sense of Definition 4.8. The integrated representation $\pi^{\mathrm{tr}} \rtimes v^{\mathrm{tr}} : F(\mathcal{G}) \to B(\ell^p(X))$, given by Proposition 5.19, satisfies the formula in the assertion and it descends to the desired representation $\Lambda_p^{\mathrm{tr}} : F^p(\mathcal{G}) \to B(\ell^p(X))$. □

We define the $C_0(X)$*-trivial representation* Λ_P^{tr} of $F^P(\mathcal{G})$ as an extension of the ℓ^∞-direct sum $\oplus_{p\in P}\Lambda_p^{\mathrm{tr}}$ of representations from Lemma 6.32.

Theorem 6.33. *Let $\mathcal{G}$ be an étale groupoid with the unit space X. Let $\emptyset \neq P \subseteq [1,\infty]$ and denote by $\Lambda^P : F^P(\mathcal{G}) \to F^P_{\mathrm{r}}(\mathcal{G})$ the canonical homomorphism. The following conditions are equivalent:*

1. *$\mathcal{G}$ is topologically free;*
2. *$C_0(X) \subseteq F^P_{\mathrm{r}}(\mathcal{G})$ is maximal abelian subalgebra;*
3. *the inclusion $C_0(X) \subseteq F^P(\mathcal{G})$ has the generalized intersection property with the hidden ideal* $\ker(\Lambda^P)$*;*
4. $\ker(\Lambda_P^{\mathrm{tr}}) \subseteq \ker(\Lambda^P)$ *where Λ_P^{tr} is the $C_0(X)$-trivial representation of $F^P(\mathcal{G})$;*
5. *$C_0(X)$ detects ideals in one (and hence all) of $F^1(\mathcal{G})$, $F^\infty(\mathcal{G})$, $F_I(\mathcal{G})$.*

Proof. (1) and (2) are equivalent by Proposition 6.23. We have $\ker(\Lambda^P) = \ker(j_P)$ where $j_P : F^P(\mathcal{G}) \to C_0(\mathcal{G})$ is the j-map for $F^P(\mathcal{G})$, as we have the commutative diagram

$$\begin{array}{ccc} F^P(\mathcal{G}) & \xrightarrow{\quad\Lambda^P\quad} & F^P_{\mathrm{r}}(\mathcal{G}) \\ & {\scriptstyle j_P}\searrow \quad \swarrow{\scriptstyle j_P^{\mathrm{r}}} & \\ & C_0(\mathcal{G}) & \end{array},$$

and j_P^{r} is injective by Proposition 6.17. Hence (1) implies (3) by Theorem 6.31.

(3) implies (4) because $\ker(\Lambda_P^{\mathrm{tr}})$ is an ideal in $F^P(\mathcal{G})$ satisfying $\ker(\Lambda_P^{\mathrm{tr}}) \cap C_0(X) = \{0\}$. To show that (4) implies (1) assume that $\mathcal{G}$ is not topologically free. Thus there is a non-empty open bisection $U \subseteq \mathcal{G}\backslash X$ with $r|_U = d|_U$. Since $U \cap X = 0$, we get $\mathbb{E}_X(f) = 0$ for all $f \in C_0(U)$. Choose any non-zero $f \in C_c(U)$ and define $f_0 \in C_c(d(U)) \subseteq C_0(X)$ by $f_0(d(\gamma)) := f(\gamma)$ for $\gamma \in U$. Since $r|_U = d|_U$, both $\Lambda_P^{\mathrm{tr}}(f)$ and $\Lambda_P^{\mathrm{tr}}(f_0)$ act on each subspace

$\ell^p(X)$, $p \in P$, by pointwise multiplication with the same function f_0. Hence $f - f_0 \in \ker \Lambda^{\mathrm{tr}}_P$ and we have $\mathbb{E}^P_X(f - f_0) = -f_0 \neq 0$. Thus $f - f_0 \notin \ker(\Lambda^P)$ so (4) fails.

This proves that (1)–(4) are equivalent. These conditions are independent of the choice of P because (1) is. For a non-empty $Q \subseteq \{1, \infty\}$ we have $\ker(\Lambda_Q) = \{0\}$, by Remark 6.18. Hence condition (5) is equivalent to (3) for $P = Q$, because $F^Q(\mathcal{G}) = F^Q_{\mathrm{r}}(\mathcal{G})$. □

COROLLARY 6.34. *The following conditions are equivalent:*

1. *$\mathcal{G}$ is topologically free and minimal;*
2. *for every unital inverse subsemigroup $S \subseteq \mathrm{Bis}(G)$ that covers $\mathcal{G}$, every S-graded reduced Banach algebra $F_{\mathcal{R}}(\mathcal{G})$ is simple;*
3. *one of the algebras $F^1(\mathcal{G})$, $F^\infty(\mathcal{G})$, $F_I(\mathcal{G})$ is simple.*

Assume in addition that $\mathcal{G}$ is second countable and amenable, then for any non-empty set $P \subseteq [1, \infty]$ the above are further equivalent to:

4. *$F^P(\mathcal{G})$ is simple.*

PROOF. By Theorem 6.31, (1) implies (2). Implication (2)⇒(3) is obvious because the algebras $F^1(\mathcal{G})$, $F^\infty(\mathcal{G})$, $F_I(\mathcal{G})$ are reduced. If $\mathcal{G}$ is second countable and amenable, then $F^P(\mathcal{G}) = F^P_{\mathrm{r}}(\mathcal{G})$, by Remark 6.18, and hence these algebras are reduced. Thus also (2)⇒(4) in the amenable case. Since simplicity implies both the intersection property and minimality, see Lemmas 6.27 and 6.30, we get that either of (3) or (4) implies (1) by Theorem 6.33. □

6.3. L^P-operator algebras associated to Renault-Deaconu groupoids

We now illustrate our results on one of the most exploited groupoid by C^*-algebraists. We fix a locally compact Hausdorff space X and a surjective local homeomorphism $\varphi : X \to X$. Thus (X, φ) is a singly generated dynamical system (SGDS) in the sense of [**Ren00**]. Recall, see Example 1.24, that the *Renault-Deaconu groupoid* associated to (X, φ) is an étale, amenable, locally compact Hausdorff groupoid where

$$\mathcal{G}(X, \varphi) := \{(y, n - m, x) : n, m \in \mathbb{N}_0,\ x, y \in X,\ \varphi^n(x) = \varphi^m(y)\},$$

and the groupoid structure is given by

$$(z, n, y)(y, m, x) := (z, n + m, x), \qquad (y, n, x)^{-1} := (x, -n, y),$$

and the topology is inherited from $X \times \mathbb{Z} \times X$. The groupoid $\mathcal{G}(X, \varphi)$ is étale as it has a basis for the topology consisting of bisections of the form

$$Z(V, n - m, U) := \{(y, n - m, x) : (y, x) \in V \times U,\ \varphi^n(x) = \varphi^m(y)\}, \tag{6.6}$$

where $U, V \subseteq X$ are open sets such that $\varphi^n|_U$ and $\varphi^m|_V$ are injective and $\varphi^n(U) = \varphi^m(V)$. We identify the unit space $Z(X, 0, X) = \{(x, 0, x) \in X \times \mathbb{Z} \times X\}$ of $\mathcal{G}(X, \varphi)$ with X via the map $X \ni x \longmapsto (x, 0, x) \in Z(X, 0, X)$. Thus the range and domain maps are given by $r(y, n, x) = y$ and $d(y, n, x) = x$.

Proposition 6.35. *The collection $S(X,\varphi)$ of bisections* (6.6) *forms a wide inverse semigroup of bisections of $\mathcal{G}(X,\varphi)$ whose canonical action is given by the formula*

$$Z(V,n-m,U)\mapsto \theta_{Z(V,n-m,U)}=(\varphi^m|_V)^{-1}\circ\varphi^n|_U.$$

In particular, we have a natural isomorphism $S(X,\varphi)\ltimes X\cong\mathcal{G}(X,\varphi)$. If X is totally disconnected then the collection $S_c(X,\varphi)$ of bisections (6.6) *with compact open U, V is also a wide inverse semigroup, and so $S_c(X,\varphi)\ltimes X\cong\mathcal{G}(X,\varphi)$.*

Proof. Let $Z(V,N,U)$ and $Z(W,M,Y)$ be bisections of the form (6.6), so $N=n-m$, $M=k-l$ where $n,m,k,l\in\mathbb{N}_0$, $\varphi^m|_V$, $\varphi^n|_U$, $\varphi^l|_W$, $\varphi^k|_Y$ are injective and $\varphi^m(V)=\varphi^n(U)$, $\varphi^l(Y)=\varphi^k(W)$. Clearly, $Z(V,N,U)^{-1}=Z(U,-N,V)$ and $\theta_{Z(V,N,U)}=(\varphi^m|_V)^{-1}\circ\varphi^n|_U:U\to V$ is a homeomorphism. Moreover,

$$Z(V,N,U)\cdot Z(W,M,Y)=Z(V',N+M,Y'),$$

where $Y'=(\varphi|_Y^k)^{-1}\circ\varphi^l(U\cap W)$ and $V'=(\varphi|_V^m)^{-1}\circ\varphi^n(U\cap W)$. More specifically, if $n\geqslant l$ then φ^{k+n-l} is injective on Y' and $\theta_{Z(V',N+M,Y')}=(\varphi^m|_{V'})^{-1}\circ\varphi^{k+n-l}|_{Y'}$ maps Y' onto V'. If $n\leqslant l$, then φ^{m+l-n} is injective on V' and $\theta_{Z(V',N+M,Y')}=(\varphi^{m+l-n}|_{V'})^{-1}\circ\varphi^k|_{Y'}$ maps Y' onto V'. Thus $S(X,\varphi)$ is an inverse semigroup. It is wide because it forms a basis for the topology of $\mathcal{G}(X,\varphi)$. If the sets U, V, W, Y above are compact open, then also Y' and V' are compact open. Hence $S_c(X,\varphi)$ is an inverse semigroup. It forms a basis for the topology of $\mathcal{G}(X,\varphi)$ when X is totally disconnected. □

Remark 6.36. Let $S(\varphi)$ be the unital inverse subsemigroup of $\mathrm{PHomeo}(X)$ generated by restrictions $\varphi|_U\in\mathrm{PHomeo}(X)$ to open sets $U\subseteq X$ where $\varphi|_U$ is injective. Proposition 6.35 implies that $S(\varphi)$ consists of partial homeomeomorphisms of the form $(\varphi|_V^m)^{-1}\circ\varphi|_U^n$, and we have the canoncial semigroup epimorphism $h:S(X,\varphi)\twoheadrightarrow S(\varphi)$. A look at the germ relation in [**Ren00**] shows that the groupoid of germs $\mathrm{Germ}(X,\varphi)$ defined there is canonically isomorphic to $S(\varphi)$. So we have a canonical groupoid epimorphism $\mathcal{G}(X,\varphi)\cong S(X,\varphi)\ltimes X\twoheadrightarrow S(\varphi)\ltimes X\cong\mathrm{Germ}(X,\varphi)$. By [**Ren00**, Proposition 2.3] this is an isomorphism if and only if $\mathcal{G}(X,\varphi)$ is topologically free.

For each $U\in\mathrm{Bis}(\mathcal{G})$ denote δ_U the indicator function on $\mathcal{G}$ corresponding U. Then $C_c(\mathcal{G})=\mathrm{span}\{a\delta_U:U\in\mathrm{Bis}(\mathcal{G}),\ a\in C_c(r(U))\}$, cf. (5.5). In terms of $\mathcal{G}(X,\varphi)$ we have

$$C_c(\mathcal{G}(X,\varphi))=\mathrm{span}\{a\delta_{(V,n-m,U)}:V,U\subseteq X,\ \varphi^n|_U,\varphi^m|_V\text{- inj.},\ \varphi^m(V)=\varphi^n(U),\ a\in C_c(V)\}.$$

We now describe general representations in Banach algebras associated to such groupoids.

Theorem 6.37. *Let $\varphi:X\to X$ be a surjective local homeomorphism and let $S\subseteq S(X,\varphi)$ be any wide inverse subsemigroup where $S(X,\varphi)$ of bisections* (6.6). *We have an inverse semigroup action $\alpha:S\to\mathrm{PAut}(C_0(X))$ given by*

$$\alpha_{Z(V,n-m,U)}(a):=a\circ(\varphi^n|_U)^{-1}\circ\varphi^m|_V,\qquad a\in C_0(U).$$

and a natural isometric isomorphism

$$F^S(\mathcal{G}(X,\varphi))\cong C_0(X)\rtimes_\alpha S.$$

Thus every representation $\psi : F^S(\mathcal{G}(X,\varphi)) \to B$ is determined by the formula

$$\psi(a\delta_{Z(V,N,U)}) = \pi(a)v_{Z(V,N,U)}, \qquad a \in C_c(V), \tag{6.7}$$

for a covariant representation (π, v) of α. Moreover,

1. *for any $p \in (1,\infty)\backslash\{2\}$ and localizable measure μ,* (6.7) *establishes a bijective correspondence between non-degenerate representations $\psi : F^p(\mathcal{G}(X,\varphi)) \to B(L^p(\mu))$ and pairs (π, v) where $\pi : C_0(X) \to B(L^p(\mu))$ is a non-degenerate representation acting by multiplication operators, $v : S \to \mathrm{SPIso}(L^p(\mu))$ is a semigroup homomorphism satisfying*
$$v_{Z(V,n-m,U)}\pi(a) = \pi(a \circ (\varphi^m|_V)^{-1} \circ \varphi^n|_U)v_{Z(V,n-m,U)} \text{ for } a \in C_0(U),$$
and $v_{Z(U,0,U)}$ is a projection onto $\overline{\pi(C_0(U))L^p(\mu)}$ for any open $U \subseteq X$;
2. *for any $p \in [1,\infty]$ there is an isometric representation $\psi : F^p(\mathcal{G}(X,\varphi)) \to B(L^p(\mu))$, for some localizable measure μ, which is determined via* (6.7) *by a pair (π, v) described in item* (1).

If S consist of compact open sets, then (1) *is valid for all $p \in [1,\infty]\backslash\{2\}$.*

Proof. By Proposition 6.35 the prescribed action is the canonical action. Hence the first part of the assertion follows from Theorem 5.21. Statements (1), (2) follow from Theorem 5.38. □

Remark 6.38. By [**SW16**, Lemma 3.5], $\mathcal{G}(X,\varphi)$ is amenable groupoid. If we additionally assume that $\mathcal{G}(X,\varphi)$ is second countable which is equivalent to assume that X is metrizable then by Remark 6.18, we obtain that $F^P(\mathcal{G}(X,\varphi)) = F^P_{\mathrm{r}}(\mathcal{G}(X,\varphi))$ for any non-empty $P \subseteq [1,\infty]$.

We define topologicall freeness of a local homeomorphism in the same way is it defined for an action of a homeomorphism, cf. [**BKL24**, Definition 8.1].

Definition 6.39. A local homeomorphism $\varphi : X \to X$ is *topologically free* if the set of periodic $\{x : \varphi^n(x) = x \text{ for some } n > 0\}$ has empty interior in X.

Lemma 6.40. *The groupoid $\mathcal{G}(X,\varphi)$ is topologically free if and only if the local homemorphism $\varphi : X \to X$ is topologically free.*

Proof. Recall that $\mathcal{G}(X,\varphi)$ is topologically free if $\mathrm{Iso}(\mathcal{G}(X,\varphi))\backslash X$ has empty interior in $\mathcal{G}(X,\varphi)$. Note that

$$\mathrm{Iso}(\mathcal{G}(X,\varphi)) = \{(x, n-m, x) : n,m \in \mathbb{N}_0,\ x \in X,\ \varphi^n(x) = \varphi^m(y)\},$$

Hence $\mathcal{G}(X,\varphi)$ is topologically free if and only if the set

$$\bigcup_{\substack{k,l\in\mathbb{N}_0\\ l<k}} \{x \in X : \varphi^k(x) = \varphi^l(x)\}$$

has empty interior in X. This clearly implies topological freeness of φ. Conversely, if the displayed set has non-empty interior, then by the Bair category theorem there is a non-empty open set $U \subseteq \{x \in X : \varphi^k(x) = \varphi^l(x)\}$ for some $l < k$. Then $V := \varphi^l(U)$

is non-empty open set contained in $\{x \in X : \varphi^n(x) = x\}$ where $n := k - l \in \mathbb{N}$, which contradicts topological freeness of φ. □

THEOREM 6.41. *Let $\varphi : X \to X$ be a surjective local homeomorphism on a locally compact metric space and let $\mathcal{G}(X,\varphi)$ be the associated Renault-Deaconu groupoid. For any $\varnothing \neq P \subseteq [1,\infty]$ the following conditions are equivalent:*

(1) *φ is topologically free;*
(2) *$C_0(X) \subseteq F^P(\mathcal{G}(X,\varphi))$ is maximal abelian subalgebra;*
(3) *$C_0(X) \subseteq F^P(\mathcal{G}(X,\varphi))$ has the intersection property.*

PROOF. We have $F^P(\mathcal{G}(X,\varphi)) = F^P_{\mathrm{r}}(\mathcal{G}(X,\varphi))$ by Remark 6.38. Hence in view of Lemma 6.40 the assertion follows from Theorem 6.33. □

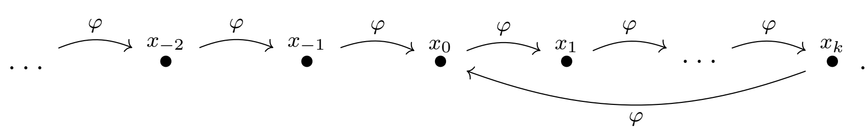
DEFINITION 6.42. Let $U \subseteq X$. We say U is *positively φ-invariant* if $\varphi(U) \subseteq U$, and that U is *φ-invariant* if $\varphi^{-1}(U) = U$. We say that φ is *minimal* if there are no non-trivial open invariant sets in X.

LEMMA 6.43. *A set $U \subseteq X$ is φ-invariant if and only if it is $\mathcal{G}(X,\varphi)$-invariant. In particular, φ is minimal if and only if $\mathcal{G}(X,\varphi)$ is minimal.*

PROOF. Clearly, $U = \varphi^{-1}(U)$ if and only if $f^{-m}f^n(U) \subseteq U$ for all $n, m \in \mathbb{N}_0$ if and only if U is $\mathcal{G}(X,\varphi)$-invariant. This shows the first part, and so the second part of the assertion follows. □

There exits minimal local homeomorphisms that are not topologically free, cf. [**BKL24**, Example 8.10]:

EXAMPLE 6.44 (Directed graphs with one circuit). We say that a surjective map $\varphi : X \to X$ is a *directed graph with one circuit*, i.e. X is a countable discrete set and there is a point $x \in X$ such that for any $y \in X$ there is $n \geqslant 1$ such that $\varphi^n(y) = x$. In particular, x is a periodic point and φ is a local homeomorphism. The name comes from the fact that graph of φ looks exactly like directed graphs considered in [**BJSS17**]. In particular, if we denote by $x_0, x_1, ..., x_k$ the orbit of x, then all points in X have to eventually land in this orbit, so the graph of φ looks like a one circuit and possibly a number of infinite tails pointed towards this circuit. For instance such a graph with one tail $\ldots, x_{-2}, x_{-1}$ looks us follows



For any such map $\varphi : X \to X$, φ is not topologically free but it is minimal, as for any $y \in X$ the smallest φ-invariant set containing y is $\bigcup_{n,m \in N_0} f^{-m}f^n(\{y\})$ and this has to be the whole of X. Also note that if X is compact (finite), then it can not have inifnite tails and so it is just a permutation of the finite set $x_0, x_1, ..., x_k$.

LEMMA 6.45. *Assume a surjective local homeomorphism $\varphi : X \to X$ is minimal. Then φ is topologically free if and only if it is not a directed graph with one circuit. If X is compact, then φ is topologically free if and only if X is infinite.*

Proof. Assume that φ is not topologically free. Then there is a non-empty open set $U \subseteq X$ for some $n \geqslant 1$ such that $\varphi^n|_U = id$ and $U, \varphi(U), ..., \varphi^{n-1}(U)$ are pairwise disjoint. For any disjoint open sets V_1, $V_2 \subseteq U$ the sets $U_i := \bigcup_{m\in\mathbb{N}, k=0,...,n} \varphi^{-m}(\varphi^k(V_i))$ are disjoint open and invariant. Thus minimality of φ forces $U = \{x\}$ to be a singleton. Hence $\{x\}$ is clopen and $\bigcup_{m\in\mathbb{N}, k=0,...,n} \varphi^{-m}(\varphi^k(\{x\})) = X$, which means that $\varphi : X \to X$ to be a directed graph with one circuit. This shows the first part of the assertion. Now assume that X is compact. Then φ is a directed graph with one circuit if and only if it is the finite circuit, which in view of minimality is equivalen to X being finite. ☐

Theorem 6.46. *Let $\varphi : X \to X$ be a surjective local homeomorphism on a locally compact metric space and let $\mathcal{G}(X, \varphi)$ be the associated Renault-Deaconu groupoid. Let $\varnothing \neq P \subseteq [1, \infty]$. Then $F^P(\mathcal{G}(X, \varphi))$ is simple if and only if φ is topologically free and not a directed graph with one circuit. If X is compact, then $F^P(\mathcal{G}(X, \varphi))$ is simple if and only if φ is minimal and X is infinite.*

Proof. We have $F^P(\mathcal{G}(X, \varphi)) = F^P_{\mathrm{r}}(\mathcal{G}(X, \varphi))$ by Remark 6.38. Hence in view of Lemma 6.40 the assertion follows from Corollary 6.34. ☐

6.4. Graph L^p-operator algebras

We will now relate the constructions and results from previous section to graph algebras.

Let $Q = (Q^0, Q^1, r_Q, s_Q)$ be a *directed graph* (sometimes also called a *quiver*). So Q^0 is the set of vertices, Q^1 is the set of edges and $r_Q, s_Q : Q^1 \to Q^0$ are the range and source maps. For the sake of simiplicty and as we want relate to graphs full maps on the space, rather then partial maps, we will assume that the graph is *regular* in the sense that

$$1 < |r_Q^{-1}(v)| < \infty \quad \text{for every } v \in Q^0.$$

Hence Q has no sources nor infinite receivers. We denote by Q^n, $n > 0$, the set of finite paths $\mu = \mu_1 \cdots \mu_n$, where $s_Q(\mu_i) = r_Q(\mu_{i+1})$ for all $i = 1, \dots, n-1$. Then $|\mu| = n$ stands for the length of μ and $Q^* = \bigcup_{n=0}^{\infty} Q^n$ is the set of all finite paths (vertices are treated as paths of length zero). We denote by Q^∞ the set of infinite paths and put $Q^{\leqslant\infty} := Q^* \cup Q^\infty$. The maps r_Q, s_Q extend naturally to Q^* and r_Q extends to Q^∞. We recall the definition of the graph L^p-operator algebra from [**CoR19**], [**CMR25**], [**BKM25**, Subsection 6.1], and [**BKM26**, Subsection 7.4]. It generalizes L^p-Cuntz algebras introduced and studied by Phillips [**Phi12**], [**Phi13b**], see also [**Gar21**].

Definition 6.47. Let E be an L^p-space for some $p \in [1, \infty]$. A *(Cuntz–Krieger) Q-family* in E is a pair (P, T) where $P = \{P_v\}_{v\in Q^0} \subseteq B(E)$ consists of pairwise orthogonal hermitian idempotents, and $T = \{T_e\}_{e\in Q^1} \subseteq B(E)$ consists of Moore-Penrose partial isometries with mutually orthogonal range projections that in addition satisfy

$$T_e^* T_e = P_{s_Q(e)} \quad \text{and} \quad P_v = \sum_{e\in r_Q^{-1}(v)} T_e T_e^*$$

for all $e \in Q^1$ and $v \in Q^0$. Here T_e^* denotes the (unique) Moore-Penrose generalized inverse of T_e. The *graph L^p-operator algebra $F^p(Q)$* is a Banacha algebra generated by p, t, t^*

where (p,t) is a universal Q-family on an L^p-space, where universality means that for any other Q-family (P,T) on an L^p-space E the maps $p_v \mapsto P_v$, $t_e \mapsto T_e$ and $t_e^* \mapsto T_e^*$, extend to a representation $F^p(Q) \to B(E)$.

Remark 6.48. When E is a Hilbert space (an L^2-space), then the above definition agrees with the usual definition of the graph C^*-algebra, see [**Rae05**]. If $E = L^p(\mu)$, for $p \neq 2$ and a localizable measure μ, then by Theorem 3.26 operators in a Q-family (P,T) are necessarily spatial partial isometries:

$$P \cup T \subseteq \mathrm{SPIso}(L^p(\mu))$$

In [**CoR19**] the algebra $F^p(Q)$ is defined using spatial partial isometries (and σ-finite measures) but it fives the same Banach algebra, cf the proof of Theorem 6.49 below.

The *boundary space* of the graph Q is the set of infinity paths

$$Q^\infty = \{\mu_1\mu_2\mu_3\cdots : \mu_i \in Q^1,\ s_Q(\mu_i) = r_Q(\mu_{i+1}) \text{ for all } i = 1,\dots\}$$

equipped with the topology generated by the 'cylinders' $Z(\mu) := \mu Q^{\leqslant\infty}$, $\mu \in Q^*$, and their relative complements. Then Q^∞ is a totally disocnnected, locally compact Hausdorff space. Namely, the sets $Z(\mu)\backslash\bigcup_{e\in F} Z(\mu e)$, where $\mu \in Q^*$ and $F \subseteq s(\mu)Q^1$ is a finite set of edges, form a basis of compact-open sets for Q^∞. Then the shift map

$$\sigma_Q(\mu_1\mu_2\mu_3\cdots) := \mu_2\mu_3\cdots$$

is a well defined proper local homeomorphism $\sigma_Q : Q^\infty \to Q^\infty$. By definition the *groupoid of the graph* Q is the Renault-Deaconu groupoid $\mathcal{G}_Q := \mathcal{G}(Q^\infty, \sigma_Q)$ of σ. Thus

$$\mathcal{G}_Q = \{(\mu x, |\mu| - |\eta|, \eta x) : \mu,\eta \in Q^*,\ x \in Q^\infty,\ s_Q(\mu) = s_Q(\nu) = r_Q(x)\}$$

is an ample Hausdorff groupoid with the topology generated by the 'cylinders' $Z(\mu,\eta) := \{(\mu x, |\mu| - |\eta|, \eta x) \in \mathcal{G}_Q\}$, for $(\mu,\nu) \in S_Q$, and their relative complements. Hence the sets $Z(\mu,\eta)\backslash\bigcup_{\alpha\in F} Z(\mu\alpha)$, where $(\mu,\nu) \in S_Q$ and $F \subseteq s(\mu)Q^*$ is finite, form a basis of compact open bisections for $\mathcal{G}_Q$. We will now use our result to improve one of the main results of [**CoR19**], [**CMR25**] - namely the *Cuntz-Krieger uniqueness for simple algebras* $F^p(Q)$.

Recall that Q is *cofinal* if for every $\mu = \mu_1\mu_2\cdots \in Q^\infty$ and every $v \in Q^0$ there is i such that $v \leqslant s(\mu_i)$. We say that a cycle $\mu = \mu_1\mu_2\cdots\mu_n$ has an *entry* if for some $i = 1,\dots,n$ there is an edge $e \in Q^1$ such that $r(e) = r(\mu_i)$ but $e \neq \mu_i$.

Theorem 6.49 (Cuntz-Krieger uniquneness theorem for simple L^p-operator graph algebras). *Let $Q = (Q^0, Q^1, r_Q, s_Q)$ be a regular directed graph and let $p \in [1,\infty]$. The maps $p_v \mapsto 1_{Z(v)}$, $t_e \mapsto 1_{Z(e,s_Q(e))}$ and $t_e^* \mapsto 1_{Z(e,s_Q(e))}$ extend to an isometric isomorphism*

$$F^p(Q) \cong F^p(\mathcal{G}_Q).$$

Moreover, the following conditions are equivalent:

1. *$F^p(Q)$ is simple;*
2. *σ_Q is minimal and not a directed graph with one circuit (Example 6.44);*
3. *Q is cofinal and the graph dual to Q is not a directed graph with one circuit;*
4. *every non-degenerate representation $F^p(Q) \to B(E)$ on an L^p-space E is automatically isometric.*

Proof. The first part is proved using our disintegration-integration results, see [**BKM25**, Corollary 6.25] and its proof for more details. Thus the equivalence (1)⇔(2) holds by Theorem 6.46. Equivalence (2)⇔(3) is straightforward. In particular, if φ is a directed graph with one circuit, then this graph is dual to Q. Now by [**CMR25**, Corollary 1.3], (1) implies (4). For the converse assume (1). By [**CMR25**, Theorem 1.1] it is equivalent to simplicity of the associated Leavitt path algebra $L(Q)$ and hence by [**CoR19**, Theorem 1.2], every Q-family (P,T) of spatial partial isometries $L^p(\mu)$ with σ-finite measure μ yields an isometrically isomorphic copy of $F^p(Q)$. We claim that this can be extended to every non-degenerate representation $\Phi : F^p(Q) \to B(E)$ on any L^p-space E. Indeed, since $F^p(Q)$ is separable, by [**Phi13b**, Proposition 1.25] there is an isometric non-degenerate representation $\Psi : \overline{\Phi(F^p(Q))} \to B(F)$ on a separable L^p-space F. By [**Lac74**, Corollary to Theorem 3 in Section 15], F is isometrically isomorphic to σ-finite L^p-space. Thus composing Φ with Ψ we may in fact assume that assume that $E = L^p(\mu)$ for a σ-finite measure μ. Then by (the last part of) Theorem 6.37, representation Φ is given by a spatial Q-family on $L^p(\mu)$. This proves the claim. □

CHAPTER 7

L^p-operator algebras associated to transfer operators and spectral properties of weighted composition operators

Transfer operators play a crucial role in thermodynamical formalism [**Rue89**], [**Wal82**], [**FJ01**], but they were also used, averaging operators, to study the properties of the Banach space $C(X)$, see [**Pel68**]. Transfer operators on general C^*-algebras and their crossed products where formally introduced by Ruy Exel [**Exe03$_1$**]. These constructions where refined and clarified in [**Kwa17**], see also [**BKL24**]. In this chapter we introduce L^p-operator algebra version of a crossed product for transfer operators of finite type, and generalize some of the main results of [**BK21**] from C^*-algebras to L^p-setting. Namely, we describe the spectrum of the associated universal weighted isometry, which gives a strong link with thermodynamical formalism, and discuss when it gives the spectrum of any of its representations.

7.1. Endomorphisms, transfer operators and local homeomorphisms

Throughout this chapter

$$X \text{ is a compact Hausdorff space.}$$

The following duality between unital endomorphisms of $C(X)$ and continuous maps $\varphi : X \to X$ is well known, cf. for instance [**KL20**, Proposition 1.1].

LEMMA 7.1. *Every unital endomorphism $\alpha : C(X) \to C(X)$ of the algebra $C(X)$ is a composition operator with a continuous map $\varphi : X \to X$, that is $\alpha(a)(x) = a(\varphi(x))$ for any $a \in C(X)$ and $x \in X$. In particular, α is necessarily $*$-preserving.*

PROOF. We may identify X with the set of unital algebra homomorphisms $C(X) \to \mathbb{C}$. Then X becomes a closed subset of the dual space $C(X)^*$ equipped with the *-weak topology. The adjoint operator $\alpha^* : C(X)^* \to C(X)^*$ restricts to a continuous map $\alpha^* : X \to X$ that we denote by φ. For any $a \in C(X)$ and $x \in X$ we have $\alpha(a)(x) = a(\varphi(x))$ by construction. □

The above lemma implies that we have one-to-one correspondence between unital endomorphisms $\alpha : C(X) \to C(X)$ and continuous maps $\varphi : X \to X$. Under this correspondence we also have

$$\begin{aligned} \alpha \text{ is a monomorphism} &\iff \varphi \text{ is surjective,}\\ \alpha \text{ is an epimorphism} &\iff \varphi \text{ is injective,}\\ \alpha \text{ is an automorphism} &\iff \varphi \text{ is a homeomorphism.} \end{aligned}$$

We extend the above correspondence by characterizing when $\varphi : X \to X$ is a surjective local homeomorphism. To this end, we will use the notion of a transfer operator.

Definition 7.2. Let $\alpha : C(X) \to C(X)$ be a unital endomorphism and let φ be the continuous map corresponding to α. A *transfer operator* for α (or for φ) is a positive linear operator $\mathcal{L} : C(X) \to C(X)$ satisfying

$$\mathcal{L}(\alpha(a)b) = a \cdot \mathcal{L}(b), \tag{7.1}$$

for all $a, b \in C(X)$.

We will be mostly interested in unital transfer operators. Note that if $\mathcal{L}(1) = 1$, then putting $b = 1$ in (7.1) we get that $\mathcal{L}(\alpha(a)) = a$, that is $\mathcal{L}$ is a left inverse to α. Therefore if α admits a unital transfer operator, then α has to be a monomorphism, equivalently φ is surjective. In general, it may happen that α does not admit non-zero transfer operators. This happens for instance when $X = [0, 1]$ and $\varphi : [0, 1] \to [0, 1]$ is the Cantor map (which graph is Devil's staircase), cf. [**Kwa12**]. However, if $\varphi : X \to X$ is a surjective local homeomorphism, then there is a canonical unital transfer operator given by the formula

$$\mathcal{L}(a)(y) = \frac{1}{|\varphi^{-1}(y)|} \sum_{x \in \varphi^{-1}(y)} a(x), \qquad a \in C(X),\ y \in X. \tag{7.2}$$

This operator is well defined by the following lemma.

Lemma 7.3. *If $\varphi : X \to X$ is a local homeomorphism, then the map $X \ni y \mapsto |\varphi^{-1}(y)| \in \mathbb{N}$ is locally constant.*

Proof. Let $y \in \varphi(X)$. Then $\varphi^{-1}(y) = \{x_1, ..., x_N\}$ is finite because X is compact. Take a neighbourhood U_i of each x_i, $i = 1, ..., N$, such that $\varphi|_{U_i} : U_i \to \varphi(U_i)$ is a homeomorphism. We may assume that $\{U_i\}_{i=1}^N$ are pairwise disjoint. Put $V := \varphi(U_1) \cap ... \cap \varphi(U_N)$ and $V_i := \varphi^{-1}(V) \cap U_i$, $i = 1, ..., N$. Then for each $i = 1, ..., N$ the set V_i is an open neighbourhood of x_i, $\varphi(V_i) = V$, and $V_i \cap V_j = \varnothing$ for $i \neq j$. Let $E := X \setminus \bigcup_{i=1}^N V_i$. The set $U := V \cap (X \setminus \varphi(E))$ is an open neighoburhood of y such that for any $y' \in U$ we have $|\varphi^{-1}(y')| = N$. Indeed, for each $y' \in U$ there are pairwise different $x_i' \in V_i$, $i = 1, ..., N$, with $\varphi(x_i') = y'$. Suppose that there exists $x' \notin \{x_1, ...x_N\}$ with $\varphi(x') = y'$. Then $x' \in E$ which contradicts with $y' \notin \varphi(E)$. ☐

The above lemma readily implies that if φ is a surjective local homeomorphism, then for any continuous function $\varrho : X \to [0, \infty)$ the formula

$$\mathcal{L}_{\varrho,\varphi}(a)(y) = \sum_{x \in \varphi^{-1}(y)} \varrho(x) a(x), \qquad a \in C(X),\ y \in X. \tag{7.3}$$

defines a transfer operator for φ. In fact any transfer operator for φ has this form.

Proposition 7.4. *Assume that $\varphi : X \to X$ is a surjective local homeomorphism. Then every transfer operator for φ is of the form $\mathcal{L}_{\varrho,\varphi}$ for a continuous function $\varrho : X \to [0, \infty)$.*

Moreover, $\mathcal{L}_\varrho$ is unital if and only if $\sum_{x \in \varphi^{-1}(y)} \varrho(x) = 1$ for each $y \in X$.

Proof. Assume that $\mathcal{L} : C(X) \to C(X)$ is a transfer operator for φ. For each $y \in X$ the map $\mu_y : C(X) \to \mathbb{C}$ given by $\mu_y(a) := \mathcal{L}(a)(y)$, $a \in C(X)$, is a positive functional on $C(X)$. By the Riesz-Markov-Kakutani representation theorem we may identify μ_y with some positive Borel measure on X which we also denote by μ_y. Note that for each $y \in X$ we have $\operatorname{supp}\mu_y \subseteq \varphi^{-1}(y)$. Indeed, suppose that there exists $x_0 \in \operatorname{supp}\mu_y$ such that $\varphi(x_0) \neq y$. There are open neighbourhoods U of $\varphi(x_0)$ and V of y such that $U \cap V = \varnothing$. Let $b \in C(X)$ be a non-zero positive function supported on $\varphi^{-1}(U)$ and $a \in C(X)$ be a positive function supported on V such that $a(y) = 1$. Then we have $\int_X a(\varphi(x))b(x)d\mu_y = 0$ and $a(y)\int_X b(x)d\mu_y \neq 0$ which contradicts with the relation $\mathcal{L}(\alpha(a)b) = a\mathcal{L}(b)$. Since X is compact and φ is a local homeomorphism, $\varphi^{-1}(y)$ is finite for any $y \in X$. We show that the map $X \ni x \mapsto \mu_{\varphi(x)}(x)$ is continuous. Let $x_0 \in X$ and $\varepsilon > 0$. Take an open neighbourhood V of x_0 such that $\varphi|_V : V \to \varphi(V)$ is a homeomorphism and open subset $V' \subset V$ containing x_0. Since X is normal, there exists $a \in C_0(V)$ such that $a(x) = 1$ for all $x \in V'$. Since $\mathcal{L}(a)$ is continuous, there is an open neighbourhood W of $\varphi(x_0)$ such that $|\mathcal{L}(a)(\varphi(x_0)) - \mathcal{L}(a)(y)| \leqslant \varepsilon$ for any $y \in W$. Then $U := V' \cap \varphi^{-1}(W)$ is an open neighbourhood of x_0 such that

$$|\mu_{\varphi(x_0)}(x_0) - \mu_{\varphi(x)}(x)| = |\mathcal{L}(a)(\varphi(x_0)) - \mathcal{L}(a)(\varphi(x))| < \varepsilon$$

for each $x \in U$. Thus, the map $X \ni x \mapsto \mu_{\varphi(x)}(x) \in [0, \infty)$ is continuous on X. For the function $\varrho(x) := \mu_{\varphi(x)}(x)$ the (7.3) holds. This is clear that $\mathcal{L}$ is unital if and only $\sum_{x \in \varphi^{-1}(y)} \varrho(x) = 1$ for each $y \in X$. □

Remark 7.5. As we have seen above, usually there are many different transfer operators for a given endomorphism $\alpha : C(X) \to C(X)$. However, the transfer operator usually determines the endomorphism. Indeed, a faithful positive linear map $\mathcal{L} : C(X) \to C(X)$ is a transfer operator for at most one endomorphism α, see [**Kwa17**, Proposition 4.18].

Consider inclusion of C^*-algebras $A \subseteq B$. Recall that a conditional expectation from A onto B is a contractive linear projection $E : B \to A$. By Tomiyama's Theorem, cf., for instance, [**Tak02**, III, Theorem 3.4, IV, Corollary 3.4], this is equivalent to saying that $E : A \to B$ is a (completely) positive linear map such that $E(ba) = E(b)a$ and $E(ab) = aE(b)$ for all $a \in A$, $b \in B$. The following simple lemma connects the notions of conditional expectation and transfer operator.

Lemma 7.6. *Let $\alpha : C(X) \to C(X)$ be a unital monomorphism. We have one-to-one correspondence between conditional expectations $E : C(X) \to \alpha(C(X))$ and unital transfer operators $\mathcal{L} : C(X) \to C(X)$ for α. It is given by relations $E = \alpha \circ \mathcal{L}$ and $\mathcal{L} = \alpha^{-1} \circ E$*

Proof. Assume that $E : C(X) \to \alpha(C(X))$ is a conditional expectation. Then $\mathcal{L} = \alpha \circ E$ is positive and unital as a composite of positive unital maps. Moreover, for all $a, b \in C(X)$ we have $\mathcal{L}(\alpha(a)b) = \alpha^{-1}(E(\alpha(a)b)) = \alpha^{-1}(\alpha(a)E(b)) = a\alpha^{-1}(E(b)) = a\mathcal{L}(b)$. Hence $\mathcal{L}$ is a transfer operator for α. Now assume that $\mathcal{L}$ is a unital transfer operator for α and put $E := \alpha \circ \mathcal{L}$. For any $a \in C(X)$ we get $E(\alpha(a)) = \alpha(\mathcal{L}(\alpha(a)1)) = \alpha(a\mathcal{L}(1)) = \alpha(a)$. Hence E is a projection onto $\alpha(C(X))$. Moreover, E is positive as a composite of two positive maps, and is a $\alpha(C(X))$-bimodule map, that is $E(\alpha(a)b) = \alpha(\mathcal{L}(\alpha(a)b) = \alpha(a\mathcal{L}(b)) = \alpha(a)E(b)$, for $a, b \in C(X)$. Hence by Tomiyama's Theorem, $E : C(X) \to \alpha(C(X))$ is a conditional expectation. $\square$

Now we use the above correspondence to describe special classes of endomorphisms and transfer operators. Recall, see [**Wat90**], that a conditional expectation $E : B \to A \subseteq B$ is said to be of *(index-)finite type* if there exists a *quasi-basis* for E, which is a pair of finite sets $\{u_1, ..., u_N\}, \{v_1, ..., v_N\} \subseteq A$ such that $a = \sum_{i=1}^N u_i E(v_i a)$, for all $a \in A$.

Definition 7.7 (cf. [**Exe03**$_2$, Definition 8.1]). Let $\alpha : C(X) \to C(X)$ be a unital monomorphism. We say that α is an *endomorphism of finite-type* if there is a conditional expectation of finite type $E : C(X) \to \alpha(C(X)) \subseteq C(X)$. Equivalently, in view of Lemma 7.6, α is of finite type if and only if there is a unital transfer operator $\mathcal{L} : C(X) \to C(X)$ and elements $\{u_1, ..., u_N\}, \{v_1, ..., v_N\} \subseteq C(X)$ such that

$$a = \sum_{i=1}^N u_i \alpha\big(\mathcal{L}(v_i a)\big), \qquad \text{for all } a \in C(X).$$

In this case we will also say that $\mathcal{L}$ is a *transfer operator of finite type.*

Proposition 7.8. *Let $\alpha : C(X) \to C(X)$ be a unital monomorphism and $\varphi : X \to X$ the associated continuous surjective map. Then α is of finite-type if and only if φ is a local homeomorphism. Moreover, a unital transfer operator $\mathcal{L} : C(X) \to C(X)$ for α is of finite type if and only if it is given by* (7.3) *where the cocycle $\varrho : X \to [0,1]$ attains strictly positive values. Then a relevant quasi-basis $\{u_n\}_{n=1}^N, \{v_n\}_{n=1}^N \subseteq C(X)$ can be defined by*

$$u_i(x) = v_i(x) = \sqrt{\frac{h_i(x)}{\varrho(x)}}, \qquad i = 1, ..., N \tag{7.4}$$

where $\{h_i\}_{i=1}^N \subseteq C(X)$ is a partition of unity subordinate to an open cover $\{U_i\}_{i=1}^N$ of X, with $\varphi : U_i \to \varphi(U_i)$ homeomorphism for each $i = 1, ..., N$.

Proof. Assume φ is a local homeomorphism. Putting $\varrho(x) := |\varphi^{-1}(x)|^{-1} > 0$ we get a continuous cocycle for φ. Moreover, for any continuous cocycle $\varrho : X \to (0,1]$, satisfying $\sum_{x \in \varphi^{-1}(y)} \varrho(x) = 1$, the transfer operator given by (7.3) is of finite type. Indeed, for any $a \in C(X)$ and $\{u_i\}_{i=1}^N = \{v_i\}_{i=1}^N \subseteq C(X)^+$ given by (7.4) we have

$$\Big[\sum_{i=1}^N u_i \alpha\big(\mathcal{L}(v_i a)\big)\Big](x) = \sum_{i=1}^N u_i(x) \sum_{\varphi(y)=\varphi(x)} u_i(y)\varrho(y)a(y)$$
$$= \sum_{i=1}^N u_i(x)^2 \varrho(x) a(x) = \sum_{i=1}^N h_i(x) a(x) = a(x).$$

Hence α is of finite type, cf. also [**EV06**, Proposition 8.2].

Now let $\mathcal{L}$ be any unital transfer operator for α which is of finite type. Fix a corresponding quasi-basis $\{u_i\}_{i=1}^N, \{v_i\}_{i=1}^N \subseteq C(X)$. Then $\mathcal{L}$ is given by (7.3) and we have

$$a(x) = \sum_{i=1}^N u_i(x) \int_{\varphi^{-1}(\varphi(x))} v_i(y) a(y) d\mu_{\varphi(x)}(y) \quad \text{for every } a \in C(X) \text{ and } x \in X. \tag{7.5}$$

This forces φ to be at most N-to-one map. Indeed, suppose on the contrary that there are $N+1$ different points $x_1, ..., x_{N+1} \subseteq \varphi^{-1}(\varphi(x))$ for a certain $x \in X$. Then putting $c_{j,i} := u_i(x_j)$ we get a $(N+1) \times N$ matrix $C = [c_{j,i}]_{j=1,i=1}^{N+1,N}$. Its range is at most N-dimensional. Hence there is a vector $y = [y_j]_{j=1}^{N+1}$ such that for every vector $\lambda = [\lambda_i]_{i=1}^{N}$ we have $C\lambda \neq y$. This contradicts (7.5) by taking $a \in C(X)$ such that $a(x_j) = y_j$ for all $j = 1, ..., N+1$, and putting $\lambda_i := \int_{\varphi^{-1}(\varphi(x))} v_i(y)a(y)d\mu_{\varphi(x)}(y)$, for $i = 1, ..., N$.

We define $\varrho(x) := \mu_{\varphi(x)}(\{x\}) \leqslant 0$, $x \in X$. Since φ is at most N-to-one, (7.5) gives

$$a(x) = \sum_{\varphi(x)=\varphi(y)} a(y)\varrho(y) \sum_{i=1}^{N} u_i(x)v_i(y) \quad \text{for every } a \in C(X) \text{ and } x \in X.$$

Plugging into this equation $a \in C(X)$ such that $a(x) = 1$ and $a(\varphi^{-1}(\varphi(x))\setminus\{x\}) = 0$ we get $1 = \varrho(x)\sum_{i=1}^{N} u_i(x)v_i(x)$, for every $x \in X$. This implies that ϱ is strictly positive and continuous. In particular, $\operatorname{supp} \mu_y = \varphi^{-1}(y)$ for every $y \in X$, and therefore φ is an open map, see, for instance, [**Kwa17**, Lemma 3.28]. Thus it suffices to show that every point $x \in X$ has an open neighborhood on which φ is injective. Note that $X \in y \mapsto \mathcal{L}(1/\varrho)(y) = |\varphi^{-1}(y)|$ is continuous. Hence there is a neighborhood V of $\varphi(x)$ such that each point in V has exactly $M \leqslant N$ elements in the pre-image. Let $\varphi^{-1}(\varphi(x)) = \{x_1, ..., x_M\}$ and take disjoint open sets $\{U_i\}_{i=1}^{N}$ such that $x_i \in U_i \subseteq \varphi^{-1}(V)$, $i = 1, ..., M$. Putting $V' := \bigcap_{i=1}^{M} \varphi(U_i)$ and $U_i' := U_i \cap \varphi^{-1}(V')$, for $i = 1, ..., M$, we get that φ restricted to each U_i' is injective. Indeed, for every $x' \in U_i'$, the M-element set $\varphi^{-1}(\varphi(x'))$ intersects each of M disjoint sets U_j', $j = 1, ..., M$. Accordingly, x belongs to some U_i', and hence φ is locally injective. □

COROLLARY 7.9. *If $\mathcal{L} : C(X) \to C(X)$ is a transfer operator of finite type, then it is faithful, there is a unique endomorphism $\alpha : C(X) \to C(X)$ such that $\mathcal{L}$ is a transfer operator for α and the map $\varphi : X \to X$ corresponding to α is a surjective local homeomorphism.*

PROOF. Combine Proposition 7.8 and Remark 7.5. □

If $\mathcal{L}$ is a transfer operator α, then for each $n \in \mathbb{N}$, the n-th iterate $\mathcal{L}^n$ is a transfer operator for α^n. Moreover, if $\mathcal{L} = \mathcal{L}_{\varrho,\varphi}$ is given by (7.3), then $\mathcal{L}^n = \mathcal{L}_{\varrho_n,\varphi^n}$. Namely,

$$\mathcal{L}^n(a)(y) = \sum_{x\in\varphi^{-n}(y)} \varrho_n(x)a(x), \qquad a \in C(X),\ y \in X,$$

where

$$\varrho_n(x) := \varrho(x)\varrho(\varphi(x)) \dots \varrho(\varphi^{n-1}(x)). \tag{7.6}$$

Note that we have the 1-cocycle identity $\varrho_{n+m} = \varrho_n \cdot \alpha^n(\varrho_m)$ for $n, m \in \mathbb{N}_0$ ($\varrho_0 \equiv 1$). Therefore ϱ is called a cocycle,as it generates this semigroup 1-cocycle $\{\varrho_n\}_{n\in\mathbb{N}_0}$.

7.2. Covariant representations of transfer operators

From now we fix a transfer operator $\mathcal{L} : C(X) \to C(X)$ of a finite type for a local homeomemorphism $\varphi : X \to X$ and we let $\alpha : C(X) \to C(X)$ be the associated endomorphism.

Definition 7.10. A *representation* of the transfer operator $\mathcal{L}$ in a unital Banach algebra B is a triple (π, T, T^*) where $\pi : C(X) \to B$ is a unital representation and $T, T^* \in B_1$ are contractive operators such that for every $a \in C(X)$ we have

(TR1) $\pi(\mathcal{L}(a)) = T^*\pi(a)T$,

(TR2) $T\pi(a) = \pi(\alpha(a))T$.

By the *range* of (π, T, T^*) we mean the Banach subalgebra $B(\pi, T, T^*)$ of B generated by $\pi(C(X)) \cup \{T, T^*\}$. We say that a (π, T, T^*) is *injective*, etc. if π is. If $B = B(E)$ for a Banach space E we speak of covariant representations on E.

Remark 7.11. Since we assume that $\mathcal{L}$ is unital and T, T^* are contractive, (TR1) implies that T is an isometry and T^* is its left inverse – "coisometry". In other words, the pair (T, T^*) is formed by mutually inverse partial isometries in the sense of Mbekhta. If B is a C^*-algebra, then T^* is necessarily the hermitian adjoint of T and relation (TR1) implies (TR2), as putting $c := T\pi(a)$ and $d := \pi(\alpha(a))T$ and using the transfer identity (7.1) one sees, that each of the expressions c^*d, d^*d, c^*c, d^*c is equal to $\pi\big(L(\alpha(a^*a))\big)$. Thus using the C^*-equality we get

$$\|T\pi(a) - \pi(\alpha(a))T\|^2 = \|\big(c^* - d^*\big)\big(c - d\big)\| = \|c^*d + d^*d + c^*c - d^*c\| = 0.$$

However, when B is not a C^*-algebra, T and T^* will usually not be Moore-Penrose partial isometries, and it does not seem that (TR2) is automatic then.

Lemma 7.12. *If (π, T, T^*) is a representation of $\mathcal{L}$, then for every $a \in C(X)$ we have*

(TR3) $\pi(a)T^* = T^*\pi(\alpha(a))$.

Moreover, for $n, m, k, l \in \mathbb{N}_0$ and $a, b, c, d \in C(X)$ we have

$$(7.7) \quad [\pi(a)T^mT^{*n}\pi(b)] \cdot [\pi(c)T^lT^{*k}\pi(d)] = \begin{cases} \pi(a)T^mT^{*n-l+k}\pi\left(\alpha^k(\mathcal{L}^l(bc))d\right) & n \geqslant l, \\ \pi\left(a\alpha^m(\mathcal{L}^n(bc))\right)T^{l-n+m}T^{*k}\pi(d) & n < l. \end{cases}$$

Therefore the range of (π, T, T^*) is

$$B(\pi, T, T^*) = \overline{\operatorname{span}}\{\pi(a)T^nT^{*m}\pi(b) : a, b \in C(X), n, m \in \mathbb{N}_0\}.$$

Proof. Using $T^*T = 1$, (TR1) and $\mathcal{L} \circ \alpha = \mathrm{id}_{C(X)}$, for any $a \in C(X)$ we get

$$\pi(a)T^* = T^*T\pi(a)T^* = T^*\pi(\alpha(a))TT^* = \pi(\mathcal{L}(\alpha(a)))T^* = \pi(a)T^*.$$

This proves (TR3). Note that for for each $n \in \mathbb{N}$, (π, T^n, T^{*m}) is a representation of $\mathcal{L}^n$. Using this and relations (TR1), (TR2), (TR3) one readily sees the displayed product formula. This implies that the products of the form $\pi(a)T^nT^{*m}\pi(b)$ form a semigroup, and their closed linear span is the Banach algebra $B(\pi, T, T^*)$. $\square$

REMARK 7.13. Let $a \in C_0(U) \subseteq C(X)$ where $\varphi|_U$ is injective, then $a \circ \varphi|_U^{-1} \in C_0(\varphi(U)) \subseteq C(X)$. Morever, if $a, b \in C_0(U)$, then $a \cdot b = \alpha(a \circ \varphi|_U^{-1})b$ and in view of (TR2) and (TR3) for any representation (π, T, T^*) of $\mathcal{L}$ we get

$$\pi(ab)T = \pi(b)T\pi(a \circ \varphi|_U^{-1}), \qquad T^*\pi(ab) = \pi(a \circ \varphi|_U^{-1})T^*\pi(b).$$

This means that (TR2) and (TR3) have their symmetrised versions but only locally and in the presence of a witness.

We will need to assume additional covariance condition on representation of $\mathcal{L}$, which can be phrased in a number of ways:

PROPOSITION 7.14. *Let (π, T, T^*) be a representation of $\mathcal{L}$ in a Banach algebra B and let $\varphi : X \to X$ be the associated local homeomorphism. The following are equivalent:*

(1) $\pi(C(X)) \subseteq \overline{\pi(C(X))TT^*\pi(C(X))}$.
(2) $\pi(a)TT^*\pi(b) = \pi(\varrho ab)$ *for* $a, b \in C(X)$ *supported on a set where* φ *is injective.*
(3) $\sum_{i=1}^n \pi(u_i)TT^*\pi(v_i) = 1$ *for every quasi-basis* $\{u_i\}_{i=1}^N$, $\{v_i\}_{i=1}^N \subseteq C(X)$.
(4) $\sum_{i=1}^n \pi(u_i)TT^*\pi(v_i) = 1$ *for some quasi-basis* $\{u_i\}_{i=1}^N$, $\{v_i\}_{i=1}^N \subseteq C(X)$.

PROOF. (1)⇒(2),(3). Assumption (1) implies $1 \in \overline{\pi(C(X))TT^*\pi(C(X))}$. Therefore $\pi(a)$, $a \in C(X)$ is determined by its action on $\pi(C(X))T$ in the sense that if $\pi(a)\pi(c)T = \pi(b)\pi(c)T$ for all $c \in C(X)$, then $\pi(a) = \pi(b)$. Let $a, b \in C(X)$ be supported on a set where φ is injective. Then for $c \in C(X)$ and every $x \in X$ we have

$$a(x)\alpha(\mathcal{L}(bc))(x) = a(x) \sum_{t \in \varphi^{-1}(\varphi(x))} \varrho(t)b(t)c(t) = \varrho(x)a(x)b(x)c(x).$$

Hence $\Big(\pi(a)TT^*\pi(b)\Big)\pi(c)T = \pi(a\alpha(\mathcal{L}(bc)))T = \pi(\varrho ab)\pi(c)T$. Thus $\pi(a)TT^*\pi(b) = \pi(\varrho ab)$. This proves (2). Now let $\{u_i\}_{i=1}^N$, $\{v_i\}_{i=1}^N \subseteq C(X)$ be a quasi-basis for $\mathcal{L}$ and let $a \in C(X)$. We have $\sum_{i=1}^n u_i\alpha(\mathcal{L}(v_ia)) = a$. Thus

$$\sum_{i=1}^n \pi(u_i)TT^*\pi(v_i)\pi(a)T = \pi\Big(\sum_{i=1}^n u_i\alpha(\mathcal{L}(v_ia))\Big)T = \pi(a)T.$$

This implies that $\sum_{i=1}^n \pi(u_i)TT^*\pi(v_i) = 1$, and so also (3).

(2)⇒(1). Let $u = v = \{u_i\}_{i=1}^n$ be the quasi-basis given by (7.4). Then for every $a \in C(X)$ we have

$$a = \sum_{i=1}^n \varrho u_i a u_i = \sum_{i=1}^n u_i a TT^* u_i \in \overline{\pi(C(X))TT^*\pi(C(X))}.$$

Hence conditions (1)-(2) are equivalent. Implications (3)⇒(4)⇒(1) are straightforward. Thus all conditions are equivalent. □

DEFINITION 7.15. We say that a representation (π, T, T^*) of $\mathcal{L}$ in a unital L^p-operator algebra, $p \in [1, \infty]$, is *covariant* if it satisfies the equivalent conditions in Proposition 7.14 and in addition for any open set $U \subseteq X$ such that $\varphi|_U$ is injective and any $h \in C_c(U)$ with $\|h\| = 1$ the operators $\pi(\varrho^{-\frac{1}{p}}h)T$, $T^*\pi(\varrho^{-\frac{1}{q}}h)$, where $\frac{1}{p} + \frac{1}{q} = 1$, are contractive.

REMARK 7.16. For $p = 2$ the last condition in the above definition is automatic. Namely, if (π, T, T^*) is a representation of $\mathcal{L}$ in a unital L^2-operator algebra B, then we may embed B as a unital subalgebra of a C^*-algebra C and so T^* is hermittian adjoint of T, cf. Remark 7.11. Moreover, for for any open set $U \subseteq X$ such that $\varphi|_U$ is injective and any $h \in C_c(U)$ with $\|h\| = 1$ using (TR1) we get

$$\|\pi(\varrho^{-\frac{1}{2}}h)T\|^2 = \|T^*\pi(\varrho^{-\frac{1}{2}}h)\pi(\varrho^{-\frac{1}{2}}h)T\| = \|h \circ \varphi|_U^{-1}\| = 1$$

and $\|T^*\pi(\varrho^{-\frac{1}{2}}h)\| = \|\pi(\varrho^{-\frac{1}{2}}h)T\|$.

Any transfer operator of finite-type admits a natural injective non-degenerate covariant representation on an L^p-space for any $p \in [1, \infty]$.

LEMMA 7.17. *Let $\mathcal{L}$ be a transfer operator of finite type for a local homeomorphism $\varphi : X \to X$ and let $p, q \in [1, \infty]$ with $\frac{1}{p} + \frac{1}{q} = 1$. We have an injective non-degenerate covariant representation (π, T^*, T) of $\mathcal{L}$ on $\ell^p(X)$, given by the formulas*

$$\pi(a)\xi(y) := a(y)\xi(y), \quad T\xi(y) := \varrho(y)^{\frac{1}{p}}\xi(\varphi(y)), \quad T^*\xi(y) := \sum_{x \in \varphi^{-1}(y)} \varrho(x)^{\frac{1}{q}}\xi(x).$$

for all $a \in C(X)$, $\xi \in \ell^p(X)$, $y \in X$.

PROOF. It is clear that π is an isometric unital representation of $C(X)$ on $\ell^p(X)$. For $\xi \in \ell^p(X)$ and $p < \infty$ we have

$$\|T\xi\|_p^p = \sum_{x \in X} \varrho(x)|\xi(\varphi(x))|^p = \sum_{y \in X} \sum_{x \in \varphi^{-1}(x)} \varrho(x)|\xi(y)|^p = \sum_{y \in X} |\xi(y)|^p = \|\xi\|_p^p.$$

Hence T is a well defined isometry. Also applying triangle and Hölder's inequalities we get

$$\|T^*\xi\|_p^p = \sum_{y \in X} | \sum_{x \in \varphi^{-1}(y)} \varrho(x)^{\frac{1}{q}}\xi(x)|^p \leqslant \sum_{y \in X} (\sum_{x \in \varphi^{-1}(y)} \varrho(x))^{\frac{1}{q}} (\sum_{x \in \varphi^{-1}(y)} |\xi(x)|^p) = \|\xi\|_p^p.$$

Hence T^* is a well defined contraction. Checking that T and T^* are contractions for $p = \infty$ is even easier and we leave it to the reader. Moreover, for any $a \in C(X)$, $\xi \in \ell^p(X)$, $y \in X$

$$\begin{aligned}(T^*\pi(a)T\xi)(y) &= \sum_{x \in \varphi^{-1}(y)} \varrho(x)^{\frac{1}{q}} a(x) \varrho(x)^{\frac{1}{p}} \xi(\varphi(x)) = \Big(\sum_{x \in \varphi^{-1}(y)} \varrho(x)a(x) \Big) \xi(y) \\ &= \pi(\mathcal{L}(a))\xi(y)\end{aligned}$$

and

$$(T\pi(a)\xi)(y) = \varrho(y)^{\frac{1}{p}} a(\varphi(y))\xi(\varphi(y)) = (\pi(\alpha(a))T\xi)(y).$$

Thus (TR1) and (TR2) hold. Let $U \subseteq X$ be an open set such that $\varphi|_U$ is injective. It is readily seen that for any $a, b \in C_c(U)$ we have $\pi(a)TT^*\pi(b) = \pi(\varrho ab)$, and if $h \in C_c(U)$ with $\|h\| = 1$ then $\pi(\varrho^{-\frac{1}{p}}h)T = \pi(h)T_{\varphi|_U}$ and $T^*\pi(h\varrho^{-\frac{1}{q}}h) = \pi(h \circ \varphi|_U^{-1})T_{\varphi|_U^{-1}}$ are contractive weighted compostion operators. Hence (π, T^*, T) is a covariant representation of $\mathcal{L}$. $\square$

We have another, slightly more sophisticated and more important for us, covariant representation of the operator $\mathcal{L}$, which is related with the regular representation of the Deacon-Renault groupoid $\mathcal{G}(X,\varphi)$, cf. Proposition 5.24 and subsection 6.3. In fact, to large extend it can be seen purely algebraically. Indeed, note that the set $\Gamma(\varphi) := \{(x,-1,\varphi(x)) : x \in X\}$, which can be identified with "the graph of φ", is a compact open subset of $\mathcal{G}(X,\varphi)$. Hence the characteristic function $\mathbb{1}_{\Gamma(\varphi)}$ is an element of $C_c(\mathcal{G}(X,\varphi))$. The function $\mathbb{1}_{\Gamma(\varphi)}$ and its adjoint $\mathbb{1}_{\Gamma(\varphi)}^*$ in $C_c(\mathcal{G}(X,\varphi))$ act by convolution on functions $\xi : \mathcal{G}(X,\varphi) \to \mathbb{C}$ as follows:

$$(7.8)\quad \mathbb{1}_{\Gamma(\varphi)} * \xi(x,N,z) = \xi(\varphi(x),N+1,z)), \quad \mathbb{1}_{\Gamma(\varphi)}^* * \xi(y,N,z) = \sum_{x\in\varphi^{-1}(y)} \xi(x,N-1,z)).$$

For each $p \in [1,\infty]$ we put

$$\mathbb{T}_p := \varrho^{\frac{1}{p}} * \mathbb{1}_{\Gamma(\varphi)} \in C_c(\mathcal{G}(X,\varphi)).$$

Lemma 7.18. *Retain the above notation and let $p,q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$. For any $a \in C(X)$ and $b,c \in C_0(U)$ where $U \subseteq X$ is open subset such that $\varphi|_U$ is injective, we have*

$$(7.9)\qquad \mathbb{T}_q^* * a * \mathbb{T}_p = \mathcal{L}(a) \qquad \mathbb{T}_p * a = \alpha(a) * \mathbb{T}_p \qquad c * \mathbb{T}_p * \mathbb{T}_q^* * b = \varrho c b.$$

Moreover, $C_c(\mathcal{G}(X,\varphi))$ is generated as an algebra by $C(X)$, $\mathbb{T}_p$ and $\mathbb{T}_q^*$.

Proof. One directly and readily checks that formulas (7.9) hold. Recall that $C_c(\mathcal{G}(X,\varphi))$ is spanned by continuous compactly supported functions living on bisections $Z(V,m-n,U)$ given by (6.6), where $V,U \subseteq X$ are open, $\varphi^m|_U, \varphi^n|_V$ are injective and $\varphi^n(V)=\varphi^m(U)$. Moreover, every such function is of the form

$$a\delta_{Z(V,m-n,U)}(x,N,y) = \begin{cases} a(x), & (x,N,y) \in Z(V,m-n,U),\\ 0, & \text{otherwise,}\end{cases}$$

where $a \in C_c(V)$. One readily checks that letting $\Gamma(\varphi^n) := \{(x,-n,\varphi^n(x)) : x \in X\}$ we have

$$a\delta_{Z(V,m-n,U)} = a\mathbb{1}_{\Gamma(\varphi^n)} * \mathbb{1}_{\Gamma(\varphi^m)}^* = \mathbb{1}_{\Gamma(\varphi^n)} * \mathbb{1}_{\Gamma(\varphi^m)}^* * a \circ \varphi^n|_V^{-1} \circ \varphi|_U^m$$

Moreover, $\mathbb{1}_{\Gamma(\varphi)}^n = \mathbb{1}_{\Gamma(\varphi^n)}$ and $\mathbb{T}_p^n = (\varrho_n)^{\frac{1}{p}}\mathbb{1}_{\Gamma(\varphi^n)}$ where ϱ_n is given by the formula (7.6). Therefore

$$a\delta_{Z(V,m-n,U)} = a\varrho_n^{-\frac{1}{p}} * \mathbb{T}_p^n * \mathbb{T}_q^{*m} * \varrho_m^{-\frac{1}{q}}.$$

This implies that $C_c(\mathcal{G}(X,\varphi))$ is spanned by elements of the form $a * \mathbb{T}_p^n * \mathbb{T}_q^{*m} * b$ for $n,m \in \mathbb{N}_0$ and $a,b \in C(X)$. Hence $C_c(\mathcal{G}(X,\varphi))$ is generated as an algebra by $C(X)$, $\mathbb{T}_p$ and $\mathbb{T}_q^*$. □

Proposition 7.19. *Let $\mathcal{L}$ be a transfer operator of finite type for a local homeomorphism $\varphi : X \to X$ and let $p,q \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{q}=1$. Consider the regular representation $\Lambda_p : F^p(\mathcal{G}(X,\varphi)) \to B(\ell^p(\mathcal{G}(X,\varphi)))$ of the Deacon-Renault groupoid $\mathcal{G}(X,\varphi)$ associated to φ. Putting*

$$\pi := \Lambda_p|_{C(X)}, \qquad T := \Lambda_p(\mathbb{T}_p), \qquad T^* := \Lambda_p(\mathbb{T}_q^*),$$

we get an injective non-degenerate covariant representation (π, T, T^*) *of* L *on* $\ell^p(\mathcal{G}(X,\varphi))$ *such that* $B(\pi, T, T^*) = \overline{\Lambda_p(F^p(\mathcal{G}(X,\varphi)))} \cong F^p_{\mathrm{r}}(\mathcal{G}(X,\varphi))$.

PROOF. It is clear that π is an isometric unital representation of $C(X)$ on $\ell^p(\mathcal{G}(X,\varphi))$. Let $\xi \in \ell^p(\mathcal{G}(X,\varphi))$. If $p < \infty$, then (in view of (7.8))

$$\begin{aligned}\|T\xi\|_p^p &= \sum_{(x,N,z)} \varrho(x)|\xi(\varphi(x), N+1, z))|^p = \sum_{(y,N+1,z)} \sum_{x\in\varphi^{-1}(y)} \varrho(x)|\xi(y, N+1, z))|^p \\ &= \sum_{(y,N+1,z)} |\xi(y, N+1, z))|^p = \|\xi\|_p.\end{aligned}$$

Also applying triangle and Hölder's inequalities (and (7.8)) we get

$$\begin{aligned}\|T^*\xi\|_p^p &= \sum_{(x,N,z)} |\sum_{y\in\varphi^{-1}(x)} \varrho(y)^{\frac{1}{q}}\xi(y, N-1, z))|^p \\ &\leqslant \sum_{(x,N,z)} \sum_{y\in\varphi^{-1}(x)} |\xi(y, N-1, z))|^p = \sum_{(y,N-1,z)} |\xi(y, N-1, z))|^p = \|\xi\|_p.\end{aligned}$$

If $p = \infty$, then

$$\|T\xi\|_\infty = \sup_{(x,N,z)} \sum_{\substack{N=n+m\\ y\in X}} |\mathbb{1}_{\Gamma(\varphi)}(x,n,y)\xi(y,m,z))| = \sup_{(x,N,z)} |\xi(\varphi(x), N+1, z))| = \|\xi\|_\infty$$

and

$$\|T^*\xi\|_\infty = \sup_{(x,N,z)} |\sum_{y\in\varphi^{-1}(x)} \varrho(x)\xi(y, N-1, z))| \leqslant \sup_{(y,N-1,z)} |\varrho(x)\xi(y, N-1, z))| = \|\xi\|_\infty.$$

Hence T and is an isometry and T^* is a contraction. In view of (7.9) and the above estimates it follows that (π, T, T^*) is a covariant representation of L on $\ell^p(\mathcal{G}(X,\varphi))$.

Since $\Lambda_p(C_c(\mathcal{G}(X,\varphi)))$ is dense in $\overline{\Lambda_p(F^p(\mathcal{G}(X,\varphi)))}$, the last part of Lemma 7.18 implies that the algebra generated by $\pi(C(X))$, T and T^* is dense in $\overline{\Lambda_p(F^p(\mathcal{G}(X,\varphi)))}$. Hence we have $B(\pi, T, T^*) = \overline{\Lambda_p(F^p(\mathcal{G}(X,\varphi)))}$. □

Recall that if X is metrizable, then $F^p(\mathcal{G}(X,\varphi)) = F^p_{\mathrm{r}}(\mathcal{G}(X,\varphi))$, see Remark 6.38, and so the above proposition says that the L^p-operator algebra $F^p(\mathcal{G}(X,\varphi))$ is generated by a unital injective covariant representation of a transfer operator for φ. Now we will show that this covariant representation is universal:

THEOREM 7.20. *Let* $\mathcal{L}$ *be a finite type transfer operator for* φ *and let* $\mathcal{G}(X,\varphi)$ *be the associated Renault-Deaconu groupoid. For each* $p \in [1,\infty]$ *and for every unital covariant representation* (π, T, T^*) *in an* L^p*-operator algebra* B *there is a representation* $\pi \rtimes T : F^p(\mathcal{G}(X,\varphi)) \to B$ *uniquely determined by*

$$\pi \rtimes T(a) = \pi(a), \qquad \pi \rtimes T(\mathbb{T}_p) = T, \qquad \pi \rtimes T(\mathbb{T}_q^*) = T^*$$

where $a \in C(X)$ *and* $1/p + 1/q = 1$.

Proof. To show that $\pi \rtimes T$ exists we will use Theorem 6.37. To this end we need to construct from the covariant representation (π, T, T^*) of $\mathcal{L}$ a representation (π, v) of the inverse semigroup action of $S(X, \varphi)$ described there. To this end we may assume that B is a unital subalgebra of $B(L^p(\mu))$ where μ is a localizable measure. If $p \neq 2$, then π acts by multiplication operator by Theorem 3.11. By Lemma 3.14, we may assume the same for $p = 2$. Then π extends to the representation $\overline{\pi} : \mathcal{B}(X) \to B(L^p(\mu))$ of the algebra of Borel bounded functions. Note that also α and $\mathcal{L}$ extend naturally (by the same formulas) to maps on $\mathcal{B}(X)$ and then $(\overline{\pi}, T, T^*)$ might be treated as the representation of $\mathcal{L} : \mathcal{B}(X) \to B(L^p(\mu))$, so in particular relations (TR1)-(TR3) hold for Borel functions.

Since the representation (π, T, T^*) of $\mathcal{L}^n$ is covariant for any non-empty open set $U \subseteq X$ where $\varphi|_U$ is injective the operators $v_U := \overline{\pi}(\varrho^{-\frac{1}{p}} 1\!\!1_U) T$ and $v_U^* := T^* \overline{\pi}(\varrho^{-\frac{1}{q}} 1\!\!1_U)$ are contractive and $v_U v_U^* = \overline{\pi}(1\!\!1_U)$ and $v_U^* v_U = \overline{\pi}(1\!\!1_{\varphi(U)})$, by (TR1). Therefore v_U and v_U^* are mutually conjugate spatial partial isometries, cf. Theorem 3.26. Moreover, since (π, T^n, T^{*n}) is a covariant representation of $\mathcal{L}$, for any non-empty open set $U \subseteq X$ such that $\varphi^n|_U$ is injective we get

$$v_U^n = \overline{\pi}(\varrho_n^{-\frac{1}{p}} 1\!\!1_U) T^n \qquad \text{and} \qquad v_U^{*n} = T^{*n} \overline{\pi}(\varrho_n^{-\frac{1}{q}} 1\!\!1_U).$$

Let $Z(V, m-n, U) \in S(X, \varphi)$, so that $\varphi^m|_U, \varphi^n|_V$ are injective and $\varphi^n(V) = \varphi^m(U)$, and put

$$v_{Z(V,m-n,U)} := v_V^n v_U^{*m} = \overline{\pi}(\varrho_n^{-\frac{1}{p}} 1\!\!1_V) T^n T^{*m} \overline{\pi}(\varrho_m^{-\frac{1}{q}} 1\!\!1_U).$$

Note that this is well defined, as if in addition $\varphi^{m+1}|_U, \varphi^{n+1}|_V$ are injective, then

$$\overline{\pi}(\varrho_{n+1}^{-\frac{1}{p}} 1\!\!1_V) T^{n+1} T^{*m+1} \overline{\pi}(\varrho_{m+1}^{-\frac{1}{q}} 1\!\!1_U) = \overline{\pi}(\varrho_n^{-\frac{1}{p}} 1\!\!1_V) T^n T^{*m} \overline{\pi}(\varrho_m^{-\frac{1}{q}} 1\!\!1_U),$$

by Proposition 7.14(2) and Remark 7.13. Also using Remark 7.13 one sees that for $a \in C_0(U)$ we get

$$v_{Z(V,m-n,U)} \pi(a) = \pi(a \circ \theta_{Z(V,m-n,U)}) v_{Z(V,m-n,U)} \tag{7.10}$$

where $\theta_{Z(V,m-n,U)} = (\varphi^m|_V)^{-1} \circ \varphi^n|_U$. This is the covariance condition (SCR1). Condition (SCR2) also readily follows, as it needs to be checked only on idempotents and idemepotents in $S(X, \varphi)$ are sets $Z(U, 0, U)$ for which we have $v_{Z(U,0,U)} = \overline{\pi}(1\!\!1_U)$ by construction.

Let us now show that the map $S(X, \varphi) \ni Z(V, m-n, U) \mapsto v_{Z(V,m-n,U)}$ is a semigroup homomorphism. Let $Z(V, N, U)$ and $Z(W, M, Y)$ be bisections of the form (6.6), so $N = n - m$, $M = k - l$ where $n, m, k, l \in \mathbb{N}_0$, $\varphi^m|_V$, $\varphi^n|_U$, $\varphi^l|_W$, $\varphi^k|_Y$ are injective and $\varphi^m(V) = \varphi^n(U)$, $\varphi^l(Y) = \varphi^k(W)$. By the proof of Proposition 6.35 we have

$$Z(V, N, U) \cdot Z(W, M, Y) = Z(V', N + M, Y'),$$

where $Y' = (\varphi|_Y^k)^{-1} \circ \varphi^l(U \cap W)$ and $V' = (\varphi|_V^m)^{-1} \circ \varphi^n(U \cap W)$. More specifically, we may assume for instance that $n \geqslant l$ (as the other case one then gets by passing taking star in the inverse semigroup $S(X, \varphi)$ and the inverse semigroup of spatial partial isometries). Then φ^{k+n-l} is injective on Y' and $\theta_{Z(V',N+M,Y')} = (\varphi|_V^m)^{-1} \circ \varphi|_{Y'}^{k+n-l}$ maps Y' onto V'.

Then using (7.7) we get

$$\begin{aligned}
v_{Z(V,N,U)} \cdot v_{Z(W,M,Y)} &= \overline{\pi}(\varrho_m^{-\frac{1}{p}} 1\!\!1_V) T^m T^{*n} \overline{\pi}(\varrho_n^{-\frac{1}{q}} 1\!\!1_U) \overline{\pi}(\varrho_l^{-\frac{1}{p}} 1\!\!1_W) T^l T^{*k} \overline{\pi}(\varrho_k^{-\frac{1}{q}} 1\!\!1_Y) \\
&= \overline{\pi}(\varrho_m^{-\frac{1}{p}} 1\!\!1_V) T^m T^{*n-k+l} \overline{\pi}\left(\alpha^k(L^l(\varrho_n^{-\frac{1}{q}} \varrho_l^{-\frac{1}{p}} 1\!\!1_{U\cap W}))\varrho_k^{-\frac{1}{q}} 1\!\!1_Y\right).
\end{aligned}$$

Moreover, for any $y \in Y$ using that $\varphi^l|_W$ is injective we get

$$\begin{aligned}
\alpha^k(L^l(\varrho_n^{-\frac{1}{q}} \varrho_l^{-\frac{1}{p}} 1\!\!1_{U\cap W})(y) &= \sum_{x\in\varphi^{-l}(\varphi^k(y))\cap U\cap W} (\varrho(\varphi^l(x)) \ldots \varrho(\varphi^{n-1}(x)))^{-\frac{1}{q}} \\
&= [\varphi^k(y) \in \varphi^l(U \cap W)](\varrho(\varphi^k(y)) \ldots \varrho(\varphi^{k+n-l-1}(y)))^{-\frac{1}{q}}
\end{aligned}$$

and therefore $\alpha^k(L^l(\varrho_n^{-\frac{1}{q}} \varrho_l^{-\frac{1}{p}} 1\!\!1_{U\cap W}))\varrho_k^{-\frac{1}{q}} 1\!\!1_Y = \varrho_{k+n-l}^{-\frac{1}{q}} 1\!\!1_{Y'}$. Thus

$$v_{Z(V,N,U)} \cdot v_{Z(W,M,Y)} = \overline{\pi}(\varrho_m^{-\frac{1}{p}} 1\!\!1_V) T^m T^{*n-k+l} \overline{\pi}\left(\varrho_{k+n-l}^{-\frac{1}{q}} 1\!\!1_{Y'}\right)$$

Using that $V' = (\varphi|_V^m)^{-1} \circ \varphi^{k+n-l}(Y')$ and the commutation relation (7.10) we may replace V by V' above, and so finally we get

$$v_{Z(V,N,U)} \cdot v_{Z(W,M,Y)} = v_{Z(V',N+M,Y')} = v_{Z(V,N,U)\cdot Z(W,M,Y)}.$$

Hence (π, v) is a covariant representation of the action of $S(X, \varphi)$ and so it integrates to a representation $\pi \rtimes T : F^p(\mathcal{G}(X, \varphi)) \to B(E)$, which clearly satisfies the relations displayed in the assertion, and so in fact $\pi \rtimes T : F^p(\mathcal{G}(X, \varphi)) \to B \subseteq B(E)$ is a representation into B. They determine $\pi \rtimes T$ uniquely, because $C(X)$, $\mathbb{T}_p$ and $\mathbb{T}_q^*$ generate $C_c(\mathcal{G}(X, \varphi))$ as an algebra, by Lemma 7.18. ☐

Corollary 7.21. *Let $\mathcal{L}$ be a finite type transfer operator for a local homeomorphism $\varphi : X \to X$ on a compact metric space X. For each $p \in [1, \infty]$, there is an L^p-operator algebra $F^p(\mathcal{L})$ such that*

1. *$F^p(\mathcal{L}) = B(\iota, \mathbf{T}_p, \mathbf{T}_p^*)$ is generated by a covariant representation $(\iota, \mathbf{T}_p, \mathbf{T}_p^*)$ of $\mathcal{L}$;*
2. *for covariant representation (π, T, T^*) in an L^p-operator algebra B there is a representation $\pi \rtimes T : F^p(\mathcal{L}) \to B$ uniquely determined by*

$$\pi \rtimes T(\iota(a)) = \pi(a),\ a \in C(X), \qquad \pi \rtimes T(\mathbf{T}_p) = T, \qquad \pi \rtimes T(\mathbf{T}_p^*) = T^*.$$

Every L^p-operator algebra satisfying the above two conditions is naturally isometrically isomorphic to $F^p(\mathcal{L})$. The representation ι is isometric and $F^p(\mathcal{L}) \cong F^p(\mathcal{G}(X, \varphi))$.

Proof. Let $F^p(\mathcal{L})$ be the algebra $F^p(\mathcal{G}(X, \varphi)) = F^p_{\mathrm{r}}(\mathcal{G}(X, \varphi))$ and put $\iota := \mathrm{id}_{C(X)} : C(X) \to F^p(\mathcal{L})$, $\mathbf{T}_p := \mathbb{T}_p$, $\mathbf{T}_p^* := \mathbb{T}_q^*$. Then (1) and (2) follow from Proposition 7.19 and Theorem 7.20 respectively. If (σ, S, S^*) is a covariant representation of $\mathcal{L}$ in an L^p-operator algebra such that $C = B(\sigma, S, S^*)$ satisfies the analogue of (2). Then we have two contractive homomorphisms $\iota \rtimes \mathbf{T}_p : F^p(\mathcal{L}) \to C$ and $\sigma \rtimes S : C \to F^p(\mathcal{L})$, which are inverse to each other. Hence $F^p(\mathcal{L}) \cong C$. ☐

Definition 7.22. We call the algebra $F^p(\mathcal{L})$ together with the covariant representation $(\iota, \mathbf{T}_p, \mathbf{T}_p^*)$ described in Corollary 7.21, the *L^p-operator algebra crossed product* of the transfer operator $\mathcal{L}$. Since ι is isometric, we will treat $C(X)$ as a subalgebra of $F^p(\mathcal{L})$.

Proposition 7.23. *The crossed product $F^p(\mathcal{L})$, $p \in [1, \infty]$, is equipped with the circle gauge action by isometric automorphisms of $F^p(\mathcal{L})$, i.e. there is a group homomorphism $\gamma : \mathbb{T} \to \mathrm{Aut}(F^p(\mathcal{L}))$ such that $\gamma_z \circ \iota = \iota$, $\gamma_z(\mathbf{T}_p) = z\mathbf{T}_p$ and $\gamma_z(\mathbf{T}_p^*) = \overline{z}\mathbf{T}_p$ for $z \in \mathbb{T}$. The following conditions are equivalent:*

(1) *φ is topologically free, see Definition 6.39;*
(2) *$C(X) \subseteq F^p(\mathcal{L})$ is maximal abelian subalgebra;*
(3) *$C(X) \subseteq F^p(\mathcal{L})$ has the intersection property.*

Moreover, $F^p(\mathcal{L})$ is simple if and only if X is infinite and φ is minimal, see Definition 6.42.

Proof. For each $z \in \mathbb{T}$ the triple $(\iota, z\mathbf{T}_p, \overline{z}\mathbf{T}_p)$ is a covariant representation of $\mathcal{L}$ in $F^p(\mathcal{L})$. Hence by universality we have a representation $\gamma_z : F^p(\mathcal{L}) \to F^p(\mathcal{L})$ such that $\gamma_z \circ \iota = \iota$, $\gamma_z(\mathbf{T}_p) = z\mathbf{T}_p$ and $\gamma_z(\mathbf{T}_p^*) = \overline{z}\mathbf{T}_p$. Moreover, we clearly have $\gamma_1 = \mathrm{id}_{F^p(\mathcal{L})}$ and $\gamma_z \circ \gamma_w = \gamma_{zw}$ for $z, w \in \mathbb{T}$. In particular, representations γ_z and $\gamma_{\overline{z}}$ are inverse to each other, and hence they are isometries. Thus $\gamma : \mathbb{T} \to \mathrm{Aut}(F^p(\mathcal{L}))$ is a group homomorphism. The remaining part of the assertion follows from Theorems 6.41 and 6.46. □

Corollary 7.24. *For any $a \in C(X)$ the spectrum of the operator $a\mathbf{T}_p \in F^p(\mathcal{L})$, $p \in [1, \infty]$, in $F^p(\mathcal{L})$ is invariant under rotations around zero.*

Proof. Let $\lambda \in \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ and $z \in \mathbb{T}$. Since $\gamma_{\overline{z}} : F^p(\mathcal{L}) \to F^p(\mathcal{L})$ is an automorphism, $(a\mathbf{T}_p - z\lambda 1) = z(a\overline{z}\mathbf{T}_p - \lambda 1) = z \cdot \gamma_{\overline{z}}(a\mathbf{T}_p - \lambda 1)$ is not invertible, that is $z\lambda \in \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$. □

7.3. Riesz projectors for universal weighted composition operators

In this section we will use our groupoid tools to prove the following result about the spectra of weighted composition operators coming from the universal representation.

Proposition 7.25. *Let $\varphi : X \to X$ be a topologically free, surjective local homeomorphism on a compact metrizable space. Let $p \in [1, \infty]$ and let $\mathbf{T}_p$, $\mathbf{T}_p^*$ be operators that together with $C(X)$ generate $F^p(\mathcal{L})$. For any $a \in C(X)$, every Riesz projector P associated to a clopen subset of the spectrum $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ belongs to $C(X)$ and hence corresponds to a clopen subset of X. Moreover, if the corresponding part of $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ does not contain zero, then $P = \pi(\mathbb{1}_{X_0})$ where $X_0 \subseteq X$ is such that $\varphi(X_0) = X_0$, $\varphi : X_0 \to X_0$ is a homeomorphism and $a|_{X_0}$ is non-zero at every point.*

Let us start by recalling that for any Banach space E an operator $b \in B(E)$ is *hyperbolic* if $\sigma(b) \cap \mathbb{T} = \varnothing$ where $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$. Then the formula

$$P := \frac{1}{2\pi i} \int_{\mathbb{T}} (\lambda - b)^{-1} d\lambda \in B(E),$$

defines a Riesz projector, cf. [**RS90**], that divides E to two parts that correspond to parts of the spectrum of b lying in $\{z \in \mathbb{C} : |z| < 1\}$ and $\{z \in \mathbb{C} : |z| > 1\}$. Namely, the spaces $E_1 = PE$ and $E_2 = \ker P = (1-P)E$ are invariant subspaces of b and $\sigma(b|_{E_1}) \subseteq \{z \in \mathbb{C} : |z| < 1\}$ and $\sigma(b|_{E_2}) \subseteq \{z \in \mathbb{C} : |z| > 1\}$. Hence there is $n_0 \in \mathbb{N}$ and $\varepsilon > 0$ such that

$$\|b^n v_1\| < (1-\varepsilon)^n \|v_1\| \qquad \text{for all } v_1 \in E_1,\ n > n_0, \tag{7.11}$$

$$\|b^n v_2\| > (1+\varepsilon)^n \|v_2\| \qquad \text{for all } v_2 \in E_2,\ n > n_0. \tag{7.12}$$

Every $v \in E$ has a unique decomposition $v = v_1 + v_2$ where $v_i \in E_i$, $i = 1, 2$. Since $\|b^n v\| = \|b^n(v_1+v_2)\| \geqslant \|b^n v_2\| - \|b^n v_1\|$, we see that (7.11), (7.12) imply that either $\lim_{n\to\infty} \|b^n v\| = \infty$ or $\lim_{n\to\infty} \|b^n v\| = 0$ depending on whether $v_2 \neq 0$ or $v_2 = 0$. Accordingly,

$$PE = \{v \in E : \lim_{n\to\infty} b^n v = 0\}. \tag{7.13}$$

We start with some technical general facts concerning Riesz projectors for operators associated to any covariant representation of $\mathcal{L}$.

Lemma 7.26. *Let (π, T, T^*) be an injective covariant representation of transfer operator $\mathcal{L}$ on a Banach space E. Assume the identification $A \cong C(X)$ and let $a \in A$ be such that $\sigma(aT) \cap \mathbb{T} = \varnothing$. Let $P = \frac{1}{2\pi i}\int_{\mathbb{T}} (\lambda - aT)^{-1} d\lambda$ and put $E_1 = PE$ and $E_2 = (1-P)E$. Then*

1. *$P \in A'$, i.e. P commutes with elements of $A = C(X)$;*
2. *$|a|T$ commutes with P;*
3. *$a|_{E_2} : E_2 \to E_2$ is invertible.*

Proof. (1). By (7.13), $E_1 = \{v \in E : \lim_{n\to\infty} (aT)^n v = 0\}$. For any $b \in A$ and $v \in E_1$ using the commutation relation (TR2) we get

$$\|(aT)^n bv\| = \|\alpha^n(b)(aT)^n v\| \leqslant \|b\| \|(aT)^n v\| \underset{n\to\infty}{\longrightarrow} 0.$$

Thus $bE_1 \subseteq E_1$ and so $AE_1 \subseteq E_1$. Let us consider now invertible $b \in A$. Let $v \in E_2$ and write $bv = v_1 + v_2$ where $v_i \in E_i$, $i = 1, 2$. Then $v = b^{-1}v_1 + b^{-1}v_2$. Since $b^{-1}v_1 \in E_1$ and $v - b^{-1}v_2 \in E_2$, we must have $b^{-1}v_1 = 0$ and therefore $v_1 = 0$. This shows that $bE_2 \subseteq E_2$ for all invertible $b \in A$. Since every element in $A = C(X)$ is a sum of four invertible elements, this implies that $AE_2 \subseteq E_2$. Thus we have proved that operators in A respect the decomposition $E = E_1 \oplus E_2$, which is equivalent to $P \in A'$.

(2). Using the Tietze theorem we may find functions $a_k \in C(X)$, $k = 1, 2, \ldots$, such that: $|a_k(x)| \leqslant 1$ and $a_k(x) = \frac{|a(x)|}{a(x)}$ when $|a(x)| \geqslant \frac{1}{k}$. Then $\lim_{k\to\infty} a_k a = |a|$. Since P commutes with aT and $a_k \in C(X)$ we get

$$(|a|T)P = \lim_{k\to\infty} a_k aTP = P \lim_{k\to\infty} a_k aT = P(|a|T).$$

(3). Using that $aT : E_2 \to E_2$ is surjective and a commutes with $(1-P)$ we get

$$E_2 = aTE_2 = aT(1-P)E = (1-P)aTE = a(1-P)TE \subseteq aE_2.$$

Hence $a : E_2 \to E_2$ is surjective. To see it is injective, it suffices to show that $|a| : E_2 \to E_2$ is injective, because $a^*a = |a|^2$. Suppose that $v \in E_2$ is such that $|a|v = 0$. By the Urysohn lemma we may find functions $c_k \in C(X)$, $k = 2, 3, ...$, such that

$$0 \leqslant c_k \leqslant 1, \qquad c_k(x) = \begin{cases} 0, & \text{when } |a(x)| \leqslant \frac{1}{k}, \\ 1, & \text{when } |a(x)| \geqslant \frac{1}{k-1}. \end{cases}$$

Let us put $|a|_k(x) := |a(x)|$, when $|a(x)| > \frac{1}{k}$, and $|a|_k(x) := \frac{1}{k}$, when $|a(x)| \leqslant \frac{1}{k}$. Then $|a|_k \in C(X)$ are invertible elements in $C(X)$ such that $|a|_k c_k = |a| c_k$. Thus $|a|_k(c_k v) = c_k(|a|v) = 0$. This implies that $c_k v = 0$, for all $k = 2, 3, \ldots$. Since the restriction of aT to E_2 is invertible there is $v_2 \in E_2$ such that $aTv_2 = v$. From the construction of c_k we have $\lim_{k\to\infty} c_k a = a$. Accordingly, we get $v = aTv_2 = \lim_{k\to\infty} c_k aTv_2 = \lim_{k\to\infty} c_k v = 0$. $\square$

Definition 7.27. Let $A \subseteq B$ be a unital inclusion of unital Banach algebras. We say that A is *inverse closed* (or *full*) subalgebra of B, if every element $a \in A$ which is invertible in B is invertible in A.

The following is probably well known to experts, but we were not able to find a reference:

Lemma 7.28. *Let B be a unital Banach algebra.*

(1) *Consider B as a subalgebra of $B(E)$ where $E := B$ and $b \in B$ acts on E by left multiplication. Then $B \subseteq B(E)$ is inverse closed. More generally, for any idempotent $p \in B$ the subalgebra $pBp \subseteq B(pE)$ is inverse closed.*
(2) *If $A \subseteq B$ is a unital maximal abelian subalgebra of B, then it is inverse closed.*

Proof. (1). Let $p \in B$ be an idempotent and denote by $\pi : pBp \to B(pB)$ the 'left regular representation', that is $\pi(b)a = b \cdot a$ for $b \in pBp$ and $a \in pB$. Note that π is isometric, and so we may use it to identify pBp with subalgebra of $B(pE)$ where $E := B$. Assume that $b \in pBp$ is such that $\pi(b)$ is invertible in $B(pE)$ and let $T : pB \to pB$ be the inverse of $\pi(b)$. Then we get that $b \cdot T(p) = [\pi(b)T](p) = p$. Hence $T(p) \in B$ is a right inverse of b in B. But on the other hand, we have $\pi(b)(p - T(p)b) = b - b \cdot T(p) \cdot b = b - pb = b - b = 0$, and since $\pi(b)$ is injective we get that $T(p)b = p$. So $T(p)$ is the inverse of b in pBp.

(2). Let $b \in B$ be an inverse of $a \in A$ in B. By commutativity of A, for any $c \in A$ we get $bc = 1bc = (ca)bc = cb(ac) = cb1 = cb$. Hence $b \in A$ by maximal abelianness of A. $\square$

Remark 7.29. We will consider maximal abelian subalgebras of the form $A = C(X)$, so that they are commutative C^*-algebras. Thus instead of Lemma 7.28(2) we may appeal to the much more general fact that every unital C^*-algebra A is inverse closed when unitaly embedded into any Banach algebra B, see [**Gol99**].

Proof of Proposition 7.25. By Lemma 7.28(1) we may assume that $F^p(\mathcal{L})$ is an inverse closed unital subalgebra of $B(E)$. Let $a \in C(X)$. By Corollary 7.24, $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ is invariant under the rotations around zero. Hence to prove the assertion it suffices to consider projections P and $1 - P$, where $P := \frac{1}{2\pi i}\int_{\{z:|z|=r\}}(\lambda - a\mathbf{T}_p)^{-1} d\lambda \in F^p(\mathcal{L})$ corresponds to parts of the spectrum of $a\mathbf{T}_p$ lying in $\{z \in \mathbb{C} : |z| < r\}$ and $\{z \in \mathbb{C} : |z| > r\}$ respectively. Moreover, by scaling, we may also assume that $r = 1$. By Lemma 7.26(1), $P \in \pi(C(X))'$. But by Proposition 7.23, $C(X)$ is a maximal abelian subalgebra of $F^p(\mathcal{L})$. Hence $P \in C(X)$, and so $P = \mathbb{1}_{X_1}$ for a clopen subset of X. Note that $1 - P = \mathbb{1}_{X_2}$ where

$X = X_1 \sqcup X_2$. Putting $E_2 := \mathbb{1}_{X_2}E$ we get that $a\mathbf{T}_p\mathbb{1}_{X_2} = a\mathbb{1}_{X_2}\mathbf{T}_p$ is invertible as an element of $B(E_2)$. By Lemma 7.26(3), $a\mathbb{1}_{X_2} \in \mathbb{1}_{X_2}F^p(\mathcal{L})\mathbb{1}_{X_2}$ is invertible in $B(E_2)$ as well. Thus by Remark 7.29, $a\mathbb{1}_{X_2}$ is invertible as an element of $C(X_2) \subseteq C(X_1) \oplus C(X_2)$, that is $a(x) \neq 0$ for $x \in X_2$. Also this implies that $\mathbb{1}_{X_2}\mathbf{T}_p = \mathbb{1}_{X_2}\mathbf{T}_p\mathbb{1}_{X_2}$ is invertible in $B(E_2)$ and therefore by Lemma 7.28(1) we get that $\mathbb{1}_{X_2}\mathbf{T}_p$ is invertible in $\mathbb{1}_{X_2}F^p(\mathcal{L})\mathbb{1}_{X_2}$. We will now infere the properties of X_2 by considering $F^p(\mathcal{G}(X,\varphi))$ as a subalgebra of $B(\ell^p(\mathcal{G}(X,\varphi)))$ using the isometric representation $\Lambda_p : F^p(\mathcal{G}(X,\varphi)) \to B(\ell^p(\mathcal{G}(X,\varphi)))$, cf. Proposition 7.19. By (7.8) for $\xi \in \ell^p(\mathcal{G}(X,\varphi))$ we have $\mathbb{1}_{X_2} * \xi(x,N,z) = [x \in X_2]\xi(x,N,z)$ and

$$\mathbb{1}_{X_2}\mathbf{T}_p\xi(x,N,z) = [x \in X_2]\varrho(x)^{\frac{1}{p}}\xi(\varphi(x),N+1,z).$$

Since $\mathbb{1}_{X_2}\mathbf{T}_p = \mathbb{1}_{X_2}\mathbf{T}_p\mathbb{1}_{X_2}$, this implies that $\varphi(X_2) \subseteq X_2$. Since $\mathbb{1}_{X_2}\mathbf{T}_p : \mathbb{1}_{X_2}\ell^p(\mathcal{G}(X,\varphi)) \to \mathbb{1}_{X_2}\ell^p(\mathcal{G}(X,\varphi))$ is invertible it follows that for every $y \in X_2$ the set $\varphi^{-1}(y) \cap X_2$ is a singletone. Hence $\varphi : X_2 \to X_2$ is a homeomorphism.

□

7.4. Spectral radii

In this section we discuss variational formulas for spectral radius abstract weighted shifts that we defined in the previous section. The main result is Theorem 7.43 that shows Ruelle's formula for general expanding maps. The basic fact is that such analysis can be reduced to the study of spectral radius of weighted transfer operators:

PROPOSITION 7.30. *Let $\mathcal{L} = \mathcal{L}_{\varrho,\varphi}$ be a transfer operator of finite type for a surjective local homeomorphism $\varphi : X \to X$ on a compact metric space. Let $p \in [1,\infty]$ and let $\mathbf{T}_p$, $\mathbf{T}_p^*$ be operators that together with $C(X)$ generate $F^p(\mathcal{L})$. If $p < \infty$, for any $a \in C(X)$*

$$r(a\mathbf{T}_p) = \sqrt[p]{r(\mathcal{L}_{|a|^p\varrho,\varphi})} = \lim_{n\to\infty} \|\mathcal{L}^n(|a|^p\alpha(|a|^p)\cdots\alpha^{n-1}(|a|^p))\|_\infty^{\frac{1}{np}},$$

that is the spectral radius of $a\mathbf{T}_p$ is equal to the p-th root of the spectral radius of the transfer operator $\mathcal{L}_{|a|^p\varrho,\varphi} : C(X) \to C(X)$ where $\mathcal{L}_{|a|^p\varrho,\varphi}(b) = \mathcal{L}(|a|^p b)$, $b \in C(X)$, see (7.3). Moreover,

$$r(a\mathbf{T}_\infty) = r(a\alpha) = \lim_{n\to\infty} \|a\alpha(a)\cdots\alpha^{n-1}(a))\|_\infty^{\frac{1}{n}},$$

that is the spectral radius of $a\mathbf{T}_\infty$ is equal to the the spectral radius of the weighted composition operator $a\alpha : C(X) \to C(X)$ where $[a\alpha](b) = a\alpha(b) = a \cdot (b \circ \varphi)$, $b \in C(X)$.

PROOF. Let $n \in \mathbb{N}$ and $p < \infty$. Since $(\mathcal{L}_{|a|^p\varrho,\varphi})^n : C(X) \to C(X)$ is a positive map, $\|(\mathcal{L}_{|a|^p\varrho,\varphi})^n\| = \|(\mathcal{L}_{|a|^p\varrho,\varphi})^n(1)\|_\infty$, see for example [**Kwa17**, Lemma 2.1]. Note that $(\mathcal{L}_{|a|^p\varrho,\varphi})^n(1)$ is equal to $\mathcal{L}^n(|a|^p\alpha(|a|^p)\cdots\alpha^{n-1}(|a|^p))$ and therefore

$$\|(\mathcal{L}_{|a|^p\varrho,\varphi})^n\| = \|\mathcal{L}^n(|a|^p\alpha(|a|^p)\cdots\alpha^{n-1}(|a|^p))\|_\infty.$$

Similarly, we get $\|(a\alpha)^n\| = \|a\alpha(a)\cdots\alpha^{n-1}(a))\|_\infty$. On the other hand, for any $p \in [1,\infty]$ we have $(a\mathbf{T}_p)^n = a\alpha(a)\cdots\alpha^{n-1}(a)\mathbf{T}_p^n$. To calculate its norm, we will use the regular representation where $\mathbf{T}_p$ acts on $\ell^p(\mathcal{G}(X,\varphi))$ by convolution with $\mathbb{T}_p$. Hence $\mathbb{T}_p^n = (\varrho_n)^{\frac{1}{p}}\mathbb{1}_{\Gamma(\varphi^n)}$

acts as a weighted composition operator on $\ell^p(\mathcal{G}(X,\varphi))$, see (7.8). For $p = \infty$ and any $b \in C(X)$ and $\xi \in \ell^p(\mathcal{G}(X,\varphi))$ we have

$$[\Lambda_\infty(b\mathbb{T}_\infty^n)]\xi(x,N,z) = b(\varphi^n(x)) \cdot \xi(\varphi^n(x), N+n, z).$$

Since φ is surjective, this implies that $\|b\mathbf{T}_\infty^n\| = \|b\|_\infty$. Hence taking $b := a\alpha(a)\cdots\alpha^{n-1}(a)$ we get

$$\|(a\mathbf{T}_\infty)^n\|_{B(\ell^\infty(\mathcal{G}(X,\varphi)))} = \|a\alpha(a)\cdots\alpha^{n-1}(a))\|_\infty = \|(a\alpha)^n\|_{B(C(X))},$$

which implies the second part of the assertion.

Assume then that $p < \infty$. Let $b \in C(X)$ and $\xi \in \ell^p(\mathcal{G}(X,\varphi))$. We get

$$\|\Lambda_p(b\mathbb{T}_p^n)\xi\|_p^p = \sum_{k\in\mathcal{G}(X,\varphi)} |b\mathbb{T}_p^n * \xi(k)|^p = \sum_{k\in\mathcal{G}(X,\varphi)} [|b|^p\mathbb{T}_1 * |\xi|^p](k) = \|\Lambda_1(|b|^p\mathbb{T}_1^n)|\xi|^p\|_1.$$

Taking supremum over norm one elements $\xi \in \ell^p(\mathcal{G}(X,\varphi))$ and using that the map $\ell^p(\mathcal{G}(X,\varphi)) \ni \xi \mapsto |\xi|^p \in \ell^1(\mathcal{G}(X,\varphi))$ is norm preserving and $\Lambda_1(|b|^p\mathbb{T}_1^n)$ is positive, we get

$$\|b\mathbf{T}_p^n\|^p = \|\Lambda_p(b\mathbb{T}_p^n)\|_{B(\ell^p(\mathcal{G}(X,\varphi)))}^p = \|\Lambda_1(|b|^p\mathbb{T}_1^n)\|_{B(\ell^1(\mathcal{G}(X,\varphi)))}.$$

Thus using the standard isomorphism $\ell^1(\mathcal{G}(X,\varphi))^* \cong \ell^\infty(\mathcal{G}(X,\varphi))$ we get

$$\|b\mathbf{T}_p^n\|^p = \|\Lambda_1(|b|^p\mathbb{T}_1^n)\|_{B(\ell^1(\mathcal{G}(X,\varphi)))} = \|\Lambda_1(|b|^p\mathbb{T}_1^n)^*\|_{B(\ell^\infty(\mathcal{G}(X,\varphi)))}.$$

In view of the pairing $\ell^1(\mathcal{G}(X,\varphi)) \times \ell^\infty(\mathcal{G}(X,\varphi)) \ni (\xi,\eta) \mapsto \sum_{k\in\mathcal{G}(X,\varphi))} \xi(k)\eta(k)$ we get $\Lambda_1(|b|^p\mathbb{T}_1^n)^* = \Lambda_\infty((|b|^p\mathbb{T}_1^n)^*)$. For $\xi \in \ell^\infty(\mathcal{G}(X,\varphi))$, using (7.8), we have

$$\Lambda_\infty((|b|^p\mathbb{T}_1^n)^*)\xi(y,N,z) = \sum_{x\in\varphi^{-n}(y)} |b(x)|^p\varrho_n(x)\xi(x,N-1,z)).$$

This implies that

$$\|\Lambda_\infty((|b|^p\mathbb{T}_1^n)^*)\|_{B(\ell^\infty(\mathcal{G}(X,\varphi)))} = \max_{y\in X} \sum_{x\in\varphi^{-n}(y)} |b(x)|^p\varrho_n(x) = \|\mathcal{L}^n(|b|^p)\|_\infty.$$

Combining these steps we get $\|b\mathbf{T}_p^n\|^p = \|\mathcal{L}^n(|b|^p)\|_\infty$. Taking $b := a\alpha(a)\cdots\alpha^{n-1}(a)$ we have $|b|^p = |a|^p\alpha(|a|^p)\cdots\alpha^{n-1}(|a|^p)$ and so

$$\|(a\mathbf{T}_p)^n\| = \|\mathcal{L}^n(|a|^p\alpha(|a|^p)\cdots\alpha^{n-1}(|a|^p))\|_\infty.$$

Comparing this with the first displayed formula in the proof gives the assertion. □

The variational formula for the spectral radius of a weighted automorphism $a\alpha : C(X) \to C(X)$ was given independently by Kitover [**Kit79**] and Lebedev [**Leb79**]. But it is known to hold for arbitrary weighted endomorphisms, see for instance [**KL20**, Subsection 5.1] and references cited there. In particular, we have

Theorem 7.31 (see, for instance, [**KL20**, Theorem 5.1]). *Let $\alpha : C(X) \to C(X)$ be a unital endomorphism and let $\varphi : X \to X$ be the associated dual continuous surjection. For any $a \in C(X)$ the spectral radius of the weighted endomorphism $a\alpha : C(X) \to C(X)$ is given by*

$$r(a\alpha) = \max_{\mu\in\mathrm{Inv}(X,\varphi)} \exp\int_X \ln|a|\,d\mu = \max_{\mu\in\mathrm{Erg}(X,\varphi)} \exp\int_X \ln|a|\,d\mu. \tag{7.14}$$

Corollary 7.32 (cf.[**Leb79**]). *Let $\alpha : C(X) \to C(X)$ be an automorphism and let $\pi : C(X) \to B(E)$ be an injective unital representaion and $T \in B(E)$ an invertible isometry such that $T\pi(a) = \pi(a \circ \alpha)T$ for all $a \in C(X)$. Then for any $a \in C(X)$ we have*

$$r(\pi(a)T) = \max_{\mu\in \mathrm{Erg}(X,\varphi)} \exp \int_X \ln |a|\, d\mu.$$

and if a is non-zero on X (invertible in $C(X)$), then

$$r((\pi(a)T)^{-1})^{-1} = \min_{\mu\in \mathrm{Erg}(X,\varphi)} \exp \int_X \ln |a|\, d\mu.$$

Proof. For each $n \geqslant 1$ we have

$$\|(\pi(a)T)^n\| = \|\pi(\prod_{k=0}^{n-1} \alpha^k(a))T^n\| = \|\pi(\prod_{k=0}^{n-1} \alpha^k(a))\| = \|\prod_{k=0}^{n-1} \alpha^k(a)\|_\infty = \|(a\alpha)^n\|.$$

Hence $r(\pi(a)T) = r(a\alpha)$. In view of (7.14), this gives the first part of the assertion. Note that $T^{-1}\pi(a) = \pi(\alpha^{-1}(a))T^{-1}$ and since $\mathrm{Erg}(X,\varphi) = \mathrm{Erg}(X,\varphi^{-1})$ and for any $\mu \in \mathrm{Erg}(X,\varphi)$ we have $\int_X \ln|a\circ p|\, d\mu = \int_X \ln |a|\, d\mu$ by φ-invariance of μ we get

$$r(\pi(a)T) = r(\pi(a)T^{-1}) = \max_{\mu\in \mathrm{Erg}(X,\varphi)} \exp \int_X \ln |a|\, d\mu.$$

If in addition a is invertible, then $\pi(a)T$ is invertible and $(\pi(a)T)^{-1} = T^{-1}\pi(a^{-1}) = \pi(\alpha^{-1}(a^{-1}))T^{-1}$. Hence

$$\begin{aligned} r((\pi(a)T)^{-1}) &= r(\pi(\alpha^{-1}(a^{-1}))T^{-1}) = \max_{\mu\in \mathrm{Erg}(X,\varphi)} \exp \int_X \ln |a^{-1}|\, d\mu \\ &= \max_{\mu\in \mathrm{Erg}(X,\varphi)} \exp \int_X -\ln |a|\, d\mu = \left(\min_{\mu\in \mathrm{Erg}(X,\varphi)} \exp \int_X \ln |a|\, d\mu\right)^{-1}. \end{aligned}$$

This proves the second part of the assertion. ☐

Spectral radius of general weighted transfer operator was investigated in [**ABL11**], where it was described using the notion of t-entropy. Here we recall a definition simplified in [**BL17**]. For a continuous surjective map $\varphi : X \to X$ we denote by $\mathrm{Inv}(X,\varphi)$ the simplex of regular probability measures that are *φ-invariant* in the sense that $\mu\circ\varphi^{-1} = \mu$ for $\mu \in \mathrm{Inv}(X,\varphi)$. The extreme points of this simplex form the set $\mathrm{Erg}(X,\varphi)$ of *ergodic* measures. Hence $\mu \in \mathrm{Erg}(X,\varphi)$ if $\mu \in \mathrm{Inv}(X,\varphi)$ and for φ-invariant Borel sets $A \subseteq X$ we have $\mu(A) \in \{0,1\}$.

Definition 7.33 ([**ABL11**], [**BL17**]). Let $\mathcal{L} : C(X) \to C(X)$ be a unital transfer operator for a continuous surjective map $\varphi : X \to X$. The *t-entropy of $\mathcal{L}$* is the functional $\tau_{\mathcal{L}} : \mathrm{Inv}(X,\varphi) \to \mathbb{R} \cup \{-\infty\}$ whose value at $\mu \in \mathrm{Inv}(X,\varphi)$ is defined by the formula:

$$\tau_{\mathcal{L}}(\mu) := \inf_{n\in\mathbb{N}} \frac{1}{n} \inf_{D(X)} \sum_{g\in D} \mu(g) \ln \frac{\mu(\mathcal{L}^n g)}{\mu(g)} \tag{7.15}$$

where $D(X)$ is the set of all partitions of unity in the algebra $C(X)$. If we have $\mu(g) = 0$ for a certain function g, then we set the corresponding summand in (7.15) to be zero. If there exists $g \in D(X)$ such that $\mathcal{L}^n g \equiv 0$ and $\mu(g) > 0$, then we set $\tau_{\mathcal{L}}(\mu) = -\infty$.

THEOREM 7.34 ([**ABL11**, Theorem 11.2]). *Let $\mathcal{L} : C(X) \to C(X)$ be a unital transfer operator for a continuous surjective map $\varphi : X \to X$. For $b \in C(X)^+$ consider the transfer operator $\mathcal{L}_b : C(X) \to C(X)$ given by $\mathcal{L}_b(a) = \mathcal{L}(ba)$, $a \in C(X)$. Then*

$$\ln r(\mathcal{L}_b) = \max_{\mu \in \mathrm{Inv}(X,\varphi)} \left(\int_X \ln |b| \, d\mu + \tau_{\mathcal{L}}(\mu) \right)$$

where $\tau_{\mathcal{L}}(\mu)$ is t-entropy for $\mathcal{L}$ (we adapt the convention that $\ln 0 = -\infty$). In particular, $\ln r(\mathcal{L}) = \max_{\mu \in \mathrm{Inv}(X,\varphi)} \tau_{\mathcal{L}}(\mu)$.

REMARK 7.35. If $\varphi : X \to X$ is a homeomorphism, then $\mathcal{L} : C(X) \to C(X)$ is the composition operator with φ^{-1}, an automorphism of $C(X)$, and $\tau_{\mathcal{L}}(\mu) = 0$ for every $\mu \in \mathrm{Inv}(X, \varphi)$, by [**ABL11**, Proposition 8.3]. In particular, then the above result implies that for every $a \in C(X)$ the spectral radius of the weighted automorphism $a\mathcal{L} : C(X) \to C(X)$ is given by

$$\ln r(a\mathcal{L}) = \max_{\mu \in \mathrm{Inv}(X,\varphi)} \int_X \ln |a| \, d\mu = \max_{\mu \in \mathrm{Erg}(X,\varphi)} \int_X \ln |a| \, d\mu,$$

which is consistent with (7.14).

COROLLARY 7.36. *Let $\mathcal{L}$ be a transfer operator of finite type for a surjective local homeomorphism $\varphi : X \to X$ on a compact metric space. For any $p \in [1, \infty]$ and $a \in C(X)$ we have*

$$\ln r(a\mathbf{T}_p) = \max_{\mu \in \mathrm{Inv}(X,\varphi)} \left(\int_X \ln |a| \, d\mu + \frac{\tau_{\mathcal{L}}(\mu)}{p} \right),$$

where $\tau_{\mathcal{L}}$ is the t-entropy for $\mathcal{L}$. If φ is injective, then $\ln r(a\mathbf{T}_p) = \max_{\mu \in \mathrm{Erg}(X,\varphi)} \int_X \ln |a| \, d\mu$.

PROOF. For $p = \infty$ combine Proposition 7.30 and Theorem 7.31. For $p < \infty$ combine Proposition 7.30 and Theorem 7.34 with $b = |a|^p$. ☐

One of the fundamental principles of thermodynamical formalism and a part of the celebrated Ruelle's Perron-Frobenius Theorem is that for expanding open maps the spectral exponent $\ln r(\mathcal{L}_{\varrho,\varphi})$ coincides with topological pressure $P(\ln \varrho, \varphi)$. A number of results in this direction are known, cf. [**Rue89**], [**LM98**], [**FJ01**], [**PU10**]. However, none of these sources considers a general case. Usually it is assumed that φ is topologically mixing, $\varrho : X \to (0, \infty)$ is Hölder continuous and strictly positive, and the space X is a finite dimensional manifold or a shift space. In [**BK21**] the author and his advisor proved that

$$\ln r(\mathcal{L}_{c,\varphi}) = P(\ln c, \varphi)$$

for an arbitrary open expanding map $\varphi : X \to X$ on a compact metric space and an arbitrary continuous $c : X \to [0, \infty)$. We will now briefly discuss this result.

Definition 7.37. A continuous map $\varphi : X \to X$ on a compact metric space (X, d) is (locally) *expanding* if there are $\varepsilon > 0$ and $\theta > 1$ such that

$$d(x, y) < \varepsilon \quad \Longrightarrow \quad d(\varphi(x), \varphi(y)) \geqslant \theta \cdot d(x, y).$$

Remark 7.38. Being expanding does not depend on the choice of a metric (compatible with the topology). By [**Red82**], $\varphi : X \to X$ is expanding if and only if $\varphi : X \to X$ is *positively expansive*, i.e. for some metric d there is $\delta > 0$ such that for any two points $x \neq y$ we have $d(\varphi^n(x), \varphi^n(y)) > \delta$ for some $n \in \mathbb{N}$. In particular, every expanding map is locally injective and so it is a local homeomorphism if and only if it is open.

Remark 7.39. By Schwartzman's theorem, see [**KR69**], a homeomorphism $\varphi : X \to X$ is expanding (positively expansive) if and only if X is finite. Thus being expanding is a notion designed for irreversible dynamics. A right notion for homeomorphisms is 'two-sided expansiveness': a homeomorphism $\varphi : X \to X$ is *expansive* if there is $\delta > 0$ such that for any two points $x \neq y$ we have $d(\varphi^n(x), \varphi^n(y)) > \delta$ for some $n \in \mathbb{Z}$. The literature is abundant in the results concerning expansive homeomorphisms, but it is well known that (one-sided) versions of such facts hold for expanding (positively expansive) maps, cf. [**Wal82**, Remark on page 145], [**KR69**, page 58]. A counterpart of [**Wal82**, Theorem 5.24], [**KR69**, Corollary 3.3] states that

any expansive map $\varphi : X \to X$ is a factor of a one-sided subshift.

Namely, if $\varphi : X \to X$ is expansive, then there is $n \geqslant 1$ and a closed subset Z of the product space $\{1, ..., n\}^{\mathbb{N}}$ with $\sigma(Z) = Z$, where $\sigma : \{1, ..., n\}^{\mathbb{N}} \to \{1, ..., n\}^{\mathbb{N}}$ is the one-sided shift

$$\sigma(\omega_1 \omega_2 \cdots) = (\omega_2 \omega_3 \cdots),$$

and there is a continuous surjective map $\pi : Z \to X$ such that the diagram

$$\begin{array}{ccc} Z & \xrightarrow{\sigma} & Z \\ {\scriptstyle \pi}\downarrow & & \downarrow{\scriptstyle \pi} \\ X & \xrightarrow{\varphi} & X \end{array}$$

commutes. Moreover, if X is zero dimensional and φ is open, then π above can be chosen to be injecitve, so expanding maps on zero-dimensional spaces are equivalent to one-sided subshifts (Z, σ).

Let $\varphi : X \to X$ be a continuous map on a compact metric space X and let $b \in C(X, \mathbb{R})$ be a continuous real valued function. The *topological pressure of* (X, φ) *with potential* b is usually defined using n-Bowen's metrics $d_n(x, y) = \max\limits_{i=0,\dots,n} d(\varphi^i(x), \varphi^i(y))$ and either separated on spanning sets using the following formulas

$$\begin{aligned} P(\varphi, b) &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \sup_{\substack{E \subseteq X \text{ is} \\ \varepsilon\text{-separated in } d_n}} \frac{1}{n} \ln \sum_{y \in E} \exp\left(\sum_{i=0}^{n-1} b(\varphi^i(y)) \right) \\ &= \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \inf_{\substack{E \subseteq X \text{ is} \\ (d_n, \varepsilon)\text{-spanning}}} \frac{1}{n} \ln \sum_{y \in E} \exp\left(\sum_{i=0}^{n-1} b(\varphi^i(y)) \right), \end{aligned}$$

see, for instance, [**Wal82**, 9.3]. It is crucial that $P(\varphi, b)$ can be also expressed in terms of the following variational principle:

$$P(\varphi, b) = \sup_{\mu\in \mathrm{Inv}(X,\varphi)} \left(\int_X b\, d\mu + h_\varphi(\mu) \right), \tag{7.16}$$

where $h_\varphi(\mu)$ is the *Kolmogorov-Sinai entropy* of a measure $\mu \in \mathrm{Inv}(X, \varphi)$, see [**Wal82**], [**PU10**]. Since we want to consider $b = \ln c$ where $c : X \to [0, \infty)$ is allowed to have zero values, we need to generalize the definition of topological pressure to the case where $b : X \to \mathbb{R} \cup \{-\infty\}$ is allowed to attain the value $-\infty$:

Definition 7.40. For a continuous map $b : X \to \mathbb{R} \cup \{-\infty\}$ we define the *topological pressure of* (X, φ) *with potential* b as the value $P(\varphi, b) \in \mathbb{R} \cup \{\pm\infty\}$ given by (7.16).

Remark 7.41. Recall that a point $x \in X$ is *non-wandering* if for every open neighborhood V of x we have $V \cap \varphi^n(V) \neq \varnothing$ for some $n \in \mathbb{N}$. The set $\Omega(\varphi)$ of all non-wandering points is a closed, forward φ-invariant subset of X and $\operatorname{supp}\mu \subseteq \Omega(\varphi)$ for every $\mu \in \mathrm{Inv}(X, \varphi)$. Thus we have

$$P(\varphi, b) = P(\varphi|_{\Omega(\varphi)}, b|_{\Omega(p)})$$

and so when calculating topological pressure we may always assume that $X = \Omega(\varphi)$.

Remark 7.42. If φ is expanding, then the entropy map $\mathrm{Inv}(X, \varphi) \ni \mu \to h_\varphi(\mu) \in [0, \infty)$ is upper semi-continuous, see [**PU10**, Theorem 3.5.6]. Also the map $\mathrm{Inv}(X, \varphi) \ni \mu \to \int_X b\, d\mu$ is always upper semi-continuous, cf. the proof of [**Rue89**, Lemma 1.4]. Therefore the supremum $\sup_{\mu\in\mathrm{Inv}(X,\varphi)} \left(\int_X b\, d\mu + h_\varphi(\mu)\right)$ is in fact a maximum. In addition, using ergodic decomposition we have

$$P(\varphi, b) = \max_{\mu\in \mathrm{Inv}(X,\varphi)} \left(\int_X b\, d\mu + h_\varphi(\mu) \right) = \max_{\mu\in \mathrm{Erg}(X,\varphi)} \left(\int_X b\, d\mu + h_\varphi(\mu) \right) \in \mathbb{R} \cup \{-\infty\},$$

cf. [**PU10**, Corollary 3.4.3] or [**Wal82**, 9.10.1]. A measure $\mu \in \mathrm{Inv}(X, \varphi)$ in which the above maximum is attained is called an *equilibrium* state for φ and b. If $b \in C(X, \mathbb{R})$ is Hölder continuous and φ is open and *topologically transitive*, i.e. for any non-empty open sets $U, V \subseteq X$ we have $\varphi^n(U) \cap V \neq \varnothing$ for some $n \in \mathbb{N}$, then there is a unique equilibrium measure $\mu_{\varphi,b}$ for φ and b. This measure is the unique *Gibbs measure* for φ and b, and for any $y \in X$ we have

$$P(\varphi, b) = \left(\int_X b\, d\mu + h_\varphi(\mu) \right) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{x\in\varphi^{-1}(y)} \exp\left(\sum_{i=0}^{n-1} b(\varphi^i(y)) \right),$$

cf. [**BK21**, Theorem 2.3], see [**PU10**, Theorem 5.3.2].

Theorem 7.43 ([**BK21**, Theorem 4.6]). *Suppose that* $\varphi : X \to X$ *is an expanding open map on a metrizable compact space* X. *For any continuous* $c : X \to [0, \infty)$ *we have*

$$\ln r(\mathcal{L}_{c,\varphi}) = \max_{\mu\in \mathrm{Inv}(X,\varphi)} \left(\int_X \ln c\, d\mu + h_\varphi(\mu) \right) = \max_{\mu\in \mathrm{Erg}(X,\varphi)} \left(\int_X \ln c\, d\mu + h_\varphi(\mu) \right),$$

that is the spectral exponent of the transfer operator $\mathcal{L}_{c,\varphi}$, given by (7.3)*, is equal to the topological pressure $P(\varphi, \ln c)$ of (X, φ) with potential* $\ln c$.

REMARK 7.44. If φ is not necessarily expansive local homeomorphism than we always have $\ln r(\mathcal{L}_c) \leqslant P(\varphi, \ln c)$ but this inequality might be strict and the difference might be arbitrarily large, see [**BK21**, Lemma 4.5 and a comment below].

REMARK 7.45. Let φ be open expanding. The set of non-wandering points $\Omega(\varphi)$ is equal to the closure of the set of periodic points, see [**PU10**, Proposition 4.3.6], and $\varphi : \Omega(\varphi) \to \Omega(\varphi)$ has *spectral decomposition*, that is $\Omega(\varphi) = \bigsqcup_{j=1}^{N} X_j$ is a union of finitely many φ-invariant disjoint clopen sets X_j, $j = 1, ..., N$, such that for each j, the map $\varphi|_{X_j} : X_j \to X_j$ is topologically transitive, see [**PU10**, Theorem 4.3.8 and Corollary 4.2.4]. By [**BK21**, Corollary 4.7], cf. Remark 7.42, if c is such that $c|_{\Omega(\varphi)} > 0$ and $\ln c|_{\Omega(\varphi)}$ is Hölder continuous, then for each pair of restrictions $\varphi|_{X_j}$ and $\ln c|_{X_j}$ there is the unique Gibbs measure μ_j and we have

$$\ln r(\mathcal{L}_{c,\varphi}) = \max_{j=1,...,N} \int_{X_j} \ln c \, d\mu_j + h(\mu_j) = \max_{j=1,...,N} \lim_{n\to\infty} \frac{1}{n} \ln \sum_{x \in \varphi^{-n}(y_j)} \prod_{i=0}^{n-1} c(\varphi^i(x))$$

where $y_j \in X_j$ are arbitrary points, for $j = 1, ..., N$. In particular, if $\varphi : \Omega(\varphi) \to \Omega(\varphi)$ is topologically transitive, then there is the unique measure $\mu \in \operatorname{Inv}(X, \varphi)$, the Gibbs measure for $\varphi|_{\Omega(\varphi)}$ and $\ln c|_{\Omega(\varphi)}$, such that for all $y \in \Omega(\varphi)$ we have

$$\ln r(\mathcal{L}_{c,\varphi}) = \int_X \ln c \, d\mu + h_\varphi(\mu) = \lim_{n\to\infty} \frac{1}{n} \ln \sum_{x \in \varphi^{-n}(y)} \prod_{i=0}^{n-1} c(\varphi^i(x)).$$

COROLLARY 7.46. *Let $\mathcal{L}$ be a transfer operator of finite type for a surjective open expansive map $\varphi : X \to X$ on a compact metric space. For any $p \in [1, \infty]$ and $a \in C(X)$ we have*

$$\ln r(a\mathbf{T}_p) = \max_{\mu \in \operatorname{Erg}(X,\varphi)} \int_X \ln |a\varrho^{\frac{1}{p}}| \, d\mu + \frac{h_\varphi(\mu)}{p},$$

where ϱ is the cocycle associated to $\mathcal{L}$, cf. (7.3)*. If in addition $p < \infty$, $a|_{\Omega(\varphi)} \neq 0$ and $\ln |a\varrho^{\frac{1}{p}}|$ is Hölder continuous on $\Omega(\varphi)$, then for each system (X_j, φ) where $\Omega(\varphi) = \bigsqcup_{j=1}^{N} X_j$ is the spectral decomposition for $(\Omega(\varphi), \varphi)$, there is the Gibbs measure μ_j and then*

$$\ln r(a\mathbf{T}_p) = \max_{j=1,...,N} \int_X \ln |a\varrho^{\frac{1}{p}}| \, d\mu_j + \frac{h_\varphi(\mu_j)}{p}.$$

In particular, if $\varphi : \Omega(\varphi) \to \Omega(\varphi)$ is topologically transitive, then the Gibbs measure μ for $\varphi|_{\Omega(\varphi)}$ with potential $\ln(|a|^p\varrho)|_{\Omega(\varphi)}$ is the unique measure in $\operatorname{Inv}(X, \varphi)$ for which $\ln r(a\mathbf{T}_p) = \int_X \ln |a\varrho^{\frac{1}{p}}| \, d\mu + \frac{h_\varphi(\mu)}{p}$.

PROOF. For $p = \infty$ the assertion follows from Proposition 7.30 and Theorem 7.31. Let then $p < \infty$. By Proposition 7.30 we have $r(a\mathbf{T}_p) = \sqrt[p]{r(\mathcal{L}_{|a|^p\varrho,\varphi})}$. Hence by Theorem 7.43,

$$\ln r(a\mathbf{T}_p) = 1/p \ln r(\mathcal{L}_{|a|^p\varrho}, \varphi) = \max_{\mu \in \operatorname{Erg}(X,\varphi)} \int_X \ln |a\varrho^{\frac{1}{p}}| \, d\mu + \frac{h_\varphi(\mu)}{p}.$$

The second part follows from Remark 7.45. □

Theorem 7.34 allows us to establish the following relationship between the t-entropy and the Kolmogorov-Sinai entropy, which in turn allows us to view variational formulas in Corollary 7.46 as a special case of those in Corollary 7.36.

COROLLARY 7.47. *Let $\mathcal{L} : C(X) \to C(X)$ be a unital transfer operator for a unital injective endomorphism whose dual map $\varphi : X \to X$ is expanding and open. Then for each $\mu \in \mathrm{Inv}(X,\varphi)$ the t-entropy $\tau_{\mathcal{L}}(\mu)$ of $\mathcal{L}$ is given by*

$$\tau_{\mathcal{L}}(\mu) = \int_X \ln \varrho \, d\mu + h_\varphi(\mu),$$

where ϱ is the cocycle associated to $\mathcal{L}$ and $h_\varphi(\mu)$ is the Kolmogorov-Sinai entropy of μ. In particular, for every $\mu \in \mathrm{Inv}(X,\varphi)$ we have

$$h_\varphi(\mu) = \inf_{b \in C(X,\mathbb{R})} \ln r(\mathcal{L}_{e^b}) - \int_X \ln(e^b \varrho) \, d\mu,$$

where $L_{e^b} : C(X) \to C(X)$ is the weighted transfer operator $L_{e^b}(a) := L(e^b a)$, $a \in C(X)$.

PROOF. By [**ABL11**, Proposition 2.2], the functional $\lambda : C(X,\mathbb{R}) \ni b \mapsto \ln r(\mathcal{L}_{e^b}) = \ln r(\mathcal{L}_{b \ln \varrho, \varphi}) \in \mathbb{R} \cup \{-\infty\}$ is convex and continuous. Hence its Legendre transform $\lambda^* : C(X,\mathbb{R})^* \to \mathbb{R} \cup \{+\infty\}$, given by $\lambda(\mu^*) = \sup_{b \in C(X,\mathbb{R})} \mu(b) - \lambda(b)$, is a unique convex and lower semicontinuous functional on $C(X)^*$ such that

$$\lambda(b) = \sup_{\mu \in C(X)^*} \mu(b) - \lambda^*(b), \qquad b \in C(X,\mathbb{R}),$$

cf. [**ABL11**, Proposition 3.1]. The authors of [**ABL11**] write $S = -\lambda^*$ and call it dual entropy map for $\mathcal{L}$. By [**ABL11**, Proposition 4.2], the effective domain of λ^* is contained in $\mathrm{Inv}(X,\varphi)$. Hence λ^* is determined by its values on $\mathrm{Inv}(X,\varphi)$. The map $\mathrm{Inv}(X,\varphi) \ni \mu \to \int_X \ln \varrho \, d\mu + h_\varphi(\mu)$ is upper semicontinuous and affine, cf. [**Wal82**, Theorem 8.1]. By Theorem 7.43 we have

$$\lambda(b) = \max_{\mu \in C(X)^*} \mu(b) + \int_X \ln \varrho \, d\mu + h_\varphi(\mu).$$

Hence $S(\mu) = -\lambda^*(\mu) = \int_X \ln \varrho \, d\mu + h_\varphi(\mu)$. On the other hand, we have $\tau_{\mathcal{L}} = -\lambda^*|_{\mathrm{Inv}(X,\varphi)}$ by [**ABL11**, Theorem 5.6]. This gives the assertion. □

7.5. The spectrum

Combining previous results we describe the spectrum of the universal weighted composition operators $a\mathbf{T}_p$, $a \in C(X)$, in the crossed product algebra $F^p(\mathcal{L})$ associated with a transfer operator $\mathcal{L}$ and $p \in [1,\infty]$.

THEOREM 7.48. *Let $\varphi : X \to X$ be a local homeomorphism on a compact metrizable space X and let $\mathcal{L}$ be a transfer operator of finite type for φ, so that so that $\mathcal{L}(a)(y) =$*

$\sum_{x\in\varphi^{-1}(y)} \varrho(x)a(x)$, *where* $\varrho > 0$. *Assume that* φ *is topologically free. For every* $p \in [1,\infty]$ *and every* $a \in C(X)$ *we have*

$$\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) = \{z \in \mathbb{C} : |z| \leqslant r_0\} \cup \bigcup_{r\in R}\{z \in \mathbb{C} : r_- \leqslant |z| \leqslant r_+\}$$

where $0 \leqslant r_0 < r_- \leqslant r_+$ *for* $r \in R$, *and Riesz projectors induce a decomposition* $X = X_0 \sqcup \bigsqcup_{r\in R} X_r$ *into disjoint closed sets where for each* $r \in R$, $\varphi|_{X_r} : X_r \to X_r$ *is a homeomorphism,* $a : X_r \to \mathbb{C}\backslash\{0\}$ *does not vanish and*

$$r_- = \min_{\mu\in\mathrm{Erg}(X_r,\varphi)} \exp\int_{X_r} \ln|a\varrho^{\frac{1}{p}}|\,d\mu, \quad r_+ = \max_{\mu\in\mathrm{Erg}(X_r,\varphi)} \exp\int_{X_r} \ln|a\varrho^{\frac{1}{p}}|\,d\mu. \tag{7.17}$$

If X_0 *is non-empty and forward* φ*-invariant, then*

$$r_0 = \max_{\mu\in\mathrm{Inv}(X_0,\varphi)} \exp\left(\int_{X_0} \ln|a|\,d\mu + \frac{\tau_{\mathcal{L}|_{C(X_0)}}(\mu)}{p}\right). \tag{7.18}$$

In particular, if X *does not contain a non-empty clopen forward* φ*-invariant set on which the weight* a *is non-zero and* φ *is a homeomorphism, then*

$$\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) = \left\{z \in \mathbb{C} : |z| \leqslant \max_{\mu\in\mathrm{Inv}(X,\varphi)} \exp\left(\int_X \ln|a|\,d\mu + \frac{\tau_{\mathcal{L}}(\mu)}{p}\right)\right\}.$$

PROOF. That $\sigma(a\mathbf{T}_p)$ is invariant under rotations over zero follows from Corollary 7.24. Hence $\sigma(a\mathbf{T}_p)$ consists of a disk $\{z \in \mathbb{C} : |z| \leqslant r_0\}$ and possibly a number of annuli $\bigsqcup_{r\in R}\{z \in \mathbb{C} : r_- \leqslant |z| \leqslant r_+\}$.

For simplicity suppose first that the disk is non-empty and that $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ has exactly two components, so that $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) = \{z \in \mathbb{C} : |z| \leqslant r_0\} \cup \{z \in \mathbb{C} : r_- \leqslant |z| \leqslant r_+\}$ where $0 \leqslant r < r_- \leqslant r_+$. By Proposition 7.25, we have a decomposition $X = X_0 \sqcup X_r$ into clopen sets where $\mathbb{1}_{X_0}, \mathbb{1}_{X_r}$ are Riesz projectors for the corresponding parts of the spectrum of $a\mathbf{T}_p$, X_r is forward φ-invariant, $\varphi : X_r \to X_r$ is a homeomorphism and $a|_{X_r}$ is invertible in $C(X_r)$. Putting $\mathcal{G}(X_r,\varphi) := \{(x,N,z) \in \mathcal{G}(X,\varphi) : x \in X_r\}$ we have $\mathbb{1}_{X_r}\ell^p(\mathcal{G}(X,\varphi)) = \ell^p(\mathcal{G}(X_r,\varphi))$. Denoting by T the composition operator with the bijection $\mathcal{G}(X_r,\varphi) \ni (x,N,z) \mapsto (\varphi(x),N+1,z) \in \mathcal{G}(X_r,\varphi)$ is a bijection we get an invertible isometry on $\ell^p(\mathcal{G}(X_r,\varphi))$. Putting $\pi(a)\xi(x,N,z) := a(x)\xi(x,N,z)$ for $a \in C(X)$ and $\xi \in \ell^p(\mathcal{G}(X_r,\varphi))$ we get an isometric unital representation $\pi : C(X) \to B(\ell^p(\mathcal{G}(X_r,\varphi)))$ such that $T\pi(a) = \pi(\alpha(a))T$ for $a \in C(X)$. Moreover, by (7.8) we have

$$\mathbb{1}_{X_r} a\mathbf{T}_p|_{\ell^p(\mathcal{G}(X_r,\varphi))} = \pi(a\varrho^{\frac{1}{p}})T$$

for $(y,N,z) \in \mathcal{G}(X_r,\varphi)$ and $\xi \in \ell^p(\mathcal{G}(X_r,\varphi))$. Hence formulas (7.17) follow from Corollary 7.32. Now note that X_0 is necessarily backward φ-invariant. Hence if it is also forward φ-invariant it is φ-invariant. Then both $\mathcal{L}$ and α restrict to $C(X_0)$ and we have $\mathbb{1}_{X_0}\ell^p(\mathcal{G}(X,\varphi)) = \ell^p(\mathcal{G}(X_0,\varphi|_{X_0}))$. Thus the formula (7.18) hold by Corollary 7.36 applied to the system restricted to X_0.

Now let us consider the general case. Let $|\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)| := \{|z| : z \in \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)\} = [0,r_0] \cup \bigcup_{r\in R}[r_-,r+]$. For each $r \in R$ we may find sequences $\{r_n^-\}_{n=1}^\infty$, $\{r_n^+\}_{n=1}^\infty$ lying outside $|\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)|$ such that $r_n^- \nearrow r_-$ and $r_n^+ \searrow r_+$. Let $P_{r_n^-,r_n^+}$ be the Riesz projector corresponding to the part of spectrum of $a\mathbf{T}_p$ lying in $\{z \in \mathbb{C} : r_n^- < |z| < r_n^+\}$. As above, using Proposition 7.25, we see that $P_{r_n^-,r_n^+} = \mathbb{1}_{X_{r_n^-,r_n^+}} \in C(X)$ for a certain non-empty clopen φ-invariant set $X_{r_n^-,r_n^+}$ such that $\varphi : X_{r_n^-,r_n^+} \to X_{r_n^-,r_n^+}$ is a homeomorphism and $a|_{X_{r_n^-,r_n^+}} \neq 0$. We put

$$X_r := \bigcap_{n\in\mathbb{N}} X_{r_n^-,r_n^+}.$$

This is a non-empty closed and φ-invariant set, which does not depend on the choice of the sequences $\{r_n^-\}_{n=1}^\infty$, $\{r_n^+\}_{n=1}^\infty$. Moreover, $\varphi : X_r \to X_r$ is a homeomorphism. As above we see that $r_n^\pm$ are given by the variants of formulas (7.17) where X_r is replaced with X_{r_n}. Hence for each $n \in \mathbb{N}$ there are measures $\mu_n^-, \mu_n^+ \in \mathrm{Erg}(X_{r_n^-,r_n^+}, \varphi)$ such that

$$r_n^- < \exp \int_{X_{r_n^-,r_n^+}} \ln|a(x)|\, d\mu_n^- \leqslant r_- \leqslant r_+ \leqslant \exp \int_{X_{r_n^-,r_n^+}} \ln|a(x)|\, d\mu_n^+ < r_n^+.$$

Taking μ_0^-, $\mu_0^+ \in \mathrm{Inv}(X,\varphi)$ to be weak* limit points of the sequences $\{\mu_n^-\}_{n\in\mathbb{N}}$, $\{\mu_n^+\}_{n\in\mathbb{N}}$, we see that supports of μ_0^- and μ_0^+ are contained in $X_r = \bigcap_{n\in\mathbb{N}} X_{r_n^-,r_n^+}$, and

$$r_- = \exp\int_{X_r} \ln|a(x)|\, d\mu_0^- = \min_{\mu\in\mathrm{Inv}(X_r,\varphi)} \exp\int_{X_r} \ln|a|\, d\mu = \min_{\mu\in\mathrm{Erg}(X_r,\varphi)} \exp\int_{X_r} \ln|a|\, d\mu,$$

$$r_+ = \exp\int_{X_r} \ln|a(x)|\, d\mu_0^+ = \max_{\mu\in\mathrm{Inv}(X_r,\varphi)} \exp\int_{X_r} \ln|a|\, d\mu = \max_{\mu\in\mathrm{Erg}(X_r,\varphi)} \exp\int_{X_r} \ln|a|\, d\mu.$$

Similarly, we define $X_0 = \bigcap_{n\in\mathbb{N}} X_{0,r_n}$ where $\{r_n\}_{n=1}^\infty$ is a sequence lying outside $|\sigma(a\mathbf{T}_p)|$ such that $r_n \searrow r_0$ and $\mathbb{1}_{X_{0,r_n}} \in C(X)$ is the Riesz projector corresponding to the part of $\sigma(a\mathbf{T}_p)$ lying in $\{z \in \mathbb{C} : |z| < r_n\}$, $n \in \mathbb{N}$. Then X_0 is closed non-empty and backward φ-invariant. Clearly, the sets X_r, $r \in R \cup \{0\}$, form a decomposition of X. Moreover, if X_0 is (forward) φ-invariant, then all sets X_r are φ-invariant. Then, as in the previous paragraph, we conclude that each r_n is given by (7.18) where X_0 is replaced with X_{0,r_n}. Choose $\mu_n \in \mathrm{Inv}(X_{0,r_n}, \varphi)$ that maximizes $\max_{\mu\in\mathrm{Inv}(X_{0,r_n},\varphi)} \exp\left(\int_{X_{0,r_n}} \ln|a|\, d\mu + \frac{\tau_L(\mu)}{p}\right)$. Then

$$r_0 \leqslant \exp\left(\int_{X_{0,r_n}} \ln|a|\, d\mu_n + \frac{\tau_L(\mu_n)}{p}\right) < r_n.$$

Let $\mu_0 \in \mathrm{Inv}(X_{0,r_n}, \varphi)$ be a weak* limit point of $\{\mu_n\}_{n\in\mathbb{N}}$. Then the support of μ_0 is contained in $X_0 = \bigcap_{n\in\mathbb{N}} X_{0,r_n}$. This and the above inequalities imply (7.18). □

We now apply the above theorem to expanding maps.

LEMMA 7.49. *An expanding map $\varphi : X \to X$ on a compact metrizable space X is topologically free if and only if X has no isolated points that are periodic.*

PROOF. If there is a periodic isolated point, then its orbit is a clopen set showing that $\varphi : X \to X$ is not topologically free. Conversely, suppose that φ is not topologically free. Then there is a non-empty open set $U \subseteq X$ such that every point $x \in U$ is of period $n > 0$ (i.e. $\varphi^n(x) = x$ and $\varphi^k(x) \neq x$ for $k = 1, ..., n-1$). Take any non-empty open set V such that $\overline{V} \subseteq U$. Then $K = \bigcup_{k=0}^{n-1} \varphi^k(\overline{V})$ is forward φ-invariant and $\varphi : K \to K$ is an expansive homeomorphism. Hence K is finite by Schwartzman's theorem. Thus V is open and finite, and therefore it consists of isolated points. ☐

Corollary 7.50. *Let $\varphi : X \to X$ be an expanding map and X has no isolated points. Let $\mathcal{L}$ be a transfer operator of finite type for φ and ϱ the associated cocycle. For every $p \in [1,\infty]$ and every $a \in C(X)$ we have*

$$\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) = \left\{ z \in \mathbb{C} : |z| \leqslant \max_{\mu \in \mathrm{Erg}(X,\varphi)} \exp\left(\int_X \ln(|a\varrho^{\frac{1}{p}}|)\, d\mu + \frac{h_\varphi(\mu)}{p} \right) \right\}.$$

Proof. By Lemma 7.49, Schwartzman's theorem and the last part of Theorem 7.48, $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ is a disk. The formula for $r(a\mathbf{T}_p)$ is given in Corollary 7.46. ☐

Importance of the above spectra calculation lies in that for any covariant representation (π, T, T^*) of $\mathcal{L}$ in a unital L^p-operator algebra B we have

$$\sigma_B(\pi(a)T) \subseteq \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$$

because we have a unital homomorphism $\pi \rtimes T : F^p(\mathcal{L}) \to B$ that maps $a\mathbf{T}_p$ to $\pi(a)T$. Thus $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ serves as an upper bound for such spectra. Moreover, we have the following two conjectures that we want now to discuss:

Conjecture 7.51. *Let $\mathcal{L}$ be a transfer operator of finite type for a topologically free local homeomorphism $\varphi : X \to X$ on a compact metric space. Let (π, T, T^*) be a unital injective covariant representation of $\mathcal{L}$ in a unital L^p-operator algebra B, $p \in [1,\infty]$. Then the representation $\pi \rtimes T : F^p(\mathcal{L}) \to B$ is injective by Proposition 7.23. We conjecture that*

1. *the representation $\pi \rtimes T : F^p(\mathcal{L}) \to B$ is in fact isometric;*
2. *for any $a \in C(X)$ the spectrum of $\pi(a)T$ in B coincides with the spectrum of $a\mathbf{T}_p$:*

$$\sigma_B(\pi(a)T) = \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p).$$

Hence the spectrum of $a\mathbf{T}_p$ does not depend on the choice of an L^p-operator algebra containing $F^p(\mathcal{L})$ as a unital subalgebra.

We give three important results that support the above conjectures. Firstly, they are valid for $p = 2$:

Theorem 7.52. *Conjectures 7.51 are true for $p = 2$, in which case Theorem 7.48 recovers* [**BK21**, Theorem 7.2]*: If $\mathcal{L}$ is a transfer operator of finite type for a topologically free $\varphi : X \to X$ on a compact metric space, then for every $a \in C(X)$ there is a decomposition*

$$X = X_0 \sqcup \bigsqcup_{r \in R} X_r$$

into disjoint closed sets where for each $r \in R$, $\varphi|_{X_r} : X_r \to X_r$ is a homeomorphism and $a : X_r \to \mathbb{C}\backslash\{0\}$ does not vanish, such that for any injective unital representation $\pi : C(X) \to B(H)$ on a Hilbert space H and any isometry $T \in B(H)$ satisfying

$$T\pi(a)T^* = \pi(\mathcal{L}(a)), \qquad T\pi(a) = \pi(a \circ \varphi)T, \qquad \overline{\pi(C(X)TH} = H$$

we have

$$\sigma(\pi(a)T) = \{z \in \mathbb{C} : |z| \leqslant r_0\} \cup \bigcup_{r\in R}\{z \in \mathbb{C} : r_- \leqslant |z| \leqslant r_+\}$$

where $r_\pm$ are given by (7.17) *and r_0, if X_0 is invariant, is given by* (7.18).

Proof. The unital L^2-algebra B may be embedded as a unital subalgebra into a unital C^*-algebra C. Then we may view (π, T, T^*) as a representation in C. By Corollary 7.21 $F^2(\mathcal{L}) \cong F^2(\mathcal{G}(X,\varphi))$ is the groupoid C^*-algebra, cf. Theorem 5.33(3). Thus the associated representation $\pi \rtimes T : F^2(\mathcal{L}) \to C$ is an injective unital $*$-homomorphism between two C^*-algebras. Hence it is isometric and $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) = \sigma_C(\pi(a)T)$. Since $\sigma_B(\pi(a)T)$ has to lie between these two it is in fact equal to $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$. This shows that Conjectures 7.51 are true. The second part now follows from Theorem 7.48 by Remarks 7.11, 7.16 and Proposition 7.14. □

Conjecture 7.51(2) is true provided Conjecture 7.51(1) is valid:

Proposition 7.53. *Retain the assumptions of Conjecture 7.51. Let $a \in C(X)$ and assume that there is no non-empty clopen forward φ-invariant subset $X_0 \subseteq X$ such that $\varphi|_{X_0} : X_0 \to X_0$ is a homeomorphism and $a : X_0 \to \mathbb{C}\backslash\{0\}$. Then conjecture* (1) *in 7.51 implies* (2). *More specifically for any isometric unital embedding $F^p(\mathcal{L}) \hookrightarrow B$ into a unital L^p-operator algebra we have*

$$\sigma_B(a\mathbf{T}_p) = \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) = \left\{z \in \mathbb{C} : |z| \leqslant \max_{\mu\in \mathrm{Inv}(X,\varphi)} \exp\left(\left(\int_X \ln|a|\, d\mu + \frac{\tau_{\mathcal{L}}(\mu)}{p}\right)\right)\right\}.$$

Proof. By Theorem 7.48, $\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ is the disk described in the assertion. By the general spectral theory $\sigma_B(a\mathbf{T}_p) \subseteq \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ and $\partial\sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p) \subseteq \partial\sigma_B(a\mathbf{T}_p)$. Hence either $\sigma_B(a\mathbf{T}_p) = \sigma_{F^p(\mathcal{L})}(a\mathbf{T}_p)$ or $\sigma_B(a\mathbf{T}_p)$ is a circle. The latter would imply that $a\mathbf{T}_p$ is invertible and then by Lemma 7.26(3), a is invertible and so $\mathbf{T}_p$ is an invertible isometry in B. But then $T^* = T^{-1}$ and (TR1) implies that $\mathcal{L}$ is an automorphism, which contradicts our assumption that φ is not a homeomorphism. □

Finally we use Theorem 6.49 to prove Conjecture 7.51 when $\varphi : X \to X$ is a shift of finite type coming from a cofinal graph. In view of Remark it seems likely that this is a strong predictor that this theorem holds for every open expanding map on a zero dimensional space (or at least all shift of finite type).

Theorem 7.54. *Let $\varphi : X \to X$ be the shift map on the path space of a finite cofinal directed graph Q without sources. Let $\mathcal{L}$ be a transfer operator of finite type for φ and ϱ the associated cocycle, and let $p \in [1,\infty]$. For any unital injective covariant representation (π, T, T^*) of $\mathcal{L}$ in a unital L^p-operator algebra B we have a natural isometric isomorphisms*

$$F^p(\mathcal{L}) \cong B(\pi(C(X)), T, T^*) \cong F^p(Q),$$

this algebra is simple and for and every $a \in C(X)$ we have

$$\sigma_B(\pi(a)T) = \left\{z \in \mathbb{C} : |z| \leqslant \max_{\mu\in \mathrm{Erg}(X,\varphi)} \exp\left(\left(\int_X \ln(|a\varrho^{\frac{1}{p}}|)\, d\mu + \frac{h_\varphi(\mu)}{p}\right)\right)\right\}.$$

PROOF. The first part of the assertion follows from Theorem 6.49. The second follows from Proposition 7.53 and Corollary 7.50. ☐

The above theorem generalizes the main result of [**Bar24**], where the full shift on a finite alphabet was considered. We illustrate it on a concrete example:

EXAMPLE 7.55. Consider the shift map $\varphi(\mu_1, \mu_2, ...) := (\mu_2, \mu_3, ...)$ on the Cantor space $\Sigma := \{1, ..., n\}^{\mathbb{N}}$, $n > 1$, and let μ be the Bernoulli measure - the product measure on Σ of uniform distributions on $\{1, ..., n\}$. Let $p \in [1, \infty]$. Then the composition operator $T\xi = \xi \circ \varphi$ is an isometry on the space $E = L^p_\mu(\Sigma)$, which has a contractive left inverse given by $(T^*\xi)(y) = \frac{1}{n}\sum_{x \in \varphi^{-1}(y)} \xi(x)$. Let $A \cong C(\Sigma)$ be the algebra of multiplication operators by continuous functions: $(a\xi)(x) = a(x)\xi(x)$, $a \in C(\Sigma)$, $\xi \in E$. It follows from Theorem 7.54 that the Banach algebra generated by A, T and T^* is isomorphic to the L^p-Cuntz algebra $\mathcal{O}^p_n$ introduced in [**Phi12**]:

$$B(A, T, T^*) \cong \mathcal{O}^p_n,$$

and for any $a \in A$ we have

$$\sigma(aT) = \left\{ z \in \mathbb{C} : |z| \leqslant \max_{\mu \in \mathrm{Erg}(\Sigma, \varphi)} \exp\left(\int_\Sigma \ln \frac{|a|}{\sqrt[p]{n}}\, d\mu + \frac{h_\varphi(\mu)}{p} \right) \right\}.$$

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